CONVERGENCE ACCELERATION FOR FOURIER SERIES

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The paper studies convergence acceleration for Fourier series based on Pade approximants. An application to expansions by eigenfunctions of a boundary value problem for a first order model differential equation with discontinuous coefficient is considered and numerical results discussed.

§1. INTRODUCTION

It is well known that approximation of a 2-periodic $f \in C^\infty(R)$ function by truncated Fourier series (partial sum)

$$S_N(f) := \sum_{n=-N}^{N} f_n e^{i\pi nx}, \quad f_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi nx} dx \tag{1}$$

is highly effective. When the approximated function has a discontinuity point, this truncation procedure leads to the Gibbs phenomena. To counter them, different solutions have been suggested in the literature (see [2], [12, 13] and the references therein). Thus A. Krylov in 1906 [14] and in 1966 by Lanczos [15] suggested subtracting a polynomial representing the discontinuities of the function and some of its derivatives. In [15] the correction polynomial was a linear combination of Bernoulli polynomials. In a series of papers [3, 5 – 8, 10, 11] Gottlieb and Eckhoff developed this method for practical realizations. Further we refer to this as Polynomial (or P-) method. Another way suggested in a general form by Cheney [4], is Fourier-Pade approximation which uses Pade approximants [1]. Other trigonometric–rational investigations were carried out in [9], [16].

In [17, 19] and [20] Pade approximants were applied to asymptotic expansion of Fourier coefficients. This approach leads to quasipolynomial approximation (QP-method) and actually generalizes the

P-method.

The present paper applies Fourier-Pade approximants to P and QP approximations for additional acceleration of convergence and applies that approach to expansions by the eigenfunctions of a model problem for some differential equation (see [21]) with a non-smooth coefficients.

§2. POLYNOMIAL-PADE (PP-) APPROXIMATIONS

First we describe P-approximation. Suppose $f \in C^q[-1,1]$, $q \geq 0$ and denote

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \ k = 0, \cdots, q.$$

According to asymptotic expansion of Fourier coefficients

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q-1} \frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^{1} f^{(q)}(x) e^{-i\pi n x} dx \tag{2}$$

the function f can be split into two parts

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{3}$$

where F is a relatively smooth function with Fourier coefficients $F_n = o(n^{-q})$, $n \to \infty$ and $B_k(x)$ are 2-periodic Bernoulli polynomials with Fourier coefficients

$$B_{k,n} = \begin{cases} 0, & n = 0 \\ \dfrac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n = \pm 1, \pm 2, \ldots \end{cases}$$

Approximation of F by $S_N(F)$ leads to *Polynomial (P-) approximation*

$$S_{q,N}(f) = S_N(F) + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{4}$$

where the Fourier coefficients of F can be found from (3)

$$F_n = f_n - \sum_{k=0}^{q-1} A_k(f) B_{k,n}.$$

We will write $S_{0,N}(f) \equiv S_N(f)$.

Now we apply Fourier-Pade approximation for additional acceleration of $S_N(F)$. Following [16] we consider a finite sequence of complex numbers $\theta := \{\theta_k\}_{|k|=1}^{p}$, $p \geq 1$ and denote

$$\Delta_n^0(\theta, F) = F_n,$$

$$\Delta_n^k(\theta, F) = \Delta_n^{k-1}(\theta, F) + \theta_{k\, sgn(n)} \Delta_{(|n|-1)sgn(n)}^{k-1}(\theta, F), \quad k \geq 1,$$

where $sgn(n) = 1$ if $n \geq 0$ and $sgn(n) = -1$ if $n < 0$.

From (3), (4), we get

$$
\begin{aligned}
&R_{q,N}(f) := f(x) - S_{q,N}(f) = F(x) - S_N(F) = R_N^+(F) + R_N^-(F), \\
&R_N^+(F) := \sum_{n=N+1}^{\infty} F_n e^{i\pi n x}, \quad R_N^-(F) := \sum_{n=-\infty}^{-N-1} F_n e^{i\pi n x}.
\end{aligned} \tag{5}
$$

