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ON THE SWEEPING OUT PROPERTY FOR CONVOLUTION OPERATORS OF DISCRETE MEASURES

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Abstract. Let μ_n be a sequence of discrete measures on the unit circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ with $\mu_n(0)=0$, and $\mu_n((-\delta,\delta))\to 1$, as $n\to\infty$. We prove that the sequence of convolution operators $(f*\mu_n)(x)$ is strong sweeping out, i.e. there exists a set $E\subset\mathbb{T}$ such that

$$\limsup_{n\to\infty}(\mathbb{I}_E*\mu_n)(x)=1,\quad \liminf_{n\to\infty}(\mathbb{I}_E*\mu_n)(x)=0,$$

almost everywhere on $\mathbb{T}$.

1. Introduction

We consider bounded discrete measures

$$\mu=\sum_k m_k\delta_{x_k},\quad \sum_k m_k<\infty,$$

on the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, where $X=\{x_k\}$ is a finite or countable set in $\mathbb{T}$ and δ_{x_k} is Dirac measure at x_k. Denote

$$S_\mu f(x)=\int_{\mathbb{R}} f(x+t)d\mu(t).$$

Let μ_n be a sequence of discrete measures satisfying

$$\mu_n(0)=0,\quad \mu_n((-\delta,\delta))\to 1,\ \text{as } n\to\infty, \tag{1.1}$$

for any $0<\delta\le 1/2$. It is clear if $f\in L^1(\mathbb{T})$ is continuous at $x\in\mathbb{T}$ then

$$S_{\mu_n}f(x)\to f(x), \tag{1.2}$$

and the convergence is uniformly if $f\in C(\mathbb{T})$. The almost everywhere convergence problem in the case of general $f\in L^1(\mathbb{T})$ is not trivial. J. Bourgain in [4] proved

Theorem 1 (J. Bourgain)**.** *If $x_k\searrow 0$ as $k\to\infty$, and*

$$\mu_n=\frac{1}{n}\sum_{k=1}^{n}\delta_{x_k},$$

then there exists a function $f\in L^\infty$, such that $S_{\mu_n}f(x)$ diverges on a set of positive measure.

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In fact, this theorem gave a negative answer to a problem due to A. Bellow [3] and the proof is based on a general theorem often referred as Bourgain's entropy principle. Applying his principle Bourgain was able to deduce an analogous theorems for Riemann sums

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right),$$

and for the operators

$$\frac{1}{n}\sum_{k=1}^{n} f(kx).$$

We note, that first theorem was earlier obtained by W. Rudin [8] by different technique, and the second by J. Marstrand in [7]. S. Kostyukovsky and A. Olevskii in [6], using the same entropy principle, extended Theorem 1 for general discrete sequences satisfying (1.1).

We found a new geometric proof for Theorem 1, as well as for the result from [6]. Moreover, the method allows to obtain a stronger divergence for the operators (1.2). So in this paper we prove

Theorem 2. *If discrete measures μ_n satisfy (1.1), then there exists a set $E \subset \mathbb{T}$, such that*

$$\limsup_{n\to\infty} S_{\mu_n}\mathbb{I}_E(x) = 1, \quad \liminf_{n\to\infty} S_{\mu_n}\mathbb{I}_E(x) = 0 \tag{1.3}$$

almost everywhere on $\mathbb{T}$.

The relations (1.3) for sequences of operators is called strong sweeping out property. These kind of operators are investigated by M. Akcoglu, A. Bellow, R. L. Jones, V. Losert, K.Reinhold-Larsson, M. Wierdl [1] and by M. Akcoglu, M. D. Ha, R. L. Jones [2]. In [1] strong sweeping out property for Riemann sums operators is obtained. In [2] authors prove a general version of Bourgain's entropy principle, which allows to deduce sweeping out properties for some operators, but the principle is not applicable for the operators S_{μ_n}. The proof of Theorem 2 is based on Lemma 6. It will be obtained from Lemma 6 simply applying a general result proved in [5].

