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ASYMPTOTIC BEHAVIOR OF ECKHOFF'S METHOD FOR FOURIER SERIES CONVERGENCE ACCELERATION

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Abstract. The current paper considers the problem of recovering a function using a limited number of its Fourier coefficients. Specifically, a method based on Bernoulli-like polynomials suggested and developed by Krylov, Lanczos, Gottlieb and Eckhoff is examined. Asymptotic behavior of approximate calculation of the so-called "jumps" is studied and asymptotic L_2 constants of the rate of convergence of the method are computed.

Key words: Fourier *series, convergence acceleration,* Bernoulli *polynomials*
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1 Introduction

Denote by f_n the Fourier coefficients of a function f defined on the closed interval $[-1,1]$

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}\mathrm{d}x, \qquad n \in \mathbb{Z}. \tag{1.1}$$

To recover the original function f, we can use the formula

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i\pi nx}, \qquad x \in [-1,1]. \tag{1.2}$$

It is well known that for $f \in L_2(-1,1)$ the series (1.2) is convergent in the L_2 norm

$$||f|| = \left(\int_{-1}^{1} |f(x)|^2 \mathrm{d}x\right)^{1/2}.$$

For practical purposes, approximations are obtained by using only a finite number of Fourier coefficients $\{f_n\}$, $|n| \le N < \infty$. As is well known, when approximating f by truncated Fourier series

$$S_N(f) := \sum_{n=-N}^{N} f_n e^{i\pi nx}, \tag{1.3}$$

the involved error is strongly dependent on the smoothness of f. Approximation of a 2-periodic function $f \in C^{\infty}(\mathbb{R})$ by S_N is highly effective for $N \gg 1$. When the approximated function has a point of discontinuity, the above mentioned approximation leads to the Gibbs phenomenon, which degrades the quality of approximation.

Different methods of convergence acceleration have been suggested in the literature (see, for example, [5]-[8], [11]-[14], [19], [21]-[24] etc). Increasing the convergence rate of Fourier series by subtracting a polynomial representing the discontinuities in the function and some of its first derivatives was suggested by A.Krylov as early as in 1906[16] and later in 1964 by Lanczos[18],[19] (see also [1],[2], [15], [20] for exposition and references). More detailed investigation of this approach was performed by Eckhoff[5]−[8].

Let us consider a function $f : [-1,1] \to \mathbb{C}$ which, for some $q \geq 0$, has up to q piecewise continuous derivatives in $[-1,1]$ with points of singularity

$$-1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_M < 1.$$

Moreover the existence of one-sided limits of derivatives up to order q at each γ_j are assumed:

$$f^{(k)}(\gamma_j + 0) = \lim_{x \to \gamma_j + 0} f^{(k)}(x), \quad f^{(k)}(\gamma_j - 0) = \lim_{x \to \gamma_j - 0} f^{(k)}(x), \qquad k = 0, \cdots, q.$$

Denote by $A_k^j(f)$ the "jump" of the k-th derivative of the function f in the singularity point γ_j:

$$A_k^j(f) = f^{(k)}(\gamma_j - 0) - f^{(k)}(\gamma_j + 0), \qquad k = 0, \cdots, q, \ j = 1, \cdots, M. \tag{1.4}$$

Following Eckhoff, let us consider the following representation:

$$f(x) = F(x) + \sum_{j=1}^{M} \sum_{k=0}^{q-1} A_k^j(f) B_k(x - \gamma_j + 1), \tag{1.5}$$

where $B_k(x)$ are 2-periodic Bernoulli-like polynomials with Fourier coefficients

$$B_{k,n} = \begin{cases} 0, & n = 0 \\ \dfrac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n = \pm 1, \pm 2, \cdots \end{cases} \tag{1.6}$$

Then F is a relatively smooth function. Approximation of F by truncated Fourier series leads to the following approximation of f:

$$\begin{aligned} S_{N,q}(f) \quad = \quad & \sum_{n=-N}^{N} \left(f_n - \sum_{j=1}^{M} \sum_{k=0}^{q-1} A_k^j(f) B_{k,n} e^{i\pi n(1-\gamma_j)} \right) \\ & + \sum_{j=1}^{M} \sum_{k=0}^{q-1} A_k^j(f) B_k(x - \gamma_j + 1). \end{aligned} \tag{1.7}$$

In a series of papers (see [5]-[7]) Eckhoff constructs methods to find the points of singularities γ_j, jumps A_k^j and the approximation $S_{N,q}(f)$ with sufficient precision.

