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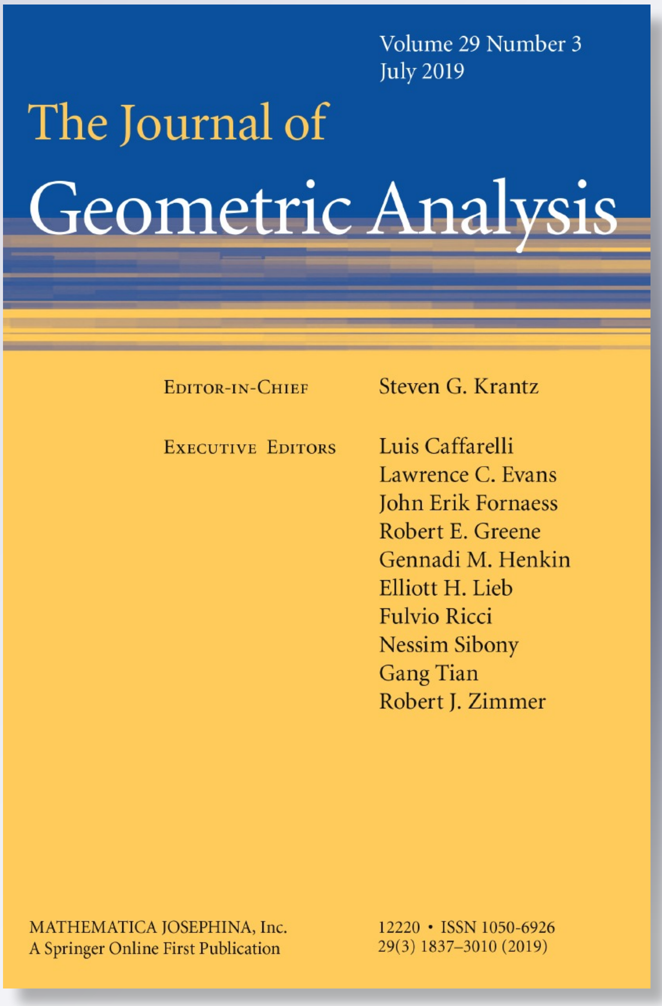

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A Flag Representation for a n-Dimensional Convex Body

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Abstract
The cosine representation of the support function of a centrally symmetric convex body plays a fundamental role in integral geometry. In this article, one new so-called *flag* representation for the support function of an origin symmetric n-dimensional convex body in terms of surface curvature functions of the convex body is found. Using the representation, we propose a sufficient condition for an origin symmetric n-dimensional convex body to be a zonoid. The condition has a local equatorial description.

Keywords Integral geometry · Convex body · Zonoid · Support function

Mathematics Subject Classification 53C45 · 52A20 · 53C65

1 Introduction

The cosine representation of the support function of a centrally symmetric convex body plays a fundamental role in integral geometry and a number of related areas (see, e.g., [7–11,16,18,21]). In this article, one new so-called *flag* representation for the support function of an origin symmetric n-dimensional convex body in terms of surface curvature functions of the convex body is found.

We denote by $\mathbf{R}^n$ $(n \geq 3)$ the Euclidean n dimensional space. Let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ with the center at the origin of $\mathbf{R}^n$, λ_{n-1} be the spherical Lebesgue measure on $\mathbf{S}^{n-1}$ $(\lambda_1 \equiv \lambda)$ and let σ_k be the total measure of $\mathbf{S}^k$ $(\lambda_k(\mathbf{S}^k) = \sigma_k)$. Denote by $\mathbf{S}_\omega \subset \mathbf{S}^{n-1}$ the great $n-2$ dimensional sphere with pole at $\omega \in \mathbf{S}^{n-1}$.

The most useful analytic description of compact convex sets is given by the support function (see [12]). The support function $H : \mathbf{R}^n \to (-\infty, \infty]$ of a convex body $\mathbf{B}$ is defined as

$$H(x) = \sup_{y \in \mathbf{B}} \langle y, x \rangle, \quad x \in \mathbf{R}^n. \tag{1.1}$$

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Throughout this article , $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbf{R}^n$. The support function of $\mathbf{B}$ is positively homogeneous and convex. Hereafer, we consider the support function H of a convex body as a function defined on the unit sphere $\mathbf{S}^{n-1}$ (because of the positive homogeneity of H, the values on $\mathbf{S}^{n-1}$ determine H completely).

It is well known that any convex body $\mathbf{B}$ is uniquely determined by its support function [12]. A convex body $\mathbf{B}$ is k-smooth if its support function $H \in \mathbf{C}^k(\mathbf{S}^{n-1})$, where $\mathbf{C}^k(\mathbf{S}^{n-1})$ denotes the space of k times continuously differentiable functions defined on $\mathbf{S}^{n-1}$.

