Asymptotic Error of Polynomial-Periodic Interpolation

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Abstract

In this paper we continue investigations started in [6] and consider parametric interpolation of smooth but non periodic function defined on the finite interval. We analyze L_2-convergence of such interpolations and obtain exact formulae for the principal term of L_2-error. An optimization problem is solved. Numerical results are presented.

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1 Introduction

Further, we use conventions: the prime on the summation indicates that the zero term is omitted; $[x]$ stands for the integer part of x; $\mathbf{Z}$ is the set of integers; then we put

$$\sum_n{}^N \cdot = \sum_{n=-[N/2]}^{-[N/2]+N-1} \cdot, \quad \sum_{n\neq 0}{}^N \cdot = \sum_{n=-[N/2]}^{-[N/2]+N-1}{}' \cdot,$$

where $N \geq 1$ is an integer.

Consider the following parametric interpolation (see [4],[5]) for $f \in C[-1,1]$

$$I_N(f,\delta) = \sum_n{}^N \frac{\check{f}_n}{d(\frac{n}{N})} \sum_{s\in\mathbf{Z}} \theta\left(\frac{n}{N}+s\right) e^{i\pi(n+sN)x} e^{-i\pi s\delta}, \qquad (1)$$

where θ is such that $\sum_{s\in Z}|\theta(x+s)|<\infty$, $d(x)=\sum_{s\in Z}\theta(x+s)\neq 0$ and

$$\check{f}_n=\frac{1}{N}\sum_k{}^N f(x_k)e^{-i\pi n x_k},\ x_k=\frac{2k+\delta}{N}$$

with $0\leq\delta\leq 2$ if N is even and $-1\leq\delta\leq 1$ if N is odd.

Acceleration problem arises if approximated function $f^q\in C[-1,1],\ q\geq 0$ has no smooth periodic continuation. One way to solve this problem is based on application of Bernoulli polynomials (see [1]-[3]). Denote

$$A_k(f)=f^{(k)}(1)-f^{(k)}(-1),\ k=0,\cdots,q$$

and consider the following *polynomial-periodic interpolation*

$$I_{q,N}(f,\delta)=I_N\left(f(x)-\sum_{k=0}^{q}A_k(f)B_k(x),\delta\right)+\sum_{k=0}^{q}A_k(f)B_k(x), \tag{2}$$

where the Bernoulli polynomials B_k are defined by the recurrent relations

$$B_0(x)=x/2,\ B_k(x)=\int B_{k-1}(x)dx,\ \int_{-1}^{1}B_k(x)dx=0,\ x\in[-1,1].$$

Our aim is to investigate the convergence of $I_{q,N}(f,\delta)$ in the framework of L_2-convergence. Some relevant results concerning splines can be found in [7].

2 Asymptotic L_2-estimates

We put

$$f_n=\frac{1}{2}\int_{-1}^{1}f(x)e^{-i\pi n x}dx.$$

Lemma 1 *If* $f\in \mathbf{C^{q+1}}[-1,1],\ f^{(q+2)}\in L_2(-1,1)$, $A_s(f)=0$, $s=0,\cdots,q+1$, $q\geq 0$, *then*

$$\sum_{r\in\mathbf{Z}}{}' f_{n+rN}e^{i\pi r\delta}=o(N^{-q-2}),\ N\to\infty,\ -[N/2]\leq n\leq -[N/2]+N-1. \tag{3}$$

Proof. Evidently,

$$f_n = \frac{\varepsilon_n}{2(i\pi n)^{q+2}},\ \varepsilon_n = \int_{-1}^{1} f^{(q+2)}(x) e^{-i\pi n x} dx,\ n \neq 0. \tag{4}$$

Taking into account the convergence of the series $\sum_{n\in\mathbf{Z}} |\varepsilon_n|^2$, we obtain

$$\sum_{n\in\mathbf{Z}} |\varepsilon_n|^2 = \sum_n{}^N \sum_{r\in\mathbf{Z}} |\varepsilon_{n+rN}|^2 = \sum_n{}^N |\varepsilon_n|^2 + \sum_n{}^N \sum_{r\in\mathbf{Z}}{}' |\varepsilon_{n+rN}|^2.$$