Suppose that the locations of the singularities $\gamma_1, \cdots, \gamma_M$ are calculated with sufficient accuracy. Now, the key problem for constructing the approximation (1.7) is the determination of singularity amplitudes. As was suggested by Eckhoff, these amplitudes can be calculated from the following minimization problem

$$\sum_{|k|=N-P}^{N} \left| f_n - \sum_{j=1}^{M} \sum_{k=0}^{q-1} A_k^j(f) B_{k,n} e^{i\pi n(1-\gamma_j)} \right|^2 \longrightarrow \min. \tag{1.8}$$

According to Eckhoff, if we choose $P \geq M(q+1)-1$, we will in most cases be guaranteed a system which at least is linearly independent and therefore determines the amplitudes uniquely. In [7] one can also find the following estimate for approximating jumps $\tilde{A}_k^j$ by solving (1.8):

$$\tilde{A}_k^j = A_k^j + O(N^{j-q-1}), \qquad N \to \infty. \tag{1.9}$$

There is no strict proof of the above estimate in the mentioned papers, though numerical tests suggest that it is true. In the current paper we will among others present a rigorous proof of the formula (1.9) in the particular case of one jump at a precisely known position.

The described method of reconstructing the function f from its Fourier coefficients f_n (we will call it Eckhoff's method) was further generalized by a number of authors (see [4], [9], [10], [17] and, for the multidimensional case, [21] and [22]).

In the paper we limit our discussion to 2-periodic functions with a single point γ_j of singularity in $[-1,1)$. Clearly, without loss of generality we may assume $\gamma_1 = -1$, i.e., f is smooth on $[-1,1]$ with up to q one-sided derivatives at the points $x = \pm 1$.

We also simplify the minimization problem (1.8) by taking exactly q equations into account for finding q unknowns. Thus (1.8) becomes a system of linear equations

$$f_n = \sum_{k=0}^{q-1} \widetilde{A}_k^1 B_{k,n}, \quad n = n_1, n_2, \cdots, n_q.$$

We investigate this system for various choices of the indices n_s and in Theorems 3.3-3.6 obtain asymptotic estimates of the error $\widetilde{A}_k^1 - A_k^1$. Using these, we deduce asymptotic estimates of the convergence rate of the approximation in this particular case. These results also provide the optimal choice of $n_1, \cdots, n_q$. Comparison to the case is given, when exact jumps are known. It is worth noting that numerical tests carried by the authors tend to agree with the theoretical estimates presented in the paper.

2 Asymptotic Behavior of Eckhoff's Method with Exact Jumps

Suppose $f:\mathbb{R}\to\mathbb{C}$ is a 2-periodic piecewise smooth function with up to q continuous derivatives on $[-1,1]$. From (1.4)-(1.7) we have

$$A_k(f)=f^{(k)}(1-0)-f^{(k)}(-1+0),\qquad k=0,\cdots,q. \tag{2.1}$$

$$f(x)=F(x)+\sum_{k=0}^{q-1}A_k(f)B_k(x), \tag{2.2}$$

$$S_{N,q}(f)=\sum_{n=-N}^{N}\left(f_n-\sum_{k=0}^{q-1}A_k(f)B_{k,n}\right)e^{i\pi nx}+\sum_{k=0}^{q-1}A_k(f)B_k(x). \tag{2.3}$$

Let us denote by $R_{N,q}$ the error of the above representation:

$$R_{N,q}(f)=f(x)-S_{N,q}(f).$$

The following theorem appears in [23]. Its proof illustrates the basic idea behind Krylov's and Lanczos's approach.

Theorem 2.1. *Suppose $f\in C^q[-1,1]$ for some $q\geq 0$ and $f^{(q)}$ is absolutely continuous in $[-1,1]$; then the following estimate holds:*

$$\lim_{N\to\infty}N^{q+\frac{1}{2}}\|R_{N,q}(f)\|=|A_q(f)|d_1(q), \tag{2.4}$$

where $d_1(q)=\dfrac{1}{\pi^{q+1}\sqrt{2q+1}}$.