We denote the class of origin symmetric convex bodies (nonempty compact convex sets) $\mathbf{B}$ in $\mathbf{R}^n$ by $\mathcal{B}^n_o$ (the so-called the *cenetred* bodies).

Zonotopes are convex bodies that are composed (in the sense of Minkowski addition) of line segments in $\mathbf{R}^n$. Zonoids are limits of zonotopes in the Hausdorff metric. Zonoids form an important subclass of centrally symmetric convex bodies (see [9,21]).

It is known that [7,8,18,21] *a convex body* $\mathbf{B} \in \mathcal{B}^n_o$ *is a (centered) zonoid if and only if the support function* H *of* $\mathbf{B}$ *admits the following representation*:

$$H(\xi) = \int_{\mathbf{S}^{n-1}} \mid \langle \xi, \Omega\rangle \mid \mu(d\Omega), \quad \xi \in \mathbf{S}^{n-1}, \tag{1.2}$$

where μ *is a positive even measure on* $\mathbf{S}^{n-1}$. *Note that* μ (*in* (1.2)) *is uniquely defined for* H *in the class of positive even measures on* $\mathbf{S}^{n-1}$.

Also it is known [8,17,21] that *the support function* H *of a sufficiently smooth origin symmetric convex body* $\mathbf{B} \in \mathcal{B}^n_o$ *has the following representation*:

$$H(\xi) = \int_{\mathbf{S}^{n-1}} \mid \langle \xi, \Omega\rangle \mid h(\Omega)\, \lambda_{n-1}(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{1.3}$$

with an even continuous function h *(not necessarily positive) defined on* $\mathbf{S}^{n-1}$. *Note that* h *in* (1.3) *is uniquely defined in the class of even continuous functions defined on* $\mathbf{S}^{n-1}$ *and is called the generating density of* $\mathbf{B}$. *Such bodies (support functions of which have the integral representation* (1.2) *with a signed even measure* μ) *are centered generalized zonoids.*

The problem of geometric characterization of zonoids was posed by Blashke (this problem was posed repeatedly (see [18])). Weil showed [20] that a local characterization of zonoids does not exist. Also in [20] Weil proposed the conjecture about local equatorial characterization of zonoids. Positive answers for even dimensions were given independently in [15] and in [9]. Finally, it was showed in [13] that the answer to the conjecture in odd dimensions is negative. In [6] was described a subclass of zonoids in $\mathbf{R}^3$ admitting a local equatorial characterization.

In this article, we find one new so-called *flag* representation for the support function of an origin symmetric n-dimensional convex body in terms of surface curvature functions of the convex body. Using the representation, we propose a sufficient condition for an origin symmetric n-dimensional convex body to be a zonoid. The condition written in terms of surface curvature functions of the convex body and has a local equatorial description.

The concept of a flag in $\mathbf{R}^n$ which naturally emerges in Combinatorial integral geometry will be of importance below. A detailed account of this concept in $\mathbf{R}^3$ is in [1,2]. By definition: a flag is an ordered pair orthogonal unit vectors in $\mathbf{R}^n$, say a_1, a_2. There are two equivalent representations of a flag:

$$(\omega, \varphi) \quad \text{and} \quad (\Omega, \Phi), \tag{1.4}$$

where $\omega \in \mathbf{S}^{n-1}$ is the spatial direction of the first vector a_1, and φ is the direction in $\mathbf{S}_\omega$ coincides with the direction of a_2, while $\Omega \in \mathbf{S}^{n-1}$ is the spatial direction of the second vector a_2, and Φ is the direction in $\mathbf{S}_\Omega$ coincides with the direction of a_1. The second representation we will write by capital letters.

Our main results are the following. Let $\mathbf{B} \in \mathcal{B}_o^n$ be an origin symmetric convex body in $\mathbf{R}^n$ with sufficiently smooth boundary. For $\Phi \in \mathrm{S}^{n-1}$, we denote by $s(\Phi)$ the point on $\partial\mathbf{B}$ the outer normal of which is Φ. By $k_i(\Phi)$, $i = 1, \ldots, n-1$, we denote the principal normal curvatures of $\partial\mathbf{B}$ at $s(\Phi)$ and let $k(\Phi, \Omega)$ be the normal curvature of $\partial\mathbf{B}$ at $s(\Phi)$ in direction $\Omega \in \mathrm{S}_\Phi$. By

$$K(\Phi) = \prod_{i=1}^{n-1} k_i(\Phi),$$

we denote the Gauss–Kronecker curvature at $s(\Phi)$. For $\Phi \in \mathrm{S}^{n-1}$ by e_Φ, we denote the hyperplane containing the origin and orthogonal to Φ and for $\xi \in \mathrm{S}^{n-1}$ (which does not collinear to Φ) by $\xi_\Phi \in \mathrm{S}_\Phi$, we denote the direction of orthogonal projection of ξ onto the plane e_Φ. In this article, we consider a convex body $\mathbf{B} \in \mathcal{B}_o^n$ with positive Gaussian curvature at every point of $\partial\mathbf{B}$. By $\widehat{(\xi, \Phi)}$, we denote the angle between two unit vectors $\Phi, \xi \in \mathrm{S}^{n-1}$.