2. Proof of theorem

Let

$$X = \{x_i : i = 1, \ldots, l\}, \quad 0 < x_1 \le x_2 \le \ldots \le x_l < 1, \tag{2.1}$$

be an arbitrary sequence of reals. Suppose

$$Y = \{y_i : i = 1, \ldots, \nu\}, \quad y_1 < y_2 < \ldots < y_\nu = x_l$$

is a maximal independent (with respect to rational numbers) subset of X containing x_l. Then we have

$$x_k = r_1^{(k)} y_1 + \ldots + r_\nu^{(k)} y_\nu, \quad k = 1, 2, \ldots, l,$$

for some rational numbers $r_i^{(k)}$. Let p be the least common multiple of the denominators of $r_i^{(k)}$. Then we get

$$x_k = \frac{n_1^{(k)} y_1 + n_2^{(k)} y_2 + \ldots + n_\nu^{(k)} y_\nu}{p}, \tag{2.2}$$

for some $n_i^{(k)} \in \mathbb{Z}$. Denote

$$\tau = \max_{i,k} |n_i^{(k)}|, \tag{2.3}$$

and

$$\begin{aligned} A_m = \quad & \Big\{ y = \tfrac{n_1 y_1 + n_2 y_2 + \ldots + n_\nu y_\nu}{p};\ n_i \in \mathbb{Z}, \\ & |n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 1,\ |n_\nu| \le \nu m\tau + 1 \Big\}. \end{aligned} \tag{2.4}$$

Lemma 1. *If (2.1) is an arbitrary sequence with $\nu \ge 2$, then for any interval $I \subset (-1, 1)$ with $|I| \le y_\nu / p$ we have*

$$\#\big(A_m \cap I\big) \sim \gamma m^{\nu-1} |I| \text{ as } m \to \infty, \tag{2.5}$$

where $\gamma = (2\tau)^{\nu-1} p / y_\nu$ is a constant depended on X.

Proof. It is easy to observe that

$$\begin{aligned} A_m \cap I = \Big\{ y = \frac{n_1 y_1 + \ldots + n_\nu y_\nu}{p} :& \\ n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu} \in \frac{p}{y_\nu} \cdot I + \mathbb{Z} \cap [-(\nu m\tau + 1), (\nu m\tau + 1)],& \\ |n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 1 \Big\}.& \end{aligned}$$

On the other hand if $y \in A_m \cap I$, then, by (2.4) we have

$$|n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu}| \le \nu m\tau.$$

Using also the relation $|I| \le y_\nu / p$, we conclude

$$\begin{aligned} A_m \cap I = \Big\{ & y = \frac{n_1 y_1 + \ldots + n_\nu y_\nu}{p} : \\ & n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu} \in \frac{p}{y_\nu} \cdot I + \mathbb{Z}, \\ & |n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 1 \Big\}. \end{aligned} \tag{2.6}$$

Since $y_1, \ldots, y_\nu$ are independent, the number

$$\theta = y_{\nu-1} / y_\nu$$

is irrational. Denoting

$$E_m = \Big\{ n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-2} \frac{y_{\nu-2}}{y_\nu} :\ |n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 2 \Big\} \tag{2.7}$$

from (2.6) we get

$$\frac{p}{y_\nu} \cdot \big(A_m \cap I\big) = \big(\{n_{\nu-1}\theta :\ |n_{\nu-1}| \le m\tau\} + E_m\big) \cap \big(\frac{p}{y_\nu} \cdot I + \mathbb{Z}\big). \tag{2.8}$$

It is well known that $n\theta + t$, $n = 1, 2, \ldots$ $(n = -1, -2, \ldots)$, is a uniformly distributed sequence. This implies

$$\frac{\#\big(\{n_{\nu-1}\theta :\ |n_{\nu-1}| \le m\tau\} + t\big) \cap \big(\frac{p}{y_\nu} \cdot I + \mathbb{Z}\big)}{2m\tau} \to \frac{p|I|}{y_\nu},\ \text{ as } m \to \infty, \tag{2.9}$$

for any $t \in \mathbb{R}$ and the convergence is uniformly. Since $y_1, \ldots, y_{\nu-1}$ are independent from (2.7) we obtain

$$|E_m| = (2m\tau + 1)^{\nu-2}.$$

Finally, using (2.8) and (2.9), we get

$$\#(A_m \cap I) = \#\left(\frac{p}{y_\nu} \cdot (A_m \cap I)\right) \sim 2m\tau \frac{p|I|}{y_\nu}|E_m| \sim (2m\tau)^{\nu-1}\frac{p|I|}{y_\nu}.$$

□

Lemma 2. *For any set (2.1) we have*

$$A_m \cap (-x_l, 0) + X \subset A_{m+1} \cap (-x_l, x_l),\ m = 1, 2, \ldots, \tag{2.10}$$

where A_m is defined in (2.4).