It is easy to check that

$$
R_N^+(F) = -\frac{\theta_1 F_N \, e^{i\pi(N+1)x}}{1+\theta_1 e^{i\pi x}} + \frac{1}{1+\theta_1 e^{i\pi x}} \sum_{n=N+1}^{\infty} \Delta_n^1(\theta, F) e^{i\pi n x}.
$$

Reiteration of this transformation p times leads to the expansion

$$
\begin{aligned}
R_N^+(F) = &- e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, F)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} + \\
&+ \frac{1}{\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, F) e^{i\pi n x}.
\end{aligned} \tag{6}
$$

Similar expansion of $R_N^-(F)$ reduces to the following *Polynomial-Pade (PP-) approximation* [16]

$$
\begin{aligned}
S_{p,q,N}(f) :=& S_N(F) - e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, F)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} - \\
&- e^{-i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_{-k} \Delta_{-N}^{k-1}(\theta, F)}{\prod_{s=1}^{k}(1+\theta_{-s} e^{-i\pi x})} + \sum_{k=0}^{q-1} A_k(f) B_k(x).
\end{aligned}
$$

It is natural to put $S_{0,q,N}(f) \equiv S_{q,N}(f)$.

There are different ways for determination of the vector θ. One option is Fourier-Pade method where the vector θ is found as a solution of the system

$$
\Delta_n^p(\theta, F) = 0, \quad n = -N-p, \cdots, -N-1, N+1, \cdots, N+p.
$$

Another option is connected with the following theorem, where $\|f\| = \left(\int_{-1}^{1} |f(x)|^2 dx\right)^{1/2}$ denotes the L_2-norm.

Theorem 1 [16]. Suppose $f \in C^{q+p}[-1,1]$, *for some* $q \geq 0$, $p \geq 1$, *and* $f^{(q+p)}$ *is absolutely continuous on* $[-1,1]$. *If*

$$
\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \; k = 1, \cdots, p, \; \tau_k > 0, \; \tau_j \neq \tau_i, \; j \neq i;
$$

then

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|f(x) - S_{p,q,N}(f)\| = |A_q(f)| c_p(q),$$

where

$$c_p(q) = \frac{1}{\pi^{q+1}} \left(\int_1^\infty |\phi_{p,q}(t)|^2 \, dt \right)^{1/2},$$

$$\phi_{p,q}(t) := \frac{(-1)^p}{t^{q+1}} - \frac{1}{q!} \sum_{j=1}^{p} \frac{e^{-\tau_j(t-1)}}{\prod_{\substack{i=1\\ i\neq j}}^{p} (\tau_i - \tau_j)} \sum_{k=0}^{p} \gamma_k(p)(-1)^{k+1} \sum_{m=0}^{p-k-1} (q+p-k-m-1)! \tau_j^m$$

and $\gamma_k(p)$ *are defined by the identity*

$$\prod_{k=1}^{p} (1 + \tau_k x) \equiv \sum_{k=0}^{p} \gamma_k(p) x^k. \tag{7}$$

It can be easily shown that for $p = 0$ and $f \in C^q[-1,1]$, for some $q \geq 0$, with absolutely continuous q-th derivative

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|f(x) - S_{q,N}(f)\| = |A_q(f)| c_0(q), \quad c_0(q) = \frac{1}{\pi^{q+1}\sqrt{2q+1}}.$$

In Table 1 we represent some results from [16] on the choice of parameters τ_k that minimize the L_2-error for $p = 3$. The ratio $c_0(q)/c_3(q)$ describes effectiveness of L_2-optimal rational approximation $S_{p,q,N}(f)$ compared to $S_{q,N}(f)$ for $N >> 1$.

q	1	2	3	4	5	6
$c_3(q)$	0.00095	0.00007	$9 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	$2 \cdot 10^{-7}$	$4 \cdot 10^{-8}$
$c_0(q)/c_3(q)$	61.3	185.1	411.6	771.8	1296.7	2017.4
τ_1	0.2510	0.6382	1.1230	1.6730	2.2699	2.9023
τ_2	1.28553	2.2362	3.2067	4.1868	5.1725	6.1617
τ_3	4.2225	5.7813	7.2573	8.6781	10.0589	11.4089

Table 1. Numerical values of $c_3(q)$ and $c_0(q)/c_3(q)$ for $1 \leq q \leq 6$ using the numerical optimal values of parameters $\tau_k, \ k = 1, 2, 3$.