Hence,

$$\lim_{N\to\infty} \sum_n{}^N \sum_{r\in\mathbf{Z}}{}' |\varepsilon_{n+rN}|^2 = 0$$

and therefore,

$$\lim_{N\to\infty} \sum_{r\in\mathbf{Z}}{}' |\varepsilon_{n+rN}|^2 = 0,\ N \to \infty,\ -[N/2] \le n \le -[N/2] + N - 1.$$

From this, we get

$$\left| \sum_{r\in\mathbf{Z}}{}' f_{n+rN} e^{i\pi r\delta} \right| \le const \sum_{r\in\mathbf{Z}}{}' \left| \frac{\varepsilon_{n+rN}}{(n+rN)^{q+2}} \right| \le$$

$$\le \frac{const}{N^{q+2}} \left(\sum_{r\in\mathbf{Z}}{}' |\varepsilon_{n+rN}|^2 \right)^{1/2} = \frac{o(1)}{N^{q+2}},\ N \to \infty. \bullet$$

By $||\cdot||$ denote the standard norm of the space $L_2(-1,1)$.
Also denote

$$d(x) = \sum_{s\in\mathbf{Z}} \theta(x+s),\ \alpha(x) = \frac{\sum_{s\in\mathbf{Z}}' \theta(x+s)}{d(x)},\ \beta(x) = \frac{\sum_{s\in\mathbf{Z}}' |\theta(x+s)|^2}{|d(x)|^2}.$$

Definition. We say that $\theta \in \mathbf{T}$ if the following conditions hold: (i) θ is piecewise continuous in R and to be definite is normalized with $\theta(x) = 1/2(\theta(x+0) + \theta(x-0))$; (ii) the sum $\sum_{s\in\mathbf{Z}} |\theta(x+s)|$ uniformly converges in $[-1/2, 1/2]$; (iii) $\alpha(x)$ and $\beta(x)$ are bounded in $[-1/2, 1/2]$; (iv) there exists monotone in $(-1/2, 0)$ as well as in $(0, 1/2)$ and integrable in $(-1/2, 1/2)$ non-negative function μ such that

$$|\alpha(x)|^2 x^{-2q-4} \le \mu(x),\ \beta(x) x^{-2q-4} \le \mu(x).$$

Lemma 2 *Suppose $\theta \in \mathbf{T}$ and the conditions of Lemma 1 hold; then*

$$||f - I_N(f,\delta)|| = o(N^{-q-1.5}), \; N \to \infty.$$

Proof. Taking into account that $\check{f}_n = \sum_{r\in\mathbf{Z}} f_{n+rN} e^{i\pi r\delta}$, we obtain

$$||f - I_N(f,\delta)|| =$$

$$= \left(2 \sum_n^N \sum_{r\in\mathbf{Z}} \left| f_{n+rN} e^{i\pi r\delta} - \frac{\theta\left(\frac{n}{N}+r\right)}{d(\frac{n}{N})} \sum_{s\in\mathbf{Z}} f_{n+sN} e^{i\pi s\delta} \right|^2 \right)^{1/2}. \tag{5}$$

We apply the triangle inequality to (5), by breaking the sums into four parts, assuming: $\{r=0, s=0\}$, $\{r\neq 0, s=0\}$, $\{r=0, s\neq 0\}$, $\{r\neq 0, s\neq 0\}$. In the case $\{r=0, s=0\}$ by Lemma 1, it follows ($t_n = \frac{n}{N}$)

$$\left(2\sum_n^N |f_n \alpha(t_n)|^2 \right)^{1/2} = \frac{const}{N^{q+2}} \left(\sum_n^N |\varepsilon_n|^2 \left| \frac{\alpha(t_n)}{t_n^{q+2}} \right|^2 \right)^{1/2} \leq$$

$$\frac{const}{N^{q+1.5}} \left(\int_{-\frac{[\sqrt{N}]}{N}}^{\frac{[\sqrt{N}]}{N}} \mu(x)dx \right)^{1/2} +$$

$$\frac{const}{N^{q+1.5}} \left(\int_{-1/2}^{1/2} \mu(x)dx \right)^{1/2} \sup_{\sqrt{N}<|n|\leq[N/2]} |\varepsilon_n| = o(N^{-q-1.5}), \; N\to\infty.$$

Application of similar arguments in other cases lead to the required estimate.•

Using Lemmas 1,2 the following theorem holds.