Proof. Take an arbitrary point $x \in\in A_m \cap (-x_l, 0)$. According to the definition of $y_1, \ldots, y_\nu$ we will have

$$x = \frac{n_1 y_1 + n_2 y_2 + \ldots + n_\nu y_\nu}{p},$$

Then suppose $x_k \in X$ has representation (2.2). Since $x \in (-x_l, 0)$ and $0 < x_k \le x_l$ we get

$$x + x_k \in (-x_l, x_l). \tag{2.11}$$

On the other hand

$$x + x_k = \frac{(n_1 + n_1^{(k)})y_1 + (n_2 + n_2^{(k)})y_2 + \ldots + (n_\nu + n_\nu^{(k)})y_\nu}{p},$$

and by (2.4) (2.3) we have

$$\begin{aligned} &|n_i + n_i^{(k)}| \le m\tau + \tau = (m+1)\tau, i = 1, 2, \ldots, \nu - 1,\\ &|n_\nu + n_\nu^{(k)}| \le \nu m\tau + 1 + \tau < \nu(m+1)\tau. \end{aligned} \tag{2.12}$$

This means $x + x_k \in A_{m+1}$. Combining (2.11) and (2.12) we get (2.10). □

Lemma 3. *For any numbers $\delta > 0$, $0 < \varepsilon < 1/3$ and measure*

$$\mu = \sum_{k=1}^{l} m_k \delta_{x_k},\ m_k > 0,\ 0 < x_1 < x_2 < \ldots < x_l, \tag{2.13}$$

there exists a real number λ, with $0 < \lambda \le \delta$, such that

$$\begin{aligned} &S_\mu \mathbb{I}_{\{t:\,\{t/\lambda\} > \varepsilon\}}(x)\\ &\qquad = \int_{\mathbb{T}} \mathbb{I}_{\{t:\,\{t/\lambda\} > \varepsilon\}}(x+t)d\mu(t) > (1 - 3\varepsilon)|\mu|, \ \text{as } \{x/\lambda\} < \varepsilon. \end{aligned} \tag{2.14}$$

Proof. Denote

$$E_t = \{\lambda > 0 :\ \{t/\lambda\} \in (\varepsilon, 1 - \varepsilon)\}, \quad t > 0. \tag{2.15}$$

It is clear

$$E_t = \bigcup_{k=0}^{\infty} \left(\frac{t}{k + 1 - \varepsilon}, \frac{t}{k + \varepsilon}\right).$$

Hence if

$$r = \min\left\{\frac{\varepsilon x_1}{2(1-\varepsilon)}, \delta\right\}$$

and $t \geq x_1$, we obtain

$$
\begin{aligned}
|E_t \cap [0,r]| &> \sum_{k>t/r} \left(\frac{t}{k+\varepsilon} - \frac{t}{k+1-\varepsilon} \right) \\
&= \sum_{k>t/r} \left(\frac{(1-2\varepsilon)t}{(k+\varepsilon)(k+1-\varepsilon)} \right) > (1-2\varepsilon)t \sum_{k>t/r} \frac{1}{(k+1)^2} \\
&> \frac{(1-2\varepsilon)tr}{t+2r} > \frac{(1-2\varepsilon)x_1 r}{x_1+2r} \geq \frac{(1-2\varepsilon)x_1 r}{x_1+\varepsilon x_1/(1-\varepsilon)} \\
&= (1-2\varepsilon)(1-\varepsilon)r > (1-3\varepsilon)r.
\end{aligned} \tag{2.16}
$$

Thus, denoting

$$F = \{t > 0 : \{t\} \in (\varepsilon, 1-\varepsilon)\},$$

by (2.15) we have

$$E_t = \{\lambda > 0 : t \in \lambda F\}$$

and therefore, using (2.16), we get

$$
\begin{aligned}
\int_0^r S_\mu \mathbb{I}_{\lambda F}(0) d\lambda &= \int_0^r \int_{\mathbb{T}} \mathbb{I}_{\lambda F}(t) d\mu(t) d\lambda \\
&= \int_{\mathbb{T}} \int_0^r \mathbb{I}_{\lambda F}(t) d\lambda d\mu(t) = \int_{\mathbb{T}} |E_t \cap [0,r]| d\mu(t) \\
&= \sum_{i=1}^{l} m_i |E_{x_i} \cap [0,r]| \geq (1-3\varepsilon) r |\mu|.
\end{aligned} \tag{2.17}
$$

This implies

$$S_\mu \mathbb{I}_{\lambda F}(0) > (1-3\varepsilon)|\mu| \tag{2.18}$$

for some $0 < \lambda \leq r \leq \delta$. From (2.18) it follows that

$$S_\mu \mathbb{I}_{\lambda F + x}(x) > (1-3\varepsilon)|\mu|, \quad x \in \mathbb{R}. \tag{2.19}$$