§3. QUASIPOLYNOMIAL–PADE (QPP-) APPROXIMATIONS

Following $[17-20]$ we consider a finite sequence of complex numbers $\eta := \{\eta_k\}_{k=1}^m$, $m \geq 1$ and denote

$$\delta_n^0(\eta, f) = A_n(f), \quad \delta_n^k(\eta, f) = \delta_n^{k-1}(\eta, f) + \eta_k \delta_{n-1}^{k-1}(\eta, f), \quad 1 \leq k \leq q.$$

If $n < 0$, we put $\delta_n^k(\eta, f) = 0, \ k = 0, 1, ...$

It can easily be checked that

$$\sum_{k=0}^{q-1} A_k(f)x^k = x^q \frac{A_{q-1}(f)\eta_1}{1+\eta_1 x} + \frac{1}{1+\eta_1 x}\sum_{k=0}^{q-1}(A_k(f)+\eta_1 A_{k-1}(f))x^k. \tag{8}$$

Note that for $\eta_1 = 0$ the sum in the left side of (8) remains unchanged. Reiteration of this transformation m times ($m \le q-1$) leads to the formula

$$\sum_{k=0}^{q-1} A_k(f)x^k = x^q \sum_{k=1}^{m} \frac{\eta_k \delta_{q-1}^{k-1}(\eta,f)}{\prod_{s=1}^{k}(1+\eta_s x)} + \frac{1}{\prod_{s=1}^{m}(1+\eta_s x)}\sum_{k=0}^{q-1}\delta_k^m(\eta,f)x^k. \tag{9}$$

Now suppose $f \in C^q[-1,1]$ for some $q \ge 1$. Applying transformation (9) to the first term of (2) with $(i\pi n)^{-1}$ instead of x we get

$$f_n = P_n + Q_n, \quad n \ne 0, \tag{10}$$

where

$$P_n = \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \sum_{k=1}^{m} \frac{\eta_k \delta_{q-1}^{k-1}(\eta,f)(i\pi n)^k}{\prod_{s=1}^{k}(i\pi n+\eta_s)}$$
$$+ \frac{(-1)^{n+1}(i\pi n)^m}{2\prod_{k=1}^{m}(i\pi n+\eta_k)} \sum_{k=q-m}^{q-1} \frac{\delta_k^m(\eta,f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q}\int_{-1}^{1} f^{(q)}(t)\, e^{-i\pi n t}dt. \tag{11}$$

and

$$Q_n = \frac{(-1)^{n+1}(i\pi n)^m}{2\prod_{s=1}^{m}(i\pi n+\eta_s)} \sum_{k=0}^{q-m-1} \frac{\delta_k^m(\eta,f)}{(i\pi n)^{k+1}}.$$

According to (10) the function f can be split into two parts

$$f(x) = P(x) + Q(x), \tag{12}$$

where

$$P(x) = \sum_{n=-\infty}^{\infty} P_n e^{i\pi n x}, \quad P_0 = f_0, \quad Q(x) = \sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty} Q_n e^{i\pi n x}.$$

Approximation of P by the truncated Fourier series leads to the following approximation [17 – 20]

$$T_{q,m,N}(f) = S_N(P) + Q(x), \tag{13}$$

where the Fourier coefficients of P can be found from (12)

$$P_n = f_n - Q_n.$$

The unknown vector η in (13) we determine from the system

$$\delta_k^m(\eta, f) = 0, \quad k = q-m, \cdots, q-1. \tag{14}$$

In [17] it was shown that the function $Q(x)$ is a quasipolynomial of the form

$$Q(x) = \sum_k a_k x^{p_k} e^{i\omega_k x},$$

where $\omega_k \in \mathbb{C}$ and $\{p_k\}$ is a set of nonnegative integers. The approximation (13), (14) we call *QP-method or QP-approximation*. It is important to note that for $\eta_1 = \eta_2 = \cdots = \eta_m = 0$ the QP-approximation coincides with P-approximation $S_{q,N}(f)$.

