ON A PARALLEL ALGORITHM FOR INTEGRAL EQUATIONS

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Abstract. In this paper we consider and investigate a parallel algorithm for numerical solution of the second kind Fredholm integral equation with prescribed accuracy. Numerical experiments are carried down on ArmCluster. The results of numerical experiments are presented and discussed.

1. Introduction

Numerical solution of integral equations attracted the interest of many investigators, so a number of different methods have been proposed (see, for example, [1]-[7] with references). The most powerful of them are based on a discretization of the integrals by means of an appropriate quadrature rule. This leads to a system of linear algebraic equations after applying it to appropriately selected collocation points. From the above system of linear equations the values of the unknown function at the integration points are obtained. In this work we use similar approach with quadrature rule that was suggested in [1]. For numerical solution of corresponding system of linear equations we apply the parallel algorithms available on ArmCluster (the package ScaLAPACK).

2. Description of the method

Consider the following integral

$$I = \int_a^b f(x)dx, \qquad a, b \in [-1, 1]. \tag{2.1}$$

By the change of variable $x = \sin^2 \frac{\pi t}{2}$ we obtain

$$I = \int_{a^*}^{b^*} F(t)dt, \tag{2.2}$$

where

$$\begin{aligned} a^* = \frac{1}{\pi} arccos(1 - 2a),\ b^* = \frac{1}{\pi} arccos(1 - 2b), \\ F(x) = \frac{\pi}{2} f(\sin^2 \frac{\pi x}{2}) \sin \pi x. \end{aligned} \tag{2.3}$$

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By the use of classical trigonometric interpolation

$$
\begin{aligned}
F(x) &\approx \sum_{n=-N}^{N} \check{F}_n e^{i\pi n x}, \\
\check{F}_n &= \frac{1}{2N+1} \sum_{k=-N}^{N} F(x_k) e^{-i\pi n x_k}, \ \ x_k = \frac{2k}{2N+1}
\end{aligned} \tag{2.4}
$$

we obtain the following quadrature formula

$$
I \approx \frac{4}{2N+1} \sum_{k=1}^{N} F(x_k) \sum_{n=1}^{N} \frac{\sin \pi n x_k}{\pi n} (\cos \pi n a^* - \cos \pi n b^*). \tag{2.5}
$$

For $f \in C^\infty[a,b]$ the accuracy of the quadrature formula (2.5) is "infinite".

Now consider the second kind Fredholm integral equation

$$
z(x) + \int_a^b K(x,t) z(t) dt = f(x), \tag{2.6}
$$

where $K(x,t), f(x)$ are smooth functions. By the change of variables $x = a + (b-a)u$, and $t = a + (b-a)v$ we derive

$$
\tilde{z}(u) + \int_0^1 \tilde{K}(u,v) \tilde{z}(v) dv = \tilde{f}(u), \tag{2.7}
$$

with

$$
\begin{aligned}
\tilde{z}(u) &= z(a + (b-a)u), \ \ \tilde{f}(u) = f(a + (b-a)u) \\
\tilde{K}(u,v) &= (b-a) K(a + (b-a)u, a + (b-a)v).
\end{aligned} \tag{2.8}
$$

By application of (2.5) to (2.8) we obtain

$$
\tilde{z}(u) + \sum_{k=1}^{N} P_k \tilde{K}\left(u, \sin^2 \frac{\pi x_k}{2}\right) \tilde{z}\left(\sin^2 \frac{\pi x_k}{2}\right) = f(u), \tag{2.9}
$$

where

$$
P_k = \frac{2\pi \sin \pi x_k}{2N+1} \sum_{n=1}^{N} \frac{\sin \pi n x_k}{\pi n} (1 - (-1)^n). \tag{2.10}
$$

By the change of variable $u = \sin^2 \frac{\pi w}{2}$ we get

$$
\tilde{\tilde{z}}(w) + \sum_{k=1}^{N} P_k \tilde{\tilde{K}}(w, x_k) \tilde{\tilde{z}}(x_k) = \tilde{\tilde{f}}(w), \tag{2.11}
$$

where

$$
\begin{aligned}
\tilde{\tilde{z}}(u) &= \tilde{z}\left(\sin^2 \frac{\pi u}{2}\right), \ \ \tilde{\tilde{f}}(u) = \tilde{f}\left(\sin^2 \frac{\pi u}{2}\right) \\
\tilde{\tilde{K}}(u,v) &= \tilde{K}\left(\sin^2 \frac{\pi u}{2}, \sin^2 \frac{\pi v}{2}\right).
\end{aligned} \tag{2.12}
$$