Theorem 1 *Suppose $\theta \in \mathbf{T}$ $f \in \mathbf{C^{q+1}}[-1,1]$, $f^{(q+2)} \in L_2(-1,1)$, $q \geq 0$; then*

$$N^{2q+3}||f - I_{q,N}(f,\delta)||^2 \to$$

$$\frac{|A_{q+1}(f)|^2}{2\pi^{2q+4}} \int_{-1/2}^{1/2} \sum_{r\in\mathbf{Z}} \left| \frac{(-1)^{r\sigma} e^{i\pi r\delta}}{(x+r)^{q+2}} - \frac{\theta(x+r)}{d(x)} \sum_{s\in\mathbf{Z}} \frac{(-1)^{s\sigma} e^{i\pi s\delta}}{(x+s)^{q+2}} \right|^2 dx, \tag{6}$$

while $N\to\infty$ remains odd or even and $\sigma = 0$ if N is even and $\sigma = 1$ if N is odd.

Proof. We use the inequality

$$||f - I_{q,N}(f,\delta)|| = ||F - I_N(F,\delta)|| \le$$

$$\le ||F_1 - I_N(F_1,\delta)|| + ||F_2 - I_N(F_2,\delta)||, \tag{7}$$

where

$$F = F_1 + F_2, \tag{8}$$

and

$$F_1(x) = {\sum_{n\in\mathbf{Z}}}' F_{1,n} e^{i\pi nx}, \;\; F_{1,n} = \frac{(-1)^{n+1}}{2} \frac{A_{q+1}(f)}{(i\pi n)^{q+2}}, \;\; n \ne 0,$$

$$F_2(x) = {\sum_{n\in\mathbf{Z}}}' F_{2,n} e^{i\pi nx}, \;\; F_{2,n} = \frac{1}{2(i\pi n)^{q+2}} \int_{-1}^{1} f^{(q+2)}(x) e^{-i\pi nx} dx, \;\; n \ne 0.$$

By Lemma 2,

$$\lim_{N\to\infty} N^{q+1.5} ||F_2 - I_N(F_2,\delta)|| = 0. \tag{9}$$

On the other hand ($t_n = n/N$),

$$||F_1 - I_N(F_1,\delta)||^2 = \frac{|A_{q+1}(f)|^2}{2(\pi N)^{2q+4}} \sum_n^N G(t_n),$$

where

$$G(x) = \sum_{r\in\mathbf{Z}} \left| \frac{(-1)^{r\sigma} e^{i\pi r\delta}}{(x+r)^{q+2}} - \frac{\theta(x+r)}{d(x)} \sum_{s\in\mathbf{Z}} \frac{(-1)^{s\sigma} e^{i\pi s\delta}}{(x+s)^{q+2}} \right|^2.$$

The assumptions of the theorem imply integrability of the function $G(x)$ in the interval $(-1/2, 1/2)$ and the existence of the limit (N remains odd or even)

$$\lim_{N\to\infty} \frac{1}{N} \sum_n^N G(t_n) = \int_{-1/2}^{1/2} G(x) dx.$$

With (9) this proves the theorem.•

3 Detail investigation of special cases

In this item we consider three examples. All calculations are executed by the MATHEMATICA package [8].

Example 1. Consider the simplest case

$$\theta(x) = \begin{cases} 1, & |x| < 1/2, \\ 1/2, & x = \pm 1/2, \\ 0, & |x| > 1/2. \end{cases}$$

Here $I_{q,N}(f, \delta)$ coincides with Bernoulli method (see [1]-[3]).