It is clear

$$\bigcup_{x:\, \{x/\lambda\} < \varepsilon} (\lambda F + x) = \{t : \{t/\lambda\} > \varepsilon\}. \tag{2.20}$$

Thus, using (2.19) and (2.20), for any x, $\{x/\lambda\} < \varepsilon$, we obtain

$$S_\mu \mathbb{I}_{\{t:\, \{t/\lambda\} > \varepsilon\}}(x) \geq S_\mu \mathbb{I}_{\lambda F + x}(x) > (1-3\varepsilon)|\mu|.$$

This implies (2.14) and lemma is proved. □

Lemma 4. *For any measure (2.13)and number $0 < \varepsilon < 1/3$ there exist finite sets $E, G \subset (-x_l, x_l)$ such that*

$$E \cap G = \varnothing, \quad \#E > \frac{\varepsilon \# G}{4}, \tag{2.21}$$

$$S_\mu \mathbb{I}_G(x) > (1-3\varepsilon)|\mu|, \quad x \in E. \tag{2.22}$$

Proof. Denote

$$U_\lambda = \{t \in (-x_l, 0) : \{t/\lambda\} < \varepsilon\}, \quad V_\lambda = \{t \in (-x_l, x_l) : \{t/\lambda\} > \varepsilon\} \tag{2.23}$$

It is clear $|U_\lambda| \to \varepsilon x_l$ and $|V_\lambda| \to 2(1-\varepsilon)x_l$ as $\lambda \to 0$. On the other hand, by Lemma 3, for λ small enough we have (2.14). So we can fix λ satisfying (2.14) and the conditions

$$0 < \lambda < x_1, \quad |V_\lambda| < 2x_l, \quad |U_\lambda| > \frac{\varepsilon x_l}{2}. \tag{2.24}$$

Denote

$$E_m = A_m \cap U_\lambda, \quad G_m = A_{m+1} \cap V_\lambda. \tag{2.25}$$

Since the sets U_λ and V_λ are finite union of intervals in $(-1, 1)$, according to Lemma 1 we have

$$\#E_m \sim \gamma m^{\mu-1}|U_\lambda|, \quad \#G_m \sim \gamma m^{\mu-1}|V_\lambda|$$

as $m \to \infty$. Hence for an integer m large enough, denoting

$$E = E_m, \quad G = G_m$$

and taking into account (2.24) we will have

$$\#E > \frac{\varepsilon \#G}{4}. \tag{2.26}$$

Besides, since $U_\lambda \cap V_\lambda = \varnothing$ we have $E \cap G = \varnothing$ and so (2.21). To show (2.22) we take an arbitrary $x \in E$. Because of (2.23) and (2.25)we will have

$$x \in A_m \cap (-x_l, 0), \quad \{x/\lambda\} < \varepsilon.$$

From Lemma 2 we get $x + X \in A_{m+1} \cap (-x_l, x_l)$. Thus we get

$$S_\mu \mathbb{I}_G(x) = S_\mu \mathbb{I}_{V_\lambda}(x) = S_\mu \mathbb{I}_{\{t:\, \{\lambda t\} > \varepsilon\}}(x)$$

and therefore, since we have $\{x/\lambda\} < \varepsilon$, from Lemma 3 we obtain (2.22). □

For an arbitrary nonempty finite set $A \subset \mathbb{R} \setminus \{0\}$ we define

$$(A) = \begin{cases} \min\{|x-y| : x, y \in A,\, x \neq y\}, & \text{if } \#A \geq 2, \\ |x|, & \text{if } A = \{x\}. \end{cases}$$

Lemma 5. *Let $A_k \subset \mathbb{R} \setminus \{0\}$, $k = 1, 2, \ldots,$ be a sequence of nonempty finite sets such that and*

$$\max A_{k+1} \leq \frac{1}{4} \cdot (A_k), \quad k = 1, 2, \ldots. \tag{2.27}$$

Then the equality

$$x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n, \quad x_i, y_i \in A_i,\ i = 1, 2, \ldots, n \tag{2.28}$$

implies $x_i = y_i$, $i = 1, 2, \ldots, n$.