For additional acceleration of QP-method we proceed as in the previous section. From (13) and (12), we have, see (5)

$$R_{q,m,N}(f) := f(x) - T_{q,m,N}(f) = R_N^+(P) + R_N^-(P),$$

where

$$\begin{aligned} R_N^+(P) :&= \sum_{n=N+1}^{\infty} P_n e^{i\pi n x} \\ &= -e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, P)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} + \frac{1}{\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, P) e^{i\pi n x}. \end{aligned}$$

Similar expansion of $R_N^-(P)$ reduces to the following *Quasipolynomial-Pade (QPP-) approximation*:

$$\begin{aligned} T_{p,q,m,N}(f) = S_N(P) &- e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, P)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} \\ &- e^{-i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_{-k} \Delta_{-N}^{k-1}(\theta, P)}{\prod_{s=1}^{k}(1+\theta_{-s} e^{-i\pi x})} + Q(x). \end{aligned}$$

Now we prove an analog of Theorem 1 for QPP-approximations.

In [16] the following lemma was proved:

Lemma 1 [16]. Suppose that the sequence P_n has the following asymptotic expansion for $q \geq 0$, $p \geq 1$:

$$P_n = \frac{(-1)^{n+1}}{2} \sum_{s=q}^{p+q} \frac{\alpha_s}{(i\pi n)^{s+1}} + o(n^{-p-q-1}), \quad n \to \infty,$$

where $\{\alpha_s\}_{s=q}^{p+q}$ are some constants. If

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \quad k = 1, \cdots, p,$$

then the asymptotic expansion

$$\Delta_n^p(\theta,P)=\alpha_q\frac{(-1)^{n+p+1}}{2(i\pi)^{q+1}q!}\sum_{k=0}^{p}\frac{(q+p-k)!(-1)^k\gamma_k(p)}{N^k(n-k)^{q+1}|n-k|^{p-k}}+o(n^{-q-p-1}),$$

holds as $N\to\infty$, $|n|\geq N+1$, *where the numbers* $\gamma_k(p)$ *are defined by (7).*

By $\mu_k(m), k=0,\cdots,m$, we denote the coefficients of the polynomial

$$\prod_{k=1}^{m}(1+\eta_k x)\equiv\sum_{k=0}^{m}\mu_k(m)x^k.$$

Note that the system (14) can be written in the following form:

$$\sum_{s=1}^{m}\mu_s(m)A_{k-s+q-m-1}(f)=-A_{k+q-m-1}(f),\quad k=1,\cdots,m, \tag{15}$$

and denote

$$U_r^m=[A_{k-s+r}(f)],\quad k,\quad s=1,\cdots,m.$$

Theorem 2. Suppose $f\in C^{q+p}[-1,1]$ *for some* $q\geq 1$, $p\geq 0$ *and* $f^{(q+p)}$ *is absolutely continuous on* $[-1,1]$. *If* $\det U_{q-m-1}^m\neq 0$, *then with* η *found from (14) and* $\theta_k=\theta_{-k}=1-\frac{\tau_k}{N}$, $k=1,\cdots,p$, $\tau_k>0$, $\tau_j\neq\tau_i$, $j\neq i$,

$$\lim_{N\to\infty}N^{q+\frac{1}{2}}\|f-T_{p,q,m,N}(f)\|=\left|\frac{\det U_{q-m}^{m+1}}{\det U_{q-m-1}^{m}}\right|c_p(q),$$

where $c_p(q)$ *is defined in theorem 1.*

Proof: From (12), (13) and (6) we have

$$f(x)-T_{p,q,m,N}(f):=R_{p,q,m,N}^{+}(P)+R_{p,q,m,N}^{-}(P),$$

where

$$R_{p,q,m,N}^{\pm}(P)=\frac{1}{\prod_{k=1}^{p}(1+\theta_{\pm k}e^{\pm i\pi x})}\sum_{n=N+1}^{\infty}\Delta_{\pm n}^{p}(\theta,P)e^{\pm i\pi nx}.$$

After simple calculations we obtain from (11)

$$P_n=\frac{(-1)^{n+1}}{2}\sum_{\ell=q}^{p+q}\frac{\alpha_\ell}{(i\pi n)^{\ell+1}}+o(n^{-p-q-1}),\quad n\to\infty,$$

where

$$\alpha_\ell=A_\ell(f)+\sum_{k=1}^{m}\eta_k\delta_{q-1}^{k-1}(\eta,f)\sum_{s=1}^{k}\frac{\eta_s^{k-1}(-1)^{\ell-q}\eta_s^{\ell-q}}{\prod_{\substack{j=1\\ j\neq s}}^{k}(\eta_s-\eta_j)}.$$