The reader will easily prove that

$$d(x) \equiv 1, \;\; x \in [-1/2, 1/2],$$

$$\alpha(x) = \begin{cases} 0, & |x| < 1/2, \\ 1/2, & x = \pm 1/2. \end{cases},$$

$$\beta(x) = \begin{cases} 0, & |x| < 1/2, \\ 1/4, & x = \pm 1/2. \end{cases}.$$

Using Theorem 1, we get

$$N^{q+1.5} ||f - I_{q,N}(f, \delta)|| \to |A_{q+1}(f)| a_q(\sigma, \delta),$$

where

$$a_q(\sigma, \delta) = \frac{1}{\pi^{q+2}} \left[\frac{2^{2q+3}}{2q+3} + \int_{-1/2}^{1/2} \left| {\sum_{s \in \mathbf{Z}}}' \frac{(-1)^{s\sigma} e^{i\pi s \delta}}{(x+s)^{q+2}} \right|^2 dx \right]^{1/2}.$$

The following relations

$$a_q(0, 1-\delta) = a_q(0, 1+\delta), \;\; a_q(1, -\delta) = a_q(1, \delta), \quad 0 \le \delta \le 1,$$

$$a_q(0, \delta) = a_q(1, \delta - 1), \quad a_q(1, \delta) = a_q(0, \delta - 1), \quad 0 \le \delta \le 1$$

simplify the analysis of $a_q(\sigma, \delta)$. Hence, for the same q, the behavior of the functions $a_q(0, \delta)$ and $a_q(1, \delta)$ are the same. On Fig.1 the graphics of the functions $a_0(0, \delta)$ and $a_1(1, \delta)$ are represented. These two graphics are enough for complete analysis of $a_q(\sigma, \delta)$ as for even q the behavior of $a_q(\sigma, \delta)$

for $\sigma = 0, 1$ is similar to the left diagram on Fig.1 otherwise to the right.

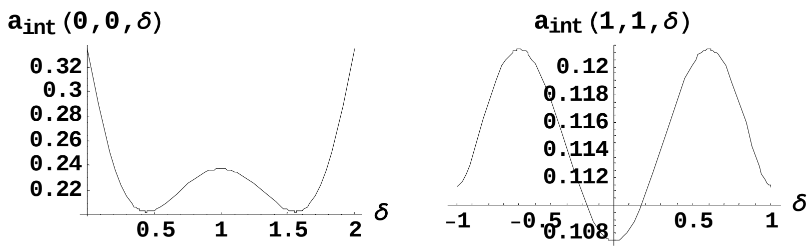

Fig.1. Functions $a_0(0,\delta)$ and $a_1(1,\delta)$.

Such behavior of $a_q(\sigma,\delta)$ allows to find an optimal value δ_{opt} of the parameter δ for fixed values of q. Table 1 contains the values of δ_{opt} and $a_q(0,\delta_{opt})$.

q	0	1	2	3
δ_{opt}	0.442400	1	0.505015	1
$a_q(0,\delta_{opt})$	0.202665	0.107417	0.060729	0.034503
q	4	5	6	7
δ_{opt}	0.521790	1	0.527234	1
$a_q(0,\delta_{opt})$	0.020017	0.011739	0.006964	0.00416521

Table 1. The optimal values δ_{opt} of the parameter δ
and the values of $a_q(0,\delta_{opt})$.

Example 2. Consider the following

$$\theta(x) = \begin{cases} \cos^s \frac{\pi}{2}x, & |x| \le 1, \\ 0, & |x| > 1, \end{cases} \qquad s > q + 1.5.$$

We have

$$d(x) = \cos^s \frac{\pi}{2}x + \sin^s \frac{\pi}{2}|x|,$$

$$\alpha(x) = \frac{\sin^s \frac{\pi}{2}|x|}{\cos^s \frac{\pi}{2}x + \sin^s \frac{\pi}{2}|x|}, \ \beta(x) = \frac{\sin^{2s} \frac{\pi}{2}|x|}{(\cos^s \frac{\pi}{2}x + \sin^s \frac{\pi}{2}|x|)^2}.$$

By Theorem 1

$$N^{q+1.5}||f - I_{q,N}(f,\delta)|| \to |A_{q+1}(f)| b_q(\sigma,\delta,s), \ N \to \infty$$

where

$$b_q(\sigma,\delta,s) = \frac{1}{\pi^{q+2}}\left[\int_0^1 \left|\frac{\alpha(x)}{x^{q+2}} - (1-\alpha(x)){\sum_{r\in\mathbf{Z}}}' \frac{(-1)^{r\sigma}e^{i\pi r\delta}}{(x+r)^{q+2}}\right|^2 dx + \frac{1}{2q+3}\right]^{1/2}.$$

On Fig. 2 the contour plots of the graphics (projection of the graphics on $s\times\delta$ plane) of the functions $b_0(1,\delta,s)$ and $b_1(1,\delta,s)$ are represented, where the lower values of b_q are marked by dark colors. It is easy to check that for fixed q, the graphics of the functions $b_q(\sigma,\delta,s),\ \sigma=0,1$ are the same. If q is even then the behavior of the $b_q(\sigma,\delta,s),\ \sigma=0,1$ is similar to the left diagram on Fig.2 otherwise to the right.