Proof. By q-fold integration by parts in (1.1) we have the following:

$$f_n=\frac{(-1)^{n+1}}{2}\sum_{k=0}^{q-1}\frac{A_k(f)}{(i\pi n)^{k+1}}+\frac{1}{2(i\pi n)^q}\int_{-1}^{1}f^{(q)}(x)e^{-i\pi nx}\mathrm{d}x. \tag{2.5}$$

Therefore,

$$R_{N,q}(f)=\sum_{|n|>N}F_ne^{i\pi nx},$$

where

$$\begin{aligned}F_n&=\frac{1}{2(i\pi n)^q}\int_{-1}^{1}f^{(q)}(x)e^{-i\pi nx}\mathrm{d}x\\&=\frac{(-1)^{n+1}}{2}\frac{A_q(f)}{(i\pi n)^{q+1}}+\frac{1}{2(i\pi n)^{q+1}}\int_{-1}^{1}f^{(q+1)}(x)e^{-i\pi nx}\mathrm{d}x.\end{aligned} \tag{2.6}$$

Note that the second term is $o(n^{-q-1})$ as $n\to\infty$ according to the Riemann-Lebesgue theorem. From (2.6) we get

$$\|R_{N,q}\|^2=2\sum_{|n|>N}|F_n|^2=\frac{|A_q|^2}{\pi^{2q+2}}\sum_{n=N+1}^{\infty}\frac{1}{n^{2q+2}}+o(N^{-2q-1}),\quad N\to\infty.$$

This concludes the proof.

3 Computation of Jumps

The approximation $S_{N,q}$ described in the previous section has a serious drawback: it assumes that we can compute jumps of the reconstructed function before we even start to reconstruct it. To avoid this problem, Eckhoff suggested in [5]-[8] to compute approximate jump values $\widetilde{A}_k$ for $A_k(f)$ directly from the Fourier coefficients. As the Fourier coefficients F_n asymptotically decay faster than the coefficients $B_{k,n}$ according to (2.6), they can be discarded for large $|n|$. Hence, from (2.2) we derive the following system of linear equations for determining approximate jumps $\widetilde{A}_k$, $k = 0, \cdots, q-1$, of the function f:

$$f_n = \sum_{k=0}^{q-1} \widetilde{A}_k B_{k,n}, \qquad n = n_1, n_2, \cdots, n_q. \tag{3.1}$$

Thus, for any given N we assume to have chosen q different integer indices

$$n_1 = n_1(N), n_2 = n_2(N), \cdots, n_q = n_q(N)$$

for evaluating the system (3.1). Solving it we get the values $\widetilde{A}_k$, which, as we later prove, approximate the jumps $A_k(f)$ for large N. Throughout the paper we will suppose that

$$\alpha N \leq |n_s| \leq N, \qquad s = 1, \cdots, q \tag{3.2}$$

for some $0 < \alpha \leq 1$.

Now rewrite (3.1) in the form

$$2(-1)^{n_s+1} f_{n_s} i\pi n_s = \sum_{k=0}^{q-1} \widetilde{A}_k x_s^k, \qquad s = 1, \cdots, q, \tag{3.3}$$

where

$$x_s = \frac{1}{i\pi n_s}, \qquad s = 1, \cdots, q. \tag{3.4}$$

Denoting

$$y_s = 2(-1)^{n_s+1} f_{n_s} x_s^{-1}, \tag{3.5}$$

we can present (3.3) in the matrix form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{q-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{q-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_q & x_q^2 & \dots & x_q^{q-1} \end{pmatrix} \begin{pmatrix} \widetilde{A}_0 \\ \widetilde{A}_1 \\ \vdots \\ \widetilde{A}_{q-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix}. \tag{3.6}$$

There are well-known algorithms for solving linear equation system with a Vandermonde matrix, see e.g. [3].

Let $P_j(x)$ be the polynomial of degree $q-1$ defined by

$$P_j(x)=\prod_{\substack{n=1\\n\neq j}}^{q}\frac{x-x_n}{x_j-x_n}=\sum_{k=1}^{q}m_{jk}x^{k-1},\qquad j=1,\cdots,q, \tag{3.7}$$

where by m_{jk} we denote the coefficients of the polynomial $P_j(x)$. From the equations

$$P_j(x_i)=\sum_{k=1}^{q}m_{jk}x_i^{k-1}=\delta_{ij},\qquad i,j=1,\cdots,q, \tag{3.8}$$

where δ_{ij} is Kronecker's symbol, we see that the transpose of (m_{jk}) is the inverse of the Vandermonde matrix (x_i^{k-1}) on the left hand side of (3.6). Therefore the solution of (3.6) can be written as

$$\widetilde{A}_{j-1}=\sum_{k=1}^{q}m_{kj}y_k,\qquad j=1,\cdots,q. \tag{3.9}$$

Let us calculate the elements m_{kj} explicitly. We have

$$\prod_{\substack{n=1\\n\neq j}}^{q}(x-x_n)=\frac{1}{x-x_j}\prod_{n=1}^{q}(x-x_n)=\frac{1}{x-x_j}\sum_{n=0}^{q}\gamma_n x^n=\sum_{n=0}^{q-1}\beta_n x^n,$$

where by β_n and γ_n we denote the coefficients of the corresponding polynomials. After simple calculation we derive

$$\beta_n=-\frac{1}{x_j^{n+1}}\sum_{s=0}^{n}\gamma_s x_j^s,\qquad n=0,\cdots,q-1. \tag{3.10}$$