Theorem 1 *The support function of an origin symmetric* 2*-smooth convex body* $\mathbf{B} \in \mathcal{B}_o^n$ *has the following representation. For* $\xi \in \mathbf{S}^{n-1}$,

$$H(\xi) = \frac{(n-1)}{2\,\sigma_{n-2}^2} \int_{\mathrm{S}^{n-1}} \int_{\mathrm{S}_\Omega} \frac{\langle \xi, \Omega\rangle^2}{\sin^{n-1}\widehat{(\xi, \Phi)}} \, \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} \, \lambda_{n-2}(d\Phi)\, \lambda_{n-1}(d\Omega). \tag{1.5}$$

Note that the inner improper integral (for $\xi \in \mathrm{S}_\Omega$) in (1.5) converges (see Lemma 3). The function

$$\rho(\Omega, \Phi, \xi) = \frac{\langle \xi, \Omega\rangle^2}{\sin^{n-1}\widehat{(\xi, \Phi)}} \tag{1.6}$$

defined for $\Omega, \xi \in \mathrm{S}^{n-1}$, $\Phi \in \mathrm{S}_\Omega$ $(\xi \neq \Phi)$ is called *the flag density* function (for $\xi = \Phi$, we assume that $\rho = 0$). In $\mathbf{R}^3$ the notion of a flag density was introduced and effectively used in the works of R. V. Ambartzumian [1,2] (see also [3,14]).

The following identity is proved (see Lemma 3). For $\Omega, \xi \in \mathbf{S}^{n-1}$, $n \geq 3$, we have

$$\int_{\mathrm{S}_\Omega} \rho(\Omega, \Phi, \xi)\, \lambda_{n-2}(d\Phi) = \frac{\sigma_{n-1}\,(n-2)!!}{2\,(n-3)!!} \mid \langle \xi, \Omega\rangle \mid . \tag{1.7}$$

Thus, the flag density function is a disintegration of the cosine function.

As a consequence of Theorem 1, we propose the following sufficient condition for an origin symmetric convex body to be a zonoid.

Theorem 2 *Let* $\mathbf{B}$ *be an origin symmetric 2-smooth convex body in* $\mathbf{R}^n$ *(*$\boldsymbol{B} \in \mathcal{B}_o^n$*). If for any* $\Omega \in \mathbf{S}^{n-1}$ *and* $\xi \in \mathbf{S}^{n-1}$ *the expression*

$$F(\Omega, \xi) = \int_{\mathrm{S}_\Omega} \frac{|\langle \xi, \Omega\rangle|}{\sin^{n-1}(\widehat{\xi, \Phi})} \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} \lambda_{n-2}(d\Phi) \tag{1.8}$$

does not depend on $\xi \in \mathbf{S}^{n-1}$ *then* $\mathbf{B}$ *is a zonoid.*

It was proved in [6] that in $\mathbf{R}^3$ a convex body boundary of which is an ellipsoid satisfy the condition (1.8). Let $\mathbf{B} \in \mathcal{B}_o^3$ be a convex body boundary of which is an ellipsoid with the semi-principal axes of length a, b, c. For $\Omega = (n_1, n_2, n_3) \in \mathbf{S}^2$ and $\xi \in \mathbf{S}^2$, we have (see [6]),

$$F(\Omega, \xi) = \frac{2\pi}{(abc)} \left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2} \right)^{-2}.$$

It follows from Theorem 1 that, there is not a convex body which are not zonoids for which $F(\Omega, \xi)$ is independent of ξ. The following problem is open: describe the subclass of zonoids satisfying the condition (1.8).

Also we have the following consequence of Theorem 2.

Theorem 3 *Let* $\mathbf{B}$ *be an origin symmetric 2-smooth convex body in* $\mathbf{R}^n$*. If for any* $\Omega \in \mathbf{S}^{n-1}$ *and* $\Phi \in \mathrm{S}_\Omega$ *the expression*

$$G(\Omega, \Phi) = \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} \tag{1.9}$$

does not depend on $\Phi \in \mathrm{S}_\Omega$ *then* $\mathbf{B}$ *is a zonoid.*

The form $G(\Omega, \Phi)$ is natural from the geometric viewpoint since for a body from the subclass of zonoids (satisfying the condition (1.9)) the function $G(\Omega, \Phi) = G(\Omega)$ is the generating density (multiplied by a constant, see (1.5), (1.7)) of the body in terms of curvatures of the body with an equatorial description. Also, it was proved in [6] that in $\mathbf{R}^3$ a convex body boundary of which is an ellipsoid satisfy the condition (1.9) (see (5.8)).