Now we can apply Lemma 1 to the sequence P_n taking into account that

$$\alpha_q = A_q(f) + \sum_{k=1}^{m} \eta_k \delta_{q-1}^{k-1}(\eta, f) = \delta_q^m(\eta, f),$$

which follows from the relation

$$\delta_q^m(\eta, f) = \delta_q^{m-1}(\eta, f) + \eta_m \delta_{q-1}^{m-1}(\eta, f) = \delta_q^{m-2}(\eta, f) + \eta_{m-1}\delta_{q-1}^{m-2}(\eta, f) + \eta_m \delta_{q-1}^{m-1}(\eta, f)$$
$$= \delta_q^0(\eta, f) + \sum_{k=1}^{m} \eta_k \delta_{q-1}^{k-1}(\eta, f) = A_q(f) + \sum_{k=1}^{m} \eta_k \delta_{q-1}^{k-1}(\eta, f).$$

By Cramer's rule, from (15) we get

$$\mu_s(m) = \frac{M_s}{\det U_{q-m-1}^m}, \quad s = 1, \cdots, m,$$

where $\{M_s\}$ are the corresponding minors. Consequently,

$$\delta_q^m(\eta, f) = A_q(f) + \sum_{s=1}^{m} \mu_s(m) A_{q-s}(f) = A_q(f) + \frac{1}{\det U_{q-m-1}^m} \sum_{s=1}^{m} M_s A_{q-s}(f)$$
$$= (-1)^m \frac{\det U_{q-m}^{m+1}}{\det U_{q-m-1}^m}.$$

To get the proof it remains to proceed as in the proof of Theorem 1 with $A_q(f)$ replaced by $(-1)^m \frac{\det U_{q-m}^{m+1}}{\det U_{q-m-1}^m}$.

Note that for $p = 0$ Theorem 2 was proved in [20].

§4. NUMERICAL RESULTS

For any given f, q and m we put

$$a_{q,m}(f) = \left| A_q(f) \frac{\det (U_{q-m-1}^m)}{\det (U_{q-m}^{m+1})} \right|.$$

The constant $a_{q,m}(f)$ describes the effectiveness of QPP-approximation compared to PP-approximation (with the same value of parameter p), assuming $N >> 1$ and Theorems 1,2 valid. Calculations show that this constant in fact describes the convergence acceleration in broader situations. Note that $a_{q,m}$ is independent of the parameter p.

Let us investigate the example

$$f(x) = \frac{\sin(5x - 0.2)}{1.1 - x}. \tag{16}$$

Figure 1 represents the graphs of $a_{q,m}(f)$ for (16) for $q = 5, 6, 7$ and $1 \le m \le q-1$. For this function the QPP-method proves to be more precise than PP-method (for the same value of parameter p) almost 15 times for $q = 5; m = 3$ and 50 times when $q = 7; m = 3$.

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Figure 1: Graphics of $a_{q,m}(f)$ for (16) for $q = 5, 6, 7$ and $1 \le m \le q-1$.

The relative effectiveness of QPP-method against PP-method can be described by the fraction

$$a_{N,q,m,p}(f) = \frac{\max\limits_{|x|\le 1} |f - S_{p,q,N}(f)|}{\max\limits_{|x|\le 1} |f - T_{p,q,m,N}(f)|}.$$

Table 1 shows approximate values of $a_{N,7,3,3}$ for (16). Calculations are carried out with 64 digits of precision by MATHEMATICA package.

N	32	64	128	256	512
$a_{N,7,3,3}$	101.17	64.36	50.91	48.58	48.57

Table 1: Approximate values of $a_{N,7,3,3}$ for different N.

Comparison with theoretical value $a_{7,3} = 49.4408$ shows that experimental and theoretical estimates are rather close for $N \ge 64$.