Proof. Suppose to the contrary in (2.28) we have $x_i = y_i$, $i < k$, and $x_k \neq y_k$. Hence we get

$$x_k + \ldots + x_n = y_k + \ldots + y_n. \tag{2.29}$$

From (2.27) and the relation

$$\max A_i \leq \frac{1}{4} \cdot (A_{i-1}) \leq \frac{1}{2} \max A_{i-1}$$

it follows that

$$(2.30)\quad |x_i|, |y_i| \le \max A_i \le \frac{1}{2}\max A_{i-1} \le \dots \le \frac{1}{2^{i-k-1}}\max A_{k+1} \le \frac{\dot{(A_k)}}{2^{i-k+1}} \le \frac{|x_k - y_k|}{2^{i-k+1}}$$

for any $i = k+1, k+2, \dots, n$. Thus, using (2.29) and (2.30), we get

$$|x_k - y_k| \le |x_{k+1}| + |y_{k+1}| + \dots + |x_n| + |y_n| < 2|x_k - y_k| \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = |x_k - y_k|$$

which is a contradiction and so $x_i = y_i$ for all $i = 1, 2, \dots, n$. □

Lemma 6. *Let μ_n be a sequence of measures, satisfying the condition (1.1). Then for any numbers $\Delta > 0$ and $0 < \delta < 1$ there exists a measurable set $A \subset \mathbb{T}$, $|A| > 0$, such that*

$$(2.31)\qquad |\{x \in \mathbb{T} : \sup_{n\in\mathbb{N}} S_{\mu_n}\mathbb{I}_A(x) > \delta\}| > \Delta \cdot |A|.$$

Proof. It is easy to observe that can be supposed each $\operatorname{supp}\mu_n$ is a finite set and moreover

$$\mu_n = \sum_{i=l(n-1)+1}^{l(n)} m_i\delta_{x_i}, \quad n = 1, 2, \dots,$$

where $0 = l(0) < l(1) < l(2) < \dots$ are integers, $1 > x_i \searrow 0$ and $m_i > 0$, $i = 1, 2, \dots$. Applying Lemma 4 with $\varepsilon = (1-\delta)/3$ we define finite sets E_n and G_n with

$$(2.32)\qquad E_n, G_n \subset (-x_{l(n)}, x_{l(n)}), \quad E_n \cap G_n = \varnothing,$$

$$(2.33)\qquad \#(E_n) > \frac{(1-\delta)\#(G_n)}{12},$$

$$(2.34)\qquad S_{\mu_n}\mathbb{I}_{G_n}(x) > \delta, \quad x \in E_n.$$

Clearly we can chose a sequence of integers n_k, $k = 1, 2, \dots$, satisfying

$$(2.35)\qquad \max(E_{n_{k+1}} \cap G_{n_{k+1}}) < \frac{\dot{(E_{n_k} \cap G_{n_k})}}{4}, \quad k = 1, 2, \dots.$$

So the sequence of sets $A_k = E_{n_k} \cup G_{n_k}$ satisfies the condition (2.27). Fix an integer

$$(2.36)\qquad m > \frac{12\Delta}{1-\delta},$$

and denote

$$(2.37)\qquad G = G_{n_1} + G_{n_2} + \dots + G_{n_m},$$

$$(2.38)\qquad F_k = \sum_{i\ne k} G_{n_i} + E_{n_k}, \quad E = \cup_{i=1}^{n} F_i.$$

Notice that the sets F_k are mutually disjoint. Indeed, suppose to the contrary $F_p \cap F_q \ne \varnothing$, $p \ne q$, and $x \in F_p \cap F_q$. We then have

$$x = x_1 + \dots + x_m = y_1 + \dots + y_m, \text{ where}$$
$$x_i, y_i \in A_i, \quad x_p \in E_{n_p}, y_p \in G_{n_p},$$

Since $G_{n_p} \cap E_{n_p} = \varnothing$ (see (2.32)), we have $x_{n_p} \neq y_{n_p}$. On the other hand because $x_i, y_i \in A_i$ and the family A_i satisfies the hypothesis of Lemma 5 we get $x_i = y_i$ for all $i = 1, 2, \ldots, m$. This is a contradiction and so F_k are mutually disjoint. Similarly we can prove that any point $x \in G$ has unique representation

$$x = x_1 + \ldots + x_m, \quad x_i \in G_{n_i},\ i = 1, 2, \ldots, m.$$

This implies

$$\#G = \prod_{i=1}^{m} \#(G_{n_i}).$$

By the same argument, using (2.33), we get

$$\#F_k = \prod_{i \neq k} \#(G_{n_i}) \cdot \#(E_{n_k}) \geq \prod_{i \neq k} \#(G_{n_i}) \cdot \frac{(1-\delta)\#(G_{n_k})}{12} = \frac{(1-\delta)\#G}{12}.$$