Note that for any $\Omega \in \mathbf{S}^{n-1}$, the expressions $F(\Omega, \xi)$ and $G(\Omega, \Phi)$ depend on boundary of $\mathbf{B}$ which consists of points where the exterior unit vector belongs to a neighborhood of the equator $\mathbf{S}_\Omega$. Hence for any $\Omega \in \mathbf{S}^{n-1}$ the expressions $F(\Omega, \xi)$ and $G(\Omega, \Phi)$ have a local equatorial description.

2 Preliminary Results

For two unit vectors $\omega, \xi \in \mathrm{S}^{n-1}$, $\omega \neq \xi$, we denote by $B(\omega, \xi)$ the image of $\mathbf{B} \in \mathcal{B}_o^n$ under orthogonal projection onto the 2-dimensional plane $e(\omega, \xi)$ containing the origin and spanned by ω and ξ; $R(\omega, \xi)$ is defined as the radius of curvature of $\partial B(\omega, \xi)$ at the point outer normal direction of which is ω, and is called the tangential radius of curvature of the body. Since $R(\omega, \xi_1) = R(\omega, \xi_2)$, where $\omega, \xi_1, \xi_2 \in \mathrm{S}^{n-1}$ are linearly dependent vectors, we assume where necessary that ξ is orthogonal to ω.

We need the following result (see also [4]): *The support function of 2-smooth origin symmetric convex body* $\mathbf{B} \in \mathcal{B}_o^n$ *has the following representation*:

$$H(\xi) = \frac{1}{2\sigma_{n-2}} \int_{\mathrm{S}^{n-1}} \frac{R(\omega, \xi)}{\sin^{n-3}\widehat{(\omega, \xi)}} \, \lambda_{n-1}(d\omega), \quad \xi \in \mathrm{S}^{n-1}. \tag{2.1}$$

Here we give a short proof of (2.1). Let $\nu \in \mathrm{S}_\xi$ be a direction perpendicular to $\xi \in \mathrm{S}^{n-1}$. We approximate $\mathrm{B}(\nu, \xi) \subset e(\nu, \xi)$ from inside by polygons that have all their vertices on $\partial \mathrm{B}(\nu, \xi)$. We denote by a_i the sides of the approximation polygon, by u_i the direction normal to a_i within $e(\nu, \xi)$; (also by u_i, we denote the angle between the normals to a_i and ξ). Let $H_{\mathrm{B}(\nu,\xi)}$ be the support function of $\mathrm{B}(\nu, \xi)$ in the plane $e(\nu, \xi)$. We have

$$4H(\xi) = 4H_{\mathrm{B}(\nu,\xi)}(\xi) = \lim \sum_i |a_i| \, \sin\widehat{(\xi, u_i)}$$

$$= \lim \sum_i R_\nu(u_i)|u_{i+1} - u_i| \, \sin\widehat{(\xi, u_i)} = \int_{\mathbf{S}^1} R_\nu(u) \, \sin\widehat{(\xi, u)} \, \lambda_1(du), \tag{2.2}$$

where $R_\nu(u)$ is the radius of curvature of $\mathrm{B}(\nu, \xi)$ at the point outer normal direction of which is u. Integrating both sides of (2.2) by $\lambda_{n-2}(d\nu)$ over $\mathbf{S}_\xi$, and taking into account

$$\lambda_{n-1}(d\omega) = \sin^{n-2} u \, du \, \lambda_{n-2}(d\nu),$$

where $\omega = (u, \nu)$ (we use the spherical coordinates for $\omega \in \mathrm{S}^{n-1}$, where u is the polar angle measured from $\xi \in \mathrm{S}^{n-1}$), we obtain (2.1) (note that $R_\nu(u) = R(\omega, \xi)$).

According to Blaschke's theorem ([8, p. 117], see also [19]), we have (for $\omega \neq \xi$)

$$R(\omega, \xi) = R(\omega, \xi_\omega) = \sum_{i=1}^{n-1} R_i(\omega) \, \langle \xi_\omega, \varphi_i \rangle^2, \tag{2.3}$$

where φ_i, $i = 1, \ldots, n-1$ are the unit vectors determining the principal directions and $R_i(\omega)$ is the principal radius of curvature corresponding to φ_i at the point of $\partial \mathbf{B}$ outer normal direction of which is ω.

Substituting (2.3) into (2.1), we get

$$H(\xi) = \frac{1}{2\sigma_{n-2}} \int_{\mathbf{S}^{n-1}} \sum_{i=1}^{n-1} \frac{R_i(\omega)\,\langle \xi_\omega, \varphi_i \rangle^2}{\sin^{n-3}\widehat{(\omega, \xi)}}\,\lambda_{n-1}(d\omega), \tag{2.4}$$

Note, that the representation (2.4) first was found by Panina [14] using other method.
We need to prove the following theorem.