	$S_{0,q,N}$	$S_{1,q,N}$	$S_{2,q,N}$	$S_{3,q,N}$
128	1.8×10^{-9}	2.1×10^{-10}	4.5×10^{-11}	1.3×10^{-11}
256	7.3×10^{-12}	8.3×10^{-13}	1.7×10^{-13}	4.7×10^{-14}

	$T_{0,q,m,N}$	$T_{1,q,m,N}$	$T_{2,q,m,N}$	$T_{3,q,m,N}$
128	3.5×10^{-11}	4.1×10^{-12}	8.8×10^{-13}	2.5×10^{-13}
256	1.52×10^{-13}	1.7×10^{-14}	3.5×10^{-15}	9.7×10^{-16}

Table 2: Uniform errors by QPP and PP methods for different values of N, p and $q = 7, m = 3$.

Figure 1 shows also the optimal values of m when parameter q is fixed. Thus we see that for $q = 7$ the optimal is $m = 3$. Table 2 presents uniform errors in approximation of (16) by QPP and PP methods for different values of N, p and $q = 7, m = 3$. Comparison shows that $S_{2,7,N}$ has the same precision as $T_{0,7,3,N}$ while $T_{1,7,3,N}$ is 3 times more precise than $S_{3,7,N}$.

All calculations are carried out by the package MATHEMATICA with 64 digits of precision.

§ 5. A MODEL PROBLEM

The approach of previous sections was generalized in [18] for expansions by eigenfunctions for one-dimensional boundary problems in the case where the coefficients of the equations are smooth. Here we consider a simple first order model differential equation with non-smooth coefficient. Some preliminary results were obtained in [21].

We pose the eigenvalue problem

$$i\frac{du}{dx} = \lambda\,\varepsilon(x)u(x), \quad x \in (-1,1), \tag{17}$$

$$u(-1) = u(1), \tag{18}$$

where $\varepsilon(x) > \delta > 0$, $\varepsilon^{(q+1)}$ is a piecewise-continuous function in $[-1,1]$ with potential points of discontinuity among $\alpha = \{\alpha_k\}_{k=0}^{\mu}$, $-1 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{\mu-1} < \alpha_\mu = 1$. We denote this class of functions by $C_\alpha^{q+1}[-1,1]$.

It is easy to calculate the eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{\phi_n\}$ of the problem (17), (18)

$$\phi_n(x) = e^{-i\eta n\int_{-1}^{x}\varepsilon(t)dt}, \quad \lambda_n = -\pi\eta n, \quad n \in \mathbb{Z},$$

where

$$\eta = \frac{2\pi}{\int_{-1}^{1}\varepsilon(t)dt}.$$

The system $\{\phi_n(x))\}_{n=-\infty}^{\infty}$ is orthogonal in the weighted space $L_2[(-1,1),\varepsilon]$.

We consider now the formal series

$$f(x) = \sum_{n=-\infty}^{\infty} f_n\phi_n(x),\ f_n = \frac{\eta}{2\pi}\int_{-1}^{1}\varepsilon(x)f(x)\overline{\phi_n(x)}dx \tag{19}$$

and the approximation formula:

$$W_N(f) = \sum_{n=-N}^{N} f_n\phi_n(x). \tag{20}$$

Lemma 2. If $f \in C_\alpha^{q+1}[-1,1]$ *then the following asymptotic expansion holds*

$$f_n = H_n + G_n, \tag{21}$$

$$H_n = \frac{\eta}{2\pi}\sum_{\ell=0}^{\mu-1}\sum_{k=0}^{q}\frac{(-1)^k A_k(f,\alpha_\ell)}{(i\eta n)^{k+1}}e^{i\eta n\int_{-1}^{\alpha_l}\varepsilon(t)dt}, \quad G_n = \frac{\eta}{2\pi}\frac{(-1)^{q+1}}{(i\eta n)^{q+1}}\int_{-1}^{1}\varepsilon(x)g_{q+1}(x)\overline{\phi_n(x)}dx,$$

where

$$g_0(x) = f(x), \quad g_k(x) = \frac{g'_{k-1}(x)}{\varepsilon(x)},$$

$$A_k(f,-1) = g_k(1) - g_k(-1), \quad A_k(f,x) = g_k(x-0) - g_k(x+0).$$

Proof: We divide the integral in (19) into parts by the jump points

$$f_n = \frac{\eta}{2\pi}\int_{-1}^{1} \varepsilon(x) f(x)\overline{\phi_n(x)}dt = \frac{\eta}{2\pi}\sum_{\ell=0}^{\mu-1}\int_{\alpha_\ell}^{\alpha_{l+1}} \varepsilon(x) f(x)\overline{\phi_n(x)}dx.$$