Substituting this into (3.7), we get

$$P_j(x)=-\frac{1}{\prod\limits_{\substack{n=1\\n\neq j}}^{q}(x_j-x_n)}\sum_{k=0}^{q-1}\frac{x^k}{x_j^{k+1}}\sum_{s=0}^{k}\gamma_s x_j^s,\qquad j=1,\cdots,q, \tag{3.11}$$

and from (3.7)

$$m_{jk}=-\frac{1}{x_j^k\prod\limits_{\substack{n=1\\n\neq j}}^{q}(x_j-x_n)}\sum_{s=0}^{k-1}\gamma_s x_j^s,\qquad k,j=1,\cdots,q. \tag{3.12}$$

From (3.9) and (3.12), we derive

$$\widetilde{A}_{j-1} = -\sum_{k=1}^{q} \frac{y_k}{x_k^j \prod_{\substack{n=1\\n\neq k}}^{q} (x_k - x_n)} \sum_{s=0}^{j-1} \gamma_s x_k^s, \qquad j = 1, \cdots, q. \tag{3.13}$$

Taking into account the definition of y_k in (3.5), we get

$$\widetilde{A}_j = 2\sum_{k=1}^{q} \frac{(-1)^{n_k} f_{n_k}}{x_k^{j+2} \prod_{\substack{n=1\\n\neq k}}^{q} (x_k - x_n)} \sum_{s=0}^{j} \gamma_s x_k^s, \qquad j = 0, \cdots, q-1. \tag{3.14}$$

Lemma 3.1. *Let $x_s = (i\pi n_s)^{-1}$ and $\alpha N \le |n_s| \le N$, $s = 1, \cdots, q$ for some $0 < \alpha \le 1$. Denote*

$$\omega_j(q) = \sum_{k=1}^{q} \frac{x_k^j}{\prod_{\substack{n=1\\n\neq k}}^{q} (x_k - x_n)}, \qquad j \ge 0; \tag{3.15}$$

then

(a) $\omega_j(q) = 0$ *for* $j = 0, \cdots, q-2$;

(b) $\omega_{q-1}(q) = 1$;

(c) $\omega_j(q) = O(N^{q-j-1})$ *when* $N \to \infty$ *forevery* $j \ge 0$.

Proof. From the fact that $(q-1)$-th order Lagrange polynomial interpolation is exact for the monomials x^k for $k = 0, \cdots, q-1$, we have

$$x^j = \sum_{k=1}^{q} x_k^j \prod_{\substack{s=1\\s\neq k}}^{q} \frac{x - x_s}{x_k - x_s} = \sum_{s=1}^{q} b_s(j) x^{s-1}, \qquad j = 0, \cdots, q-1, \tag{3.16}$$

where $b_s(j)$ are the coefficients of the corresponding polynomial. Now note that $b_q(j) = \omega_j(q)$. This observation concludes claims (a) and (b) of the lemma. For the claim (c), note that

$$\omega_j(q) = \sum_{k=1}^{q} \underset{z=x_k}{res}\, \phi_j(z) = \frac{1}{2\pi i} \int_{|z|=\frac{1}{\alpha N}} \phi_j(z) \mathrm{d}z, \tag{3.17}$$

where

$$\phi_j(z) = \frac{z^j}{\prod_{s=1}^{q} (z - x_s)}.$$

From (3.17), we have

$$|\omega_j(q)| \le \frac{1}{2\pi (\alpha N)^{j+1}} \int_0^{2\pi} \frac{d\varphi}{\prod_{s=1}^{q} \left| \frac{e^{i\varphi}}{\alpha N} - \frac{1}{i\pi n_s} \right|} = O(N^{q-j-1}), \quad N \to \infty.$$

We will often use the following trivial fact:

Lemma 3.2. *Suppose that the indices n_s satisfy the condition* (3.2) *and sthat γ_j is the j-th coefficient of the polynomial $\prod_{s=1}^{q}(x-x_s)=\sum_{j=0}^{q}\gamma_j x^j$. Then*

$$\gamma_j = O(N^{-q+j}), \quad j=0,\cdots,q-1, \quad N\to\infty. \tag{3.18}$$

Proof. It easily follows from Viet's formula.