Combining this and (2.36) we conclude

$$\#E = \sum_{k=1}^{m} \#F_k > \frac{m(1-\delta)\#G}{12} > \Delta \cdot \#G. \tag{2.39}$$

To prove (2.31), we take an arbitrary $x \in E$. We have $x \in F_k$ for some $1 \leq k \leq m$ and so

$$x = x_1 + \ldots + x_m, \quad x_i \in G_{n_i},\ i \neq k,\ x_k \in E_{n_k}.$$

From (2.37) it follows that $G_{n_k} \subset G - \sum_{i \neq k} x_i$. Therefore, by (2.34), we get

$$S_{\mu_{n_k}} \mathbb{I}_G(x) = S_{\mu_{n_k}} \mathbb{I}_{G - \sum_{i\neq k} x_i}(x_k) \geq S_{\mu_{n_k}} \mathbb{I}_{G_{n_k}}(x_k) > \delta.$$

Hence we have

$$\sup_k S_{\mu_{n_k}} \mathbb{I}_G(x) > \delta, \quad x \in E, \tag{2.40}$$

Finally we let $\varepsilon = \dot{(}G \cup E)/2$ and denote

$$A = G + (-\varepsilon, \varepsilon), \quad B = E + (-\varepsilon, \varepsilon).$$

It is clear that the intervals $t + (-\varepsilon, \varepsilon)$, $t \in G \cup E$, are pairwise disjoint. Hence

$$|A| = 2\varepsilon\#G, \quad |B| = 2\varepsilon\#E,$$

and so, by (2.39) we conclude

$$|B| > \Delta|A|. \tag{2.41}$$

Then for an arbitrary $x \in B$ we have $x = t + y$ where $t \in E$ and $|y| < \varepsilon$. Hence, using (2.40), we get

$$\sup_k S_{\mu_{n_k}} \mathbb{I}_A(x) \geq \sup_k S_{\mu_{n_k}} \mathbb{I}_{G+y}(x) = \sup_k S_{\mu_{n_k}} \mathbb{I}_G(t) > \delta, \quad x \in B. \tag{2.42}$$

Collecting (2.41) and (2.42) we obtain (2.31). Lemma is proved. □

Definition. A sequence of linear operators

$$U_n : L^1(\mathbb{T}) \to \{ \text{ measurable functions on } \mathbb{T}\}.$$

is said to be strong sweeping out, if given $\varepsilon > 0$ there is a set E with $mE < \varepsilon$ such that $\limsup_{n\to\infty} U_n \mathbb{I}_E(x) = 1$ and $\liminf_{n\to\infty} U_n \mathbb{I}_E(x) = 0$ a.e..

To prove the theorem we need to show that the sequence S_{μ_n} is strong sweeping out. The following theorem gives a sufficient condition for a sequence of operators to be strong sweeping out.

Theorem 3 ([5], §7, Theorem 6)**.** *If the sequence of positive translation invariant operators* U_n *satisfies the conditions*

a: $U_n(\mathbb{I}_{\mathbb{T}}) \to 1$ *as* $n \to \infty$*,*

b: *for any* $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *there exists a number* $\delta = \delta(\varepsilon, n) > 0$*, such that if* $G \subset \mathbb{T}$ *and* $m(G) < \delta$ *then*

$$m\{x \in \mathbb{T} : U_n \mathbb{I}_G(x) > \varepsilon\} < \varepsilon, \tag{2.43}$$

c: *for any* $0 < \delta < 1$ *we have*

$$\sup_{G \subset \mathbb{T},\, |G| > 0} \frac{|\{x \in X : \sup_{n \in \mathbb{N}} U_n \mathbb{I}_G(x) \geq \delta\}|}{|G|} = \infty.$$

then it is strong sweeping out.

Observe, that each S_{μ_n} is positive translation invariant. The conditions (**a**) follows from (1.1). To show (**b**) we simply note

$$\int_{\mathbb{T}} S_{\mu_n} \mathbb{I}_G(x) dx = \int_{\mathbb{T}} \int_{\mathbb{T}} \mathbb{I}_G(x + t) dt dx = |\mu_n| \cdot |G|,$$

and therefore, by Chebishev inequality, we will have (2.43) provided $|G| < \delta = |\mu_n|/\varepsilon$. The condition (**c**) immediately follows from Lemma 6. Theorem is proved

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