Now we put $w = x_s$ and derive the following system of linear equations

$$\sum_{k=1}^{N} m_{sk}\tilde{\tilde{z}}(x_k) = \tilde{\tilde{f}}(x_s),\ \ s = 1, \cdots, N \tag{2.13}$$

with

$$m_{sk} = \delta_{sk} + \tilde{\tilde{K}}(x_s, x_k)P_k, \tag{2.14}$$

where $\delta_{ss} = 1$ and $\delta_{sk} = 0$ for $s \neq k$. Hence

$$\tilde{\tilde{z}}(x_k) = \sum_{s=1}^{N} m_{sk}^{-1}\tilde{\tilde{f}}(x_s),\ \ k = 1, \cdots, N, \tag{2.15}$$

where by m_{sk}^{-1} we denote the elements of the inverse matrix. Now note that from (2.6) we have

$$z(x) = f(x) - (b-a)\int_0^1 K\left(x, a + (b-a)v\right) z\left(a + (b-a)v\right) dv. \tag{2.16}$$

Application of (2.5) to (2.16) leads to the following representation

$$z(x) \approx f(x) - (b-a)\sum_{k=1}^{N} P_k\, K\left(x, a + (b-a)\sin^2\frac{\pi x_k}{2}\right)\ \tilde{\tilde{z}}(x_k). \tag{2.17}$$

Substituting (2.15) into (2.17) we derive the symbolic approximate solution of (2.6) in the form of a functional sum.

This approach can be applied also for the numerical solution of integral equations of the type

$$z(x) + \sum_{s=1}^{m}\int_{\alpha_s(x)}^{\beta_s(x)} K_s(x,t)z(t)dt = f(x),\ \ x \in [-1, 1], \tag{2.18}$$

where $\alpha_s(x), \beta_s(x), f(x), K_s(x,t)$ are smooth functions, and $|\alpha_s(x)|$, $|\beta_s(x)| \leq 1$. System of integral equations can also be considered similarly.

3. Example

Consider the following integral equation

$$z(x) = e^x + \int_0^1 K(x,t)z(t),\ \ x \in [0, 1], \tag{3.1}$$

where

$$K(x,t) = \begin{cases} \frac{1}{sh1} sh\, x\ sh(t-1),\ \ 0 \leq x \leq t \\ \frac{1}{sh1} sh\, t\ sh(x-1),\ \ t \leq x \leq 1 \end{cases} \tag{3.2}$$

The exact solution of (3.1) is $\frac{1}{sh\sqrt{2}} sh(\sqrt{2}(1-x)) + e\, sh(\sqrt{2}x)$. The program is working with automatic precision check scheme (with required accuracy) when the system (2.13) must be repeatedly solved for different values of N and therefore parallelization of this part of algorithm is

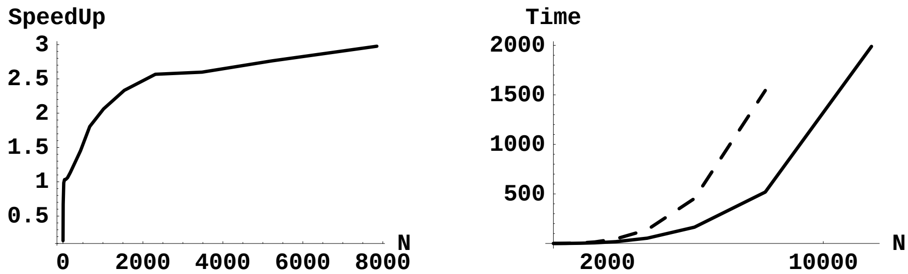

FIGURE 1. The left: speedup of parallelization for 4 computers. The right: full line is the working time of the algorithm for 4 computers and the dashed line for 1 computer.

extremely efficient. For example, we have obtained the numerical solution with absolute error 9×10^{-7} for 91 seconds on one computer and 58 seconds for 4 computers ($N = 700$). Comparison results are shown on the diagram. The left diagram shows the speedup of parallelization in dependence on N for 4 computers. As we see, for $N > 5000$ we have acceleration almost 3 times. On the right diagram the working time of the program is presented: the full line for 4 computers and the dashed line for 1 computer.

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