Theorem 4 *Let* $\mathbf{B}$ *be an origin symmetric 2-smooth convex body in* $\mathbf{R}^n$ *(*$\boldsymbol{B} \in \mathcal{B}_o^n$*). For* $\omega, \xi \in \mathrm{S}^{n-1}$*,* $\omega \neq \xi$*, we have*

$$\sum_{i=1}^{n-1} R_i(\omega)\langle \xi_\omega, \varphi_i \rangle^2 = \frac{(n-1)}{\sigma_{n-2}} \int_{\mathbf{S}_\omega} \langle \xi_\omega, \varphi \rangle^2 \frac{\sqrt{K(\omega)}}{(k(\omega, \varphi))^{\frac{n+1}{2}}} \lambda_{n-2}(d\varphi), \tag{2.5}$$

here $k(\omega, \varphi)$ *is the normal curvature of* $\partial\mathbf{B}$ *at* $s(\omega)$ *in direction* $\varphi \in \mathrm{S}_\omega$*,* φ_i *is the* i*th principal direction and* $R_i(\omega)$ *is the principal radius of curvature corresponding to* φ_i *at* $s(\omega)$ $(i = 1, \ldots, n-1)$*.*

To prove Theorem 4, we need to calculate the following integrals.

3 Two Integrals Over the *n*-Dimensional Unit Sphere

Let us consider the following integral over $\mathbf{S}^{n-1}$, $n \geq 3$:

$$I_n(k_1, \ldots, k_n) = \int_{\mathbf{S}^{n-1}} \frac{1}{(\sum_{i=1}^n k_i x_i^2)^{n/2}} \lambda_{n-1}(d\omega), \tag{3.1}$$

here x_i is the ith Cartesian coordinate of $\omega \in \mathbf{S}^{n-1}$ $(\omega = (x_1, x_2, \ldots, x_n))$, $k_i > 0$, $i = 1, \ldots, n$ is a positive real number. For $\omega \in \mathbf{S}^{n-1}$ we use the spherical coordinates $\omega = (\nu, \omega')$, where ν is the polar angle measured from nth Cdirection (the zenith direction) and $\omega' \in \mathbf{S}^{n-2}$ is the direction of orthogonal projection of ω onto the hyperplane that passes through the origin and is orthogonal to the zenith direction. From (3.1) using the spherical coordinates and making the substitution $t = \tan\nu$, we obtain

$$\begin{aligned} I_n(k_1, \ldots, k_n) &= 2 \int_{\mathbf{S}^{n-2}} \int_0^{\pi/2} \frac{\sin^{n-2}\nu\, d\nu\, \lambda_{n-2}(d\omega')}{(k_n \cos^2\nu + \sin^2\nu(\sum_{i=1}^{n-1} k_i {x'}_i^2))^{n/2}} \\ &= 2 \int_{\mathbf{S}^{n-2}} \int_0^{\infty} \frac{t^{n-2}\, dt\, \lambda_{n-2}(d\omega')}{(k_n + t^2(\sum_{i=1}^{n-1} k_i {x'}_i^2))^{n/2}}, \end{aligned} \tag{3.2}$$

where ${x'}_i$ is the ith Cartesian coordinate of $\omega' \in \mathbf{S}^{n-2}$ $i = 1, \ldots, n-1$.

For the case $n = 2k$ which is an even number, applying integration by parts $k-1$ times, we get

$$I_n(k_1,\dots,k_n) = \int_{\mathbf{S}^{n-2}} \frac{2(n-3)!!}{(\sum_{i=1}^{n-1} k_i x'^2_i)^{(k-1)}(n-2)!!} \int_0^\infty \frac{dt\, \lambda_{n-2}(d\omega')}{(k_n + t^2(\sum_{i=1}^{n-1} k_i x'^2_i))}$$
$$= \frac{\pi(n-3)!!}{{k_n}^{1/2}(n-2)!!} I_{n-1}(k_1,\dots,k_{n-1}). \tag{3.3}$$

For the case $n = 2k+1$, which is an odd number, applying integration by parts $k-1$ times in (3.2), we get

$$I_n(k_1,\dots,k_n) = \int_{\mathbf{S}^{n-2}} \frac{2(n-3)!!}{(\sum_{i=1}^{n-1} k_i x'^2_i)^{(k-1)}(n-2)!!} \int_0^\infty \frac{t\, dt\, \lambda_{n-2}(d\omega')}{(k_n + t^2(\sum_{i=1}^{n-1} k_i x'^2_i))^{3/2}}$$
$$= \frac{2(n-3)!!}{{k_n}^{1/2}(n-2)!!} I_{n-1}(k_1,\dots,k_{n-1}). \tag{3.4}$$

Finally using the recurrent relations (3.3) and (3.4), we obtain

Lemma 1 *Let $k_i > 0$, $i = 1,\dots,n$ be n positive real numbers, where $n \geq 2$. We have*

$$\int_{\mathbf{S}^{n-1}} \frac{1}{(\sum_{i=1}^{n} k_i x_i^2)^{n/2}} \lambda_{n-1}(d\omega) = \frac{\sigma_{n-1}}{\sqrt{k_1 k_2 \cdots k_n}}, \tag{3.5}$$

here x_i is the ith Cartesian coordinate of $\omega \in \mathbf{S}^{n-1} (\omega = (x_1, x_2, \dots, x_n))$.