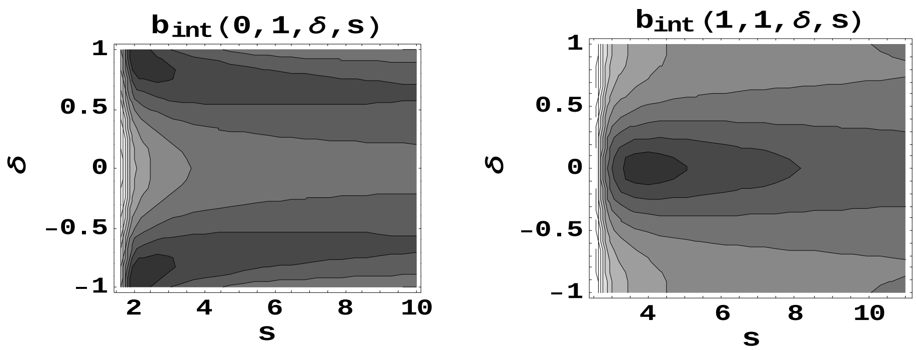

Fig.2. Contour plots of the graphics of the functions $b_0(1,\delta,s)$ and $b_1(1,\delta,s)$.

Table 2 contains the optimal values s_{opt}, δ_{opt} of parameters s, δ and the values of $b(q,0,\delta_{opt},s_{opt})$.

q	s_{opt}	δ_{opt}	$b_q(0,\delta_{opt},s_{opt})$	a_q/b_q
0	2.166471	0.097154	0.095751	2.1
1	3.842676	1	0.018596	5.8
2	4.809502	0	0.007590	8
3	6.295909	1	0.001700	20
4	7.503024	0	0.000941	21
5	8.835812	1	0.000252	46
6	10.106305	0	0.000137	51
7	11.397712	1	0.000049	85

Table 2. The optimal values of s_{opt}, δ_{opt} and the values of $b(q,0,\delta_{opt},s_{opt})$.

The last column in Table 2 describes the efficiency of optimal interpolation relatively to Example 1.

Example 3. Finally consider the following

$$\frac{\theta(x)}{d(x)} = \frac{1}{x^{q+2}}\left[\sum_{r\in\mathbf{Z}}\frac{(-1)^{r\sigma}e^{i\pi r\delta}}{(x+r)^{q+2}}\right]^{-1}, \tag{10}$$

where $\sigma=0$, $0\le\delta\le 2$ if N is even and $\sigma=1$, $-1\le\delta\le 1$ if is odd.

Evidently,

$$\theta(0)/d(0)=1,\ \theta(s)/d(0)=0\ \ if\ \ s\in\mathbf{Z}/\{0\}$$

$$\alpha(x) = \left(\sum_{r\in\mathbf{Z}}\frac{(-1)^{r\sigma}e^{i\pi r\delta}}{(x+r)^{q+2}}\right)^{-1}\sum_{r\in\mathbf{Z}}{}'\frac{(-1)^{r\sigma}e^{i\pi r\delta}}{(x+r)^{q+2}},$$

$$\beta(x) = \left|\sum_{r\in\mathbf{Z}}\frac{(-1)^{r\sigma}e^{i\pi r\delta}}{(x+r)^{q+2}}\right|^{-2}\sum_{r\in\mathbf{Z}}{}'\frac{1}{(x+r)^{2q+4}}.$$

Here the integral in (6) vanishes and we have more rapid convergence

$$||f-I_{q,N}(f,\delta)|| = o(N^{-q-1.5}),\ \ N\to\infty.$$

Now we represent a more exact estimate, displaying the principal term of asymptotic of $||f-I_{q,N}(f,\delta)||$.