The following theorems show how well the approximate values $\tilde{A}_k$ determined from (3.1) to jumps $A_k(f)$ of f dependclosely on the choice of the indices n_s. By the multiplicity of some number x in a sequence $x_1,\ldots,x_m$ we mean the number of indices i for which $x_i=x$.

Theorem 3.3. *Suppose the indices $n_s=n_s(N)$ are chosen so that*

$$\lim_{N\to\infty}\frac{n_s}{N}=c_s\neq 0, \qquad s=1,\cdots,q. \tag{3.19}$$

Let α be the greatest multiplicity of the number in the sequence $c_1,c_2,\cdots,c_q$.

Now, for $f\in C^{q+\alpha-1}[-1,1]$ such that $f^{(q+\alpha-1)}$ is absolutely continuous on $[-1,1]$, the following estimate holds

$$\widetilde{A}_j(f)=A_j(f)-A_q(f)\frac{\chi_j}{(i\pi N)^{q-j}}+o(N^{-q+j}), \qquad N\to\infty, \quad j=0,\cdots,q-1, \tag{3.20}$$

where χ_j are the coefficients of the polynomial

$$\prod_{s=1}^{q}\left(x-\frac{1}{c_s}\right)=\sum_{s=0}^{q}\chi_s x^s. \tag{3.21}$$

Proof. By $(q+\alpha)$-fold integration by parts in (1.1), we have the following:

$$\begin{aligned} f_{n_k} &= \frac{(-1)^{n_k+1}}{2}\sum_{s=0}^{q-1}\frac{A_s}{(i\pi n_k)^{s+1}}+\frac{A_q(-1)^{n_k+1}}{2(i\pi n_k)^{q+1}} \\ &\quad +\frac{(-1)^{n_k+1}}{2}\sum_{s=q+1}^{q+\alpha-1}\frac{A_s}{(i\pi n_k)^{s+1}}+\frac{1}{2(i\pi n_k)^{q+\alpha}}\int_{-1}^{1}f^{(q+\alpha)}(x)e^{-i\pi n_k x}\mathrm{d}x. \end{aligned} \tag{3.22}$$

Substituting this into (3.14) and taking into account the definition of ω_j in Lemma 3.1, we get

$$\begin{aligned} \widetilde{A}_j &= A_j - A_q\sum_{\ell=0}^{j}\gamma_\ell\omega_{q-j+\ell-1}(q)-\sum_{s=q+1}^{q+\alpha-1}A_s\sum_{\ell=0}^{j}\gamma_\ell\omega_{s-j+\ell-1}(q) \\ &\quad +\sum_{k=1}^{q}\frac{(-1)^{n_k}\varepsilon_{n_k}x_k^{q+\alpha-2-j}}{\prod_{s=1, s\neq k}^{q}(x_k-x_s)}\sum_{\ell=0}^{j}\gamma_\ell x_k^\ell, \end{aligned} \tag{3.23}$$

where

$$\varepsilon_{n_k} = \int_{-1}^{1} f^{(q+\alpha)}(x) e^{-i\pi n_k x} \mathrm{d}x = o(1), \qquad N \to \infty.$$

Now, according to claims (a) and (b) of Lemma 3.1,

$$A_q \sum_{\ell=0}^{j} \gamma_\ell \omega_{q-j+\ell-1}(q) = A_q \gamma_j, \qquad j = 0, \cdots, q-1.$$

For the second sum in (3.23), we use Lemma 3.2 and the third claim of Lemma 3.1 to obtain

$$\sum_{s=q+1}^{q+\alpha-1} A_s \sum_{\ell=0}^{j} \gamma_\ell \omega_{s-j+\ell-1}(q) = O(N^{-q+j-1}), \qquad N \to \infty.$$

For the third sum note that

$$\left| \frac{1}{\prod\limits_{\substack{s=1\\ s\neq k}}^{q} (x_k - x_s)} \right| \leq \frac{\pi^{q-1} N^{2q-2}}{\prod\limits_{\substack{s=1\\ s\neq k}}^{q} |n_k - n_s|} \leq \text{const} \frac{N^{2q-2}}{N^{q-\alpha}} = O(N^{q+\alpha-2}), \qquad N \to \infty \tag{3.24}$$

as $|n_k - n_s| \geq 1$ whenever k and s differ and $|n_k - n_s| \geq CN$ whenever c_s is different from c_k, which happens at least for $q-\alpha$ indices s. Also, from Lemma 3.2, we have $\gamma_l x_k^l = O(N^{-q})$. Therefore, the third sum is $o(N^{-q+j})$ as $N \to \infty$. Collecting all of the above estimates we obtain

$$\widetilde{A}_j(f) = A_j(f) - A_q(f)\gamma_j + o(N^{-q+j}).$$

Now note that according to Viet's formula,

$$N^{q-j}\gamma_j - \frac{\chi_j}{(i\pi)^{q-j}} = \left(-\frac{1}{i\pi}\right)^{q-j} \sum_{1 \leq s_1 < \cdots < s_{q-j} \leq q} \left(\frac{N}{n_{s_1}} \cdots \frac{N}{n_{s_{q-j}}} - \frac{1}{c_{s_1}} \cdots \frac{1}{c_{s_{q-j}}} \right),$$

which tends to 0 as $N \to \infty$.