Integration by parts yields

$$\frac{\eta}{2\pi}\int_{\alpha_l}^{\alpha_{l+1}} \varepsilon(x) f(x)\overline{\phi_n(x)}dx = \frac{\eta}{2\pi}\int_{\alpha_l}^{\alpha_{l+1}} f(x)\varepsilon(x)e^{i\eta n\int_{-1}^{x}\varepsilon(t)dt}dx =$$

$$= \frac{g_0(x)e^{i\eta n\int_{-1}^{x}\varepsilon(t)dt}}{2i\pi n}\Big|_{\alpha_l}^{\alpha_{l+1}} - \frac{1}{2i\pi n}\int_{\alpha_l}^{\alpha_{l+1}} g_1(x)\varepsilon(x)e^{i\eta n\int_{-1}^{x}\varepsilon(t)dt}dx. \tag{22}$$

From (22) we get

$$\frac{\eta}{2\pi}\int_{-1}^{1} \varepsilon(x) f(x)\overline{\phi_n(x)}dt = \sum_{l=0}^{\mu-1}\frac{A_0(f,\alpha_l)}{2i\pi n}e^{i\eta n\int_{-1}^{\alpha_l}\varepsilon(t)dt} - \frac{1}{2i\pi n}\int_{-1}^{1}\varepsilon(x) g_1(x)\overline{\phi_n(x)}dx.$$

Integration by parts $q-1$ times leads to (21). The proof is complete.

For convergence acceleration of (20) we use the idea of P-approximation. According to Lemma 2 we split the function f into two parts

$$f(x) = H(x) + G(x)$$

where $G(x)$ is smooth as compared with $H(x)$, and consider the following analog of P-method:

$$W_{q,N}(f) = H(x) + \sum_{n=-N}^{N} G_n\phi_n(x).$$

Theorem 3. If $f \in C_\alpha^{q+1}[-1,1]$, then

$$f - W_{q,N}(f) = o(N^{-q}), \quad N \to \infty.$$

Proof: immediately follows from the formulas

$$f - W_{q,N}(f) = G - W_N(G) = \sum_{|n|>N} G_n\phi_n(x),$$

$$G_n = \frac{(-1)^q\eta}{2\pi(i\eta n)^{q+1}}\int_{-1}^{1} g_{q+1}(x)\varepsilon(x)e^{i\eta n\int_{-1}^{x}\varepsilon(t)dt}dx, \tag{23}$$

and from the fact that integral in the right hand side of (23) is $o(1)$, $n \to \infty$, according to the Riemann–Lebesgue Theorem.

The analogs of PP and QPP approximations can be constructed similarly. By $W_{q,p,N}(f)$ and $W_{q,p,m,N}(f)$ we denote the analogs of PP and correspondingly QPP approximations for (20).

Consider the function

$$f(x) = \frac{1}{1.1 - x}. \tag{24}$$

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Figure 2: Uniform errors in logarithm scale while approximating (24) by $W_{q,p,N}(f)$ and $W_{q,p,m,N}(f)$ for $q = 7$, $m = 3$, $p = 0, 2$ and ε as in (25).

In Figure 2, the uniform errors in logarithm scale are shown while approximating (24) by $W_{q,p,N}(f)$ and $W_{q,p,m,N}(f)$ for $q = 7$, $m = 3$, $p = 0, 2$ and

$$\varepsilon(x) = \begin{cases} 1, & x < 1/3 \\ 4, & x \geq 1/3. \end{cases} \tag{25}$$

We see that $W_{7,2,N}(f)$ and $W_{7,2,3,N}(f)$ are 4 to 10 times more precise (the difference is higher for greater values of the parameter N) compared to approximations $W_{7,0,N}(f)$ and $W_{7,0,3,N}(f)$ correspondingly. Approximations $W_{7,0,3,N}(f)$ and $W_{7,2,N}(f)$ show the same precision (see also remarks to Table 2).

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