Theorem 2 *Let $\theta(x)/d(x)$ is defined by* (10) *and $f\in Lip(q+2.5)$, $q\ge 0$; then*

$$N^{2q+4}||f-I_N^q(f,\delta)||^2 \to \frac{1}{2\pi^{2q+4}}\left(2\zeta(2q+4)+\left|\sum_{s\in\mathbf{Z}}{}'\frac{(-1)^{s\sigma}e^{i\pi s\delta}}{s^{q+2}}\right|^2\right)\times$$
$$\left(|A_{q+1}|^2+2\int_{-1}^{1}|f^{(q+2)}(z)|^2dz-\left|\int_{-1}^{1}f^{(q+2)}(z)dz\right|^2\right), \tag{11}$$

while $N\to\infty$ remains odd or even; $\zeta(s)=\sum_{r=1}^{\infty}r^{-s}$ is the Riemann function; σ,δ are the same as in Theorem 1.

Proof. We start as in Theorem 1. It is easy to check that (see (8))

$$F_{1,n+rN}e^{i\pi r\delta} - \frac{\theta_{n+rN}}{d_n}\sum_{s\in\mathbf{Z}} F_{1,n+sN}e^{i\pi s\delta} = 0,\ \ s\in\mathbf{Z},\ \ n\neq 0. \tag{12}$$

Now for $||F_2 - I_N(F_2,\delta)||$ we note that $\sum_{r\in z}{}' F_{2,n+rN} = o(N^{-q-2}),\ \ N\to\infty$ and proceed as in the proof of Lemma 1. In the case $\{r=0, s=0\}$ we have $(t_n = n/N)$

$$\lim_{N\to\infty} 2\sum_{n\neq 0}{}^N |F_{2,n}\alpha(t_n)|^2 =$$

$$= \lim_{N\to\infty}\frac{1}{2\pi^{2q+4}}\sum_{n\neq 0}{}^N \frac{|\alpha(t_n)|^2}{t_n^{2q+4}}\left|\int_{-1}^{1} f^{(q+2)}(x)e^{-i\pi nx}dx\right|^2 =$$

$$\frac{1}{2\pi^{2q+4}}\left|\sum_{r\in\mathbf{Z}}{}' \frac{(-1)^{r\sigma}e^{i\pi r\delta}}{r^{q+2}}\right|^2 \sum_{n\in\mathbf{Z}}{}' \left|\int_{-1}^{1} f^{(q+2)}(x)e^{-i\pi nx}dx\right|^2.$$

Arguing as above, we see that

$$N^{2q+4}||f - I_N^q(f,\delta)||^2 \to \frac{1}{2\pi^{2q+4}}\left(2\zeta(2q+4) + \left|\sum_{s\in\mathbf{Z}}{}' \frac{(-1)^{s\sigma}e^{i\pi s\delta}}{s^{q+2}}\right|^2\right)\times$$

$$\left(|A_{q+1}|^2 + \sum_{n\in\mathbf{Z}}{}' \left|\int_{-1}^{1} f^{(q+2)}e^{-i\pi nz}(z)dz\right|^2\right).$$

This completes the proof by Parseval equality.•

As an application of Theorem 2 consider the well known case of the shifted B-splines

$$\theta(x) = \left(\frac{\sin\pi x}{\pi x}\right)^{q+2} e^{i\pi\delta x}. \tag{13}$$

It is easy to check that the function (13) generates a sequence satisfying (10). Hence, we can apply Theorem 2.

4 Conclusion

Approximation of smooth in the finite interval but non periodic function by classical trigonometric interpolation ($I_N(f,\delta)$ with θ as in Example 1) is non efficient due to slow L_2-convergence. Acceleration problem can be solved by Bernoulli method that corresponds to approximation $I_{q,N}(f,\delta)$ with θ as in Example 1.

Our investigations show that parametric interpolation $I_N(f,\delta)$ in combination with Bernoulli method (in our notation $I_{q,N}(f,\delta)$) provides more efficient approximation. In Example 2 we get 20 times more precise approximation for $q = 3$ compared with Bernoulli method. Moreover, Example 3 gives $\sqrt{N}$ times more rapid rate of convergence compared with Examples 1,2.

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