The following theorem somewhat strengthens the claim of Theorem 3.3, making the error estimates $O(N^{-q+j-1})$ instead of $o(N^{-q+j})$. This is possible to do if we require higher order derivatives for the function f and if we impose a rate of convergence of $\frac{1}{N}$ on the sequences $\frac{n_s}{N}$. The proof is omitted as it mimics the proof of Theorem 3.3.

Theorem 3.4. *Suppose the indices $n_s = n_s(N)$ are chosen so that*

$$\frac{n_s}{N} = c_s + O\left(\frac{1}{N}\right), \qquad N \to \infty \tag{3.25}$$

for each $s = 1, \cdots, q$, where c_s are non-zero constants. Let α be the greatest multiplicity of athe number in the sequence $c_1, c_2, \cdots, c_q$.

For $f \in C^{q+\alpha}[-1,1]$ with absolutely continuous derivative $f^{(q+\alpha)}$ on $[-1,1]$, the following estimate holds

$$\widetilde{A}_j(f) = A_j(f) - A_q(f)\frac{\chi_j}{(i\pi N)^{q-j}} + O(N^{-q+j-1}), \quad N \to \infty,\ j = 0, \cdots, q-1, \tag{3.26}$$

where the constants χ_j are defined as in Theorem 3.3.

If we omit any conditions on the indices n_s besides (3.2), we can still calculate the rate of convergence of $\widetilde{A}_j$. The following theorem so does exactly.

Theorem 3.5. *Suppose that $f \in C^{2q-1}[-1,1]$ and $f^{(2q-1)}$ is absolutely continuous on $[-1,1]$ for some $q \geq 1$. Then, if (3.2) is true, the following estimate holds:*

$$\widetilde{A}_j = A_j - A_q\frac{\chi_j}{(i\pi N)^{q-j}} + o(N^{-q+j}), \qquad j = 0, \cdots, q-1, \quad N \to \infty, \tag{3.27}$$

where the constants χ_j are defined as in Theorem 3.3.

Proof. We proceed as in the proof of Theorem 3.3 by taking $\alpha = q$. When estimating the third sum in (3.23) we use the following:

$$\left|\frac{1}{\prod\limits_{\substack{s=1\\ s\neq k}}^{q}(x_k - x_s)}\right| \leq \frac{\pi^{q-1}N^{2q-2}}{\prod\limits_{\substack{s=1\\ s\neq k}}^{q}|n_k - n_s|} \leq \pi^{q-1}N^{2q-2} = O(N^{2q-2}), \qquad N \to \infty$$

as $|n_k - n_s| \geq 1$ whenever k and s differ.

Again, we may replace $o(N^{-q+j})$ in (3.27) by $O(N^{-q+j-1})$ for functions f with an absolutely continuous derivative $f^{(2q)}$ on $[-1,1]$. This leads to the following:

Theorem 3.6. *Suppose that $f \in C^{2q}[-1,1]$ and $f^{(2q)}$ is absolutely continuous on $[-1,1]$ for some $q \geq 1$. Then, if (3.2) is true, the following estimate holds:*

$$\widetilde{A}_j = A_j - A_q\frac{\chi_j}{(i\pi N)^{q-j}} + O(N^{-q+j-1}), \qquad j = 0, \cdots, q-1, \quad N \to \infty. \tag{3.28}$$

4 Asymptotic L_2 Error Estimates

As in (2.3), let us denote

$$\widetilde{S}_{N,q}(f) = \sum_{n=-N}^{N}\left(f_n - \sum_{k=0}^{q-1}\widetilde{A}_k B_{k,n}\right)e^{i\pi nx} + \sum_{k=0}^{q-1}\widetilde{A}_k B_k(x) \tag{4.1}$$

and

$$\widetilde{R}_{N,q}(f) = f(x) - \widetilde{S}_{N,q}(f).$$

In the current section we will study the asymptotic behavior of the approximation (4.1) for different choices of the points n_s where the system (3.1) is evaluated.