Also we need to consider the following integral:

$$II_n(k_1,\dots,k_n) = \int_{\mathbf{S}^{n-1}} \frac{x_n^2}{(\sum_{i=1}^{n} k_i x_i^2)^{(n+2)/2}} \lambda_{n-1}(d\omega), \tag{3.6}$$

here x_i is the ith Cartesian coordinate of $\omega \in \mathbf{S}^{n-1}$ $(\omega = (x_1, x_2, \dots, x_n))$ and $k_i > 0$, $i = 1,\dots,n$ is a positive real number. By the same way (see (3.1)) from (3.6) using the spherical coordinates $\omega = (\nu, \omega')$ and making the substitution $t = \tan\nu$, we obtain

$$II_n(k_1,\dots,k_n) = 2\int_{\mathbf{S}^{n-2}} \int_0^{\pi/2} \frac{\cos^2\nu\, \sin^{n-2}\nu\, d\nu\, \lambda_{n-2}(d\omega')}{(k_n\cos^2\nu + \sin^2\nu(\sum_{i=1}^{n-1} k_i x'^2_i))^{(n+2)/2}}$$
$$= 2\int_{\mathbf{S}^{n-2}} \int_0^\infty \frac{t^{n-2}\, dt\, \lambda_{n-2}(d\omega')}{(k_n + t^2(\sum_{i=1}^{n-1}, k_i x'^2_i))^{(n+2)/2}}, \tag{3.7}$$

where ${x'}_i$ is the ith Cartesian coordinate of $\omega' \in \mathbf{S}^{n-2}$ $i = 1,\dots,n-1$.

For the case $n = 2k$, which is an even number, applying integration by parts $k-1$ times, we obtain (see Lemma 1)

$$\begin{aligned} II_n(k_1,\ldots,k_n) &= \int_{\mathbf{S}^{n-2}} \frac{4(n-3)!!}{(\sum_{i=1}^{n-1} k_i x'^2_i)^{(k-1)} n!!} \int_0^\infty \frac{dt\,\lambda_{n-2}(d\omega')}{(k_n + t^2(\sum_{i=1}^{n-1} k_i x'^2_i))^2} \\ &= \frac{\pi(n-3)!!}{k_n{}^{3/2} n!!} \int_{\mathbf{S}^{n-2}} \frac{\lambda_{n-2}(d\omega')}{(\sum_{i=1}^{n-1} k_i x'^2_i)^{(n-1)/2}} = \frac{(2\pi)^{n/2}}{k_n{}^{3/2} n!! \sqrt{k_1 k_2 \cdots k_{n-1}}}. \end{aligned} \tag{3.8}$$

For the case $n = 2k+1$, whih is an odd number, applying integration by parts $k-1$ times, we obtain (see Lemma 1)

$$\begin{aligned} II_n(k_1,\ldots,k_n) &= \int_{\mathbf{S}^{n-2}} \frac{6(n-3)!!}{(\sum_{i=1}^{n-1} k_i x'^2_i)^{(k-1)} n!!} \int_0^\infty \frac{t\,dt\,\lambda_{n-2}(d\omega')}{(k_n + t^2(\sum_{i=1}^{n-1} k_i x'^2_i))^{5/2}} \\ &= \frac{2(n-3)!!}{k_n{}^{3/2} n!!} \int_{\mathbf{S}^{n-2}} \frac{\lambda_{n-2}(d\omega')}{(\sum_{i=1}^{n-1} k_i x'^2_i)^{(n-1)/2}} = \frac{2(2\pi)^{[n/2]}}{k_n{}^{3/2} n!! \sqrt{k_1 k_2 \cdots k_{n-1}}}. \end{aligned} \tag{3.9}$$

Finally, we using (3.8) and (3.9), we get the following lemma:

Lemma 2 *Let $k_i > 0$, $i = 1,\ldots,n$ be n positive real numbers, where $n \geq 2$. We have*

$$\int_{\mathbf{S}^{n-1}} \frac{x_n^2}{\left(\sum_{i=1}^n k_i x_i^2\right)^{(n+2)/2}} \lambda_{n-1}(d\omega) = \frac{\sigma_{n-1}}{n\, k_n{}^{3/2} \sqrt{k_1 k_2 \cdots k_{n-1}}}, \tag{3.10}$$

here x_i is the ith Cartesian coordinate of $\omega \in \mathbf{S}^{n-1}$ $(\omega = (x_1, x_2, \ldots, x_n))$.