Theorem 4.1. *Suppose that the conditions of Theorem* 3.3 *are valid. Then the following estimate holds:*

$$\lim_{N\to\infty} N^{q+\frac{1}{2}}\|\widetilde{R}_{N,q}(f)\| = |A_q(f)|d_2(q), \tag{4.2}$$

where

$$d_2(q) = \frac{1}{\sqrt{2}\,\pi^{q+1}}\left(\int_{-1}^{1}\prod_{s=1}^{q}\left(x-\frac{1}{c_s}\right)^2 \mathrm{d}x\right)^{1/2}. \tag{4.3}$$

Proof. It is easy to check (see the proof of Theorem 2.1) that

$$\begin{aligned}\widetilde{R}_{N,q}(f) &= R_{N,q}(f)+\sum_{|n|>N} e^{i\pi nx}\sum_{k=0}^{q-1}\left(A_k-\widetilde{A}_k\right)B_{k,n}\\ &= \sum_{|n|>N} e^{i\pi nx}\sum_{k=0}^{q-1}\left(A_k-\widetilde{A}_k\right)B_{k,n}+A_q\sum_{|n|>N}B_{q,n}e^{i\pi nx}+\theta_N(x),\end{aligned} \tag{4.4}$$

where $\|\theta_N\| = o(N^{-q-1/2})$. On the other hand, for $n>N$ we have

$$\begin{aligned}\sum_{k=0}^{q-1}(A_k-\widetilde{A}_k)B_{k,n} &= \frac{A_q(-1)^{n+1}}{2i\pi n}\sum_{k=0}^{q-1}\frac{\gamma_k}{(i\pi n)^k}+\frac{o(N^{-q})}{n}\\ &= \frac{A_q(-1)^{n+1}}{2i\pi n}\left(\prod_{k=1}^{q}\left(\frac{1}{i\pi n}-\frac{1}{i\pi n_k}\right)-\frac{1}{(i\pi n)^q}\right)+\frac{o(N^{-q})}{n}\\ &= \frac{A_q(-1)^{n+1}}{2(i\pi n)^{q+1}}\left(\prod_{k=1}^{q}\left(1-\frac{n}{n_k}\right)-1\right)+\frac{o(N^{-q})}{n}.\end{aligned} \tag{4.5}$$

From (4.5) we have

$$\|\widetilde{R}_{N,q}(f)\|^2 = \frac{|A_q|^2}{2\pi^{2q+2}}\sum_{|n|>N}\left|\frac{1}{n^{q+1}}\prod_{k=1}^{q}\left(1-\frac{n}{n_k}\right)\right|^2+o(N^{-2q-1}), \qquad N\to\infty.$$

After multiplying the above equation by N^{2q+1}, the sum on the right hand side is exactly the Riemann sum of the following integral

$$\frac{1}{2\pi^{2q+2}}\int_{\mathbb{R}\setminus(-1,1)}\frac{1}{t^{2q+2}}\prod_{s=1}^{q}\left(1-\frac{t}{c_s}\right)^2\mathrm{d}t. \tag{4.6}$$

Now, substituting $t=1/x$ we get (4.3).

Remark 4.2. Note that the rate of convergence in Theorem 2.1 is the same as in Theorem 4.1, i.e., approximate calculation of jumps does not degrade the rate of convergence.

In [1], the following values are suggested for the indices n_s:

$$N, -N, \frac{1}{2}N, -\frac{1}{2}N, \frac{2}{3}N, -\frac{2}{3}N, \frac{3}{4}N, -\frac{3}{4}N, \cdots.$$

In Table 1, we show the numerical values of the ratio $d_2(q)/d_1(q)$ for different values of q for the above choice of the values n_s, calculated according to Theorems 2.1 and 4.1. The numbers in the table show the deficiency of the KGE method with the given choice of n_s as compared to the case when exact jumps are known.

Table 1 Numerical values of the ratio $d_2(q)/d_1(q)$ for different values of q

q	1	2	3	4	5	6	7	8
$d_2(q)/d_1(q)$	2	1.6	3.9	8.5	14.4	21.6	31.9	41.4

From the representation (4.3), we see that the minimal value of $d_2(q)$ and hence the best asymptotic approximation is obtained when $c_s = \pm 1$ for each $s = 1, \cdots, q$. The following theorem explicitly gives the value of $d_2(q)$ in that case.

Theorem 4.3. *Suppose the conditions of Theorem* 3.3. *are valid and moreover,* $c_s = \pm 1$ *for all* $s = 1, \cdots, q$. *Let* n *be the multiplicity of* 1 *in the sequence* $c_1, \cdots, c_q$. *Then*

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|\widetilde{R}_{N,q}(f)\| = |A_q(f)| d_3(q), \tag{4.7}$$

where

$$d_3(q) = \frac{2^q}{\pi^{q+1}\sqrt{(2q+1)\binom{2q}{2n}}}. \tag{4.8}$$

Proof. Applying Theorem 4.1 we obtain

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|\widetilde{R}_{N,q}(f)\| = |A_q(f)| d_3(q),$$

where

$$\begin{aligned} d_3(q) &= \frac{1}{\sqrt{2}\,\pi^{q+1}} \left(\int_{-1}^{1} \prod_{s=1}^{q} \left(x - \frac{1}{c_s}\right)^2 \mathrm{d}x \right)^{1/2} \\ &= \frac{1}{\sqrt{2}\,\pi^{q+1}} \left(\int_{-1}^{1} (x-1)^{2n}(x+1)^{2(q-n)} \mathrm{d}x \right)^{1/2} \\ &= \frac{1}{\sqrt{2}\,\pi^{q+1}} \left(\frac{2^{2q+1}(2n)!(2q-2n)!}{(2q+1)!} \right)^{1/2}. \end{aligned}$$

We will consider a few choices of the indices n_s which are used in literature for recovering the jumps of the function f (see [1], [5],[6], [21] etc). The first choice is the simplest one:

$$N-c \le n_s \le N, \qquad s=1,\cdots,q \tag{4.9}$$

for some constant c. In this case we get the following approximation constant:

$$d_4(q) = \frac{1}{\pi^{q+1}}\frac{2^q}{\sqrt{2q+1}}. \tag{4.10}$$

Comparing with the case when exact jumps are known gives

$$\frac{d_4(q)}{d_1(q)} = 2^q.$$

The second choice is more symmetric. Denote $m = \left[\frac{q}{2}\right]$ and take

$$\begin{aligned} -N \le n_s \le -N+c, \qquad & s=1,\cdots,m, \\ N-c \le n_s \le N, \qquad & s=m+1,\cdots,q, \end{aligned} \tag{4.11}$$

where c is an arbitrary constant. For this choice we get the approximation constant

$$d_5(q) = \begin{cases} \dfrac{2^q q!}{\pi^{q+1}}\dfrac{1}{\sqrt{(2q+1)!}}, & q=2m, \\ \dfrac{2^q q!}{\pi^{q+1}}\dfrac{\sqrt{q+1}}{\sqrt{q(2q+1)!}}, & q=2m+1. \end{cases} \tag{4.12}$$

The ratio d_5/d_1, which shows the efficiency of the approximation $S_{N,q}$ compared to $\widetilde{S}_{N,q}$ for the choice (4.11) of the indices n_s, can be estimated using Stirling's formula and it gives

$$\frac{d_5(q)}{d_1(q)} \approx \begin{cases} (\pi q)^{1/4}, & q=2m, \\ (\pi q)^{1/4}\sqrt{1+\frac{1}{q}}, & q=2m+1. \end{cases}$$

It is interesting to note that the above estimate provides an accuracy of the order 10^{-1} starting from $q=1$.

The choice of indices n_s given by (4.11) is an optimal one. The following theorem states the fact in more precise terms.

Theorem 4.4. *Suppose that $f \in C^{2q-1}[-1,1]$ and $f^{(2q-1)}$ is absolutely continuous for some $q \ge 1$. Then, if (3.2) is true, the following estimate holds:*

$$||\widetilde{R}_{N,q}(f)|| = O(N^{-q-1/2}), \qquad N \to \infty. \tag{4.13}$$

On the other hand,

$$\liminf_{N\to\infty} N^{q+\frac{1}{2}} \|\widetilde{R}_{N,q}(f)\| \geq |A_q(f)|\, d_5(q),$$

where $d_5(q)$ *is defined as in* (4.12).

Proof. Taking into account (3.27) and Lemma 3.2, we obtain

$$\sum_{k=0}^{q-1}(A_k - \widetilde{A}_k)B_{k,n} = \frac{O(N^{-q})}{n}, \quad |n| > N, \quad N \to \infty.$$

Substituting this into (4.4) we get (4.13). To prove the second claim of the theorem it is sufficient to realize that for any sequence N_k such that $N_k^{q+\frac{1}{2}} \|\widetilde{R}_{N_k,q}(f)\|$ converges there is a subsequence N_{k_j} for which $\dfrac{n_s(N_{k_j})}{N_{k_j}}$ converges for all s. We now use the Theorems 4.1 and 4.3 to get the desired inequality.

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