4 Proof of Theorem 4

For $\omega, \xi \in \mathrm{S}^{n-1}$, $\omega \neq \xi$, we have

$$\langle \xi_\omega, \varphi \rangle = \sum_{i=1}^{n-1} \langle \varphi, \varphi_i \rangle \, \langle \xi_\omega, \varphi_i \rangle, \tag{4.1}$$

here $\varphi \in \mathrm{S}_\omega$, φ_i is the unit vector at ith principal direction $(i = 1,\ldots,n-1)$, ξ_ω is the unit vector of the projection ξ onto the plane e_ω. Substituting (4.1) into the left side of (2.5), we get

$$\begin{aligned} &\frac{(n-1)}{\sigma_{n-2}} \int_{\mathbf{S}_\omega} \langle \xi_\omega, \varphi \rangle^2 \frac{\sqrt{K(\omega)}}{(k(\omega,\varphi))^{\frac{n+1}{2}}} \lambda_{n-2}(d\varphi) \\ &= \frac{(n-1)}{\sigma_{n-2}} \sum_{i=1}^{n-1} \langle \xi_\omega, \varphi_i \rangle^2 \int_{\mathbf{S}_\omega} \langle \varphi_i, \varphi \rangle^2 \frac{\sqrt{K(\omega)}}{(k(\omega,\varphi))^{\frac{n+1}{2}}} \lambda_{n-2}(d\varphi), \end{aligned} \tag{4.2}$$

note that the other summands equal 0. It follows from Lemma 2 that

$$\frac{(n-1)}{\sigma_{n-2}} \int_{\mathbf{S}_\omega} \langle \varphi_i, \varphi \rangle^2 \frac{\sqrt{K(\omega)}}{\left(\sum_{i=1}^{n-1} k_i \langle \varphi_i, \varphi \rangle^2\right)^{\frac{n+1}{2}}} \lambda_{n-2}(d\varphi) = \frac{1}{k_i}. \tag{4.3}$$

Substituting (4.3) into (4.2), we get (2.5). Theorem 4 is proved.

Substituting (2.5) into (2.1)and taking into account Blaschke's theorem, we get the following theorem.

Theorem 5 *The support function of 2-smooth origin symmetric convex body $\mathbf{B} \in \mathcal{B}_o^n$ has the following representation. For $\xi \in \mathbf{S}^{n-1}$*

$$H(\xi) = \frac{(n-1)}{2\,\sigma_{n-2}^2} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}_\omega} \frac{\langle \xi_\omega, \varphi \rangle^2}{\sin^{n-3}\widehat{(\omega, \xi)}} \frac{\sqrt{K(\omega)}}{(k(\omega, \varphi))^{\frac{n+1}{2}}} \lambda_{n-2}(d\varphi)\, \lambda_{n-1}(d\omega). \tag{4.4}$$

Note that in $\mathbf{R}^3$ the representation (4.4) first was found by stochastic approximation of $\mathbf{B} \in \mathcal{B}_o^3$ (see [5]).

Now we are going to use the dual representation for a flag. We consider the following transform:

$$(\omega, \varphi) \longrightarrow (\Omega, \Phi)$$

(we write the flag (ω, φ) in dual coordinates (Ω, Φ)). It is known that the Jacobian of the transform equals 1 (see [2]):

$$\lambda_{n-2}(d\varphi)\, \lambda_{n-1}(d\omega) = \lambda_{n-2}(d\Phi)\, \lambda_{n-1}(d\Omega). \tag{4.5}$$

After change of variables in (4.4) using (4.5) for $\xi \in \mathbf{S}^{n-1}$, we obtain

$$H(\xi) = \frac{(n-1)}{2\,\sigma_{n-2}^2} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}_\Omega} \frac{\langle \xi_\Phi, \Omega \rangle^2}{\sin^{n-3}\widehat{(\xi, \Phi)}} \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} \lambda_{n-2}(d\Phi)\, \lambda_{n-1}(d\Omega). \tag{4.6}$$

We consider the right spherical triangle with the vertices ξ, Ω and ξ_Φ (where ξ_Φ is the unit vector of the projection ξ onto the plane e_Φ) and the angle at ξ_Φ is equal to $\pi/2$. Applying the spherical cosine rule for the right spherical triangle, we get

$$\cos\widehat{(\xi, \Omega)} = \cos\widehat{(\xi_\Phi, \Omega)}\, \cos\widehat{(\xi_\Phi, \xi)} = \cos\widehat{(\xi_\Phi, \Omega)}\, \sin\widehat{(\xi, \Phi)}. \tag{4.7}$$

Substituting (4.7) into (4.6), we obtain (1.5). Theorem 1 is proved.

5 A Sufficient Condition to be a Zonoid

Theorem 2 follows immediately from Theorem 1. In this section, we are going to prove Theorem 3. Let $\mathbf{B} \in \mathcal{B}_o^n$ be an origin symmetric 2-smooth convex body in $\mathbf{R}^n$

and for any $\Omega \in \mathbf{S}^{n-1}$ and $\Phi \in \mathrm{S}_\Omega$ the expression

$$G(\Omega, \Phi) = \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} = G(\Omega) \tag{5.1}$$

does not depend on $\Phi \in \mathrm{S}_\Omega$. Applying Fubini's theorem for the inner integral of (1.5), we obtain

$$H(\xi) = \frac{(n-1)}{2\,\sigma_{n-2}^2} \int_{\mathbf{S}^{n-1}} \left[\int_{\mathrm{S}_\Omega} \frac{\langle \xi, \Omega \rangle^2}{\sin^{n-1}(\widehat{\xi, \Phi})}\, \lambda_{n-2}(d\Phi) \right] G(\Omega)\, \lambda_{n-1}(d\Omega), \qquad \xi \in \mathbf{S}^{n-1}. \tag{5.2}$$

The following lemma is valid.

Lemma 3 *For $\Omega, \xi \in \mathrm{S}^{n-1} (n \geq 3)$, we have*

$$\int_{\mathrm{S}_\Omega} \frac{\langle \xi, \Omega \rangle^2}{\sin^{n-1}(\widehat{\xi, \Phi})}\, \lambda_{n-2}(d\Phi) = \frac{\sigma_{n-1}\,(n-2)!!}{2\,(n-3)!!} \mid \langle \xi, \Omega \rangle \mid . \tag{5.3}$$

We consider the right spherical triangle with the vertices ξ, Φ and ξ_Ω (where ξ_Ω is the unit vector of the projection ξ onto the plane e_Ω) and the angle at ξ_Ω is equal to $\pi/2$. Applying the spherical cosine rule for the right spherical triangle, we get

$$\cos(\widehat{\xi, \Phi}) = \cos(\widehat{\xi, \xi_\Omega})\, \cos(\widehat{\xi_\Omega, \Phi}) \tag{5.4}$$

and from which we have,

$$\begin{aligned} \sin(\widehat{\xi, \Phi}) &= \sqrt{1 - \sin^2(\widehat{\xi, \Omega})\, \cos^2(\widehat{\xi_\Omega, \Phi})} \\ &= \sqrt{\sin^2(\widehat{\xi_\Omega, \Phi}) + \cos^2(\widehat{\xi, \Omega})\, \cos^2(\widehat{\xi_\Omega, \Phi})}. \end{aligned} \tag{5.5}$$

Using the spherical coordinates for $\Phi \in \mathbf{S}^{n-2}$ we have $\Phi = (\nu, \Phi')$ where $\nu = (\widehat{\xi_\Omega, \Phi})$ is the polar angle measured from ξ_Ω (the zenith direction) and $\Phi' \in \mathbf{S}^{n-3}$ is the direction of orthogonal projection of Φ onto the hyperplane that passes through the origin and is orthogonal to the zenith.

Substituting (5.5) into (5.3), we obtain

$$\begin{aligned} &\cos^2(\widehat{\xi, \Omega}) \int_{\mathrm{S}^{n-3}} \int_0^\pi \frac{\sin^{n-3}\nu}{(\sin^2\nu + \cos^2(\widehat{\xi, \Omega}) \cos^2\nu)^{(n-1)/2}} d\nu\, \lambda_{n-3}(d\Phi') \\ &= \frac{\sigma_{n-1}\,(n-2)!!}{2\,(n-3)!!} \mid \langle \xi, \Omega \rangle \mid . \end{aligned} \tag{5.6}$$

Lemma 3 is proved.

Substituting (5.3) into (5.2), we obtain

$$H(\xi) = \frac{(n-1)\sigma_{n-1}\,(n-2)!!}{4\,\sigma_{n-2}^2\,(n-3)!!}\int_{\mathbf{S}^{n-1}} |\,\langle \xi, \Omega\rangle\,|\; G(\Omega)\,\lambda_{n-1}(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{5.7}$$

Thus the support function of $\mathbf{B}$ admits the zonoid representation (1.2) with positive even measure; hence $\mathbf{B}$ is a zonoid. Theorem 3 is proved.

It was proved in [6] that in $\mathbf{R}^3$ a convex body boundary of which is an ellipsoid satisfy the condition (1.9) (Theorem 3). Let $\mathbf{B} \in \mathcal{B}_o^3$ be a convex body boundary of which is an ellipsoid with the semi-principal axes of length a, b, c. For $\Omega = (n_1, n_2, n_3) \in \mathbf{S}^2$ and and $\Phi \in \mathrm{S}_\Omega$, we have (see [6])

$$G(\Omega, \Phi) = G(\Omega) = \frac{1}{(abc)}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right)^{-2}. \tag{5.8}$$

Note that here $G(\Omega)$ is the generating density (multiplied by a constant, see (1.5), (1.7)) of $\mathbf{B}$.

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