ON THE SWEEPING OUT PROPERTY FOR CONVOLUTION OPERATORS OF DISCRETE MEASURES

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Abstract. Let \( \mu_n \) be a sequence of discrete measures on the unit circle \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) with \( \mu_n(0) = 0 \), and \( \mu_n((-\delta, \delta)) \to 1 \), as \( n \to \infty \). We prove that the sequence of convolution operators \( (f * \mu_n)(x) \) is strong sweeping out, i.e. there exists a set \( E \subseteq \mathbb{T} \) such that

\[
\limsup_{n \to \infty} (I_E * \mu_n)(x) = 1, \quad \liminf_{n \to \infty} (I_E * \mu_n)(x) = 0,
\]

almost everywhere on \( \mathbb{T} \).

1. Introduction

We consider bounded discrete measures

\[
\mu = \sum_k m_k \delta_{x_k}, \quad \sum_k m_k < \infty,
\]

on the circle \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \), where \( X = \{x_k\} \) is a finite or countable set in \( \mathbb{T} \) and \( \delta_{x_k} \) is Dirac measure at \( x_k \). Denote

\[
S_\mu f(x) = \int \mathbb{R} f(x + t) d\mu(t).
\]

Let \( \mu_n \) be a sequence of discrete measures satisfying

\[
\mu_n(0) = 0, \quad \mu_n((-\delta, \delta)) \to 1, \quad \text{as } n \to \infty,
\]

for any \( 0 < \delta \leq 1/2 \). It is clear if \( f \in L^1(\mathbb{T}) \) is continuous at \( x \in \mathbb{T} \) then

\[
S_{\mu_n} f(x) \to f(x),
\]

and the convergence is uniformly if \( f \in C(\mathbb{T}) \). The almost everywhere convergence problem in the case of general \( f \in L^1(\mathbb{T}) \) is not trivial. J. Bourgain in [4] proved

Theorem 1 (J. Bourgain). If \( x_k \downarrow 0 \) as \( k \to \infty \), and

\[
\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{x_k},
\]

then there exists a function \( f \in L^\infty \), such that \( S_{\mu_n} f(x) \) diverges on a set of positive measure.

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In fact, this theorem gave a negative answer to a problem due to A. Bellow [3] and the proof is based on a general theorem often referred as Bourgain’s entropy principle. Applying his principle Bourgain was able to deduce an analogous theorems for Riemann sums
\[ \frac{1}{n} \sum_{k=0}^{n-1} f\left( x + \frac{k}{n} \right), \]
and for the operators
\[ \frac{1}{n} \sum_{k=1}^{n} f(kx). \]
We note, that first theorem was earlier obtained by W. Rudin [8] by different technique, and the second by J. Marstrand in [7]. S. Kostyukovsky and A. Olevskii in [6], using the same entropy principle, extended Theorem 1 for general discrete sequences satisfying (1.1).

We found a new geometric proof for Theorem 1, as well as for the result from [6]. Moreover, the method allows to obtain a stronger divergence for the operators (1.2). So in this paper we prove

**Theorem 2.** If discrete measures \( \mu_n \) satisfy (1.1), then there exists a set \( E \subset \mathbb{T} \), such that
\[
\limsup_{n \to \infty} S_{\mu_n} \mathbb{1}_E(x) = 1, \quad \liminf_{n \to \infty} S_{\mu_n} \mathbb{1}_E(x) = 0
\]
almost everywhere on \( \mathbb{T} \).

The relations (1.3) for sequences of operators is called strong sweeping out property. These kind of operators are investigated by M. Akcoglu, A. Bellow, R. L. Jones, V. Losert, K. Reinhold-Larsson, M. Wierdl [1] and by M. Akcoglu, M. D. Ha, R. L. Jones [2]. In [1] strong sweeping out property for Riemann sums operators is obtained. In [2] authors prove a general version of Bourgain’s entropy principle, which allows to deduce sweeping out properties for some operators, but the principle is not applicable for the operators \( S_{\mu_n} \). The proof of Theorem 2 is based on Lemma 6. It will be obtained from Lemma 6 simply applying a general result proved in [5].

**2. Proof of theorem**

Let
\[
X = \{ x_i : i = 1, \ldots, l \}, \quad 0 < x_1 \leq x_2 \leq \ldots \leq x_l < 1,
\]
be an arbitrary sequence of reals. Suppose
\[
Y = \{ y_i : i = 1, \ldots, \nu \}, \quad y_1 < y_2 < \ldots < y_{\nu} = x_l
\]
is a maximal independent (with respect to rational numbers) subset of \( X \) containing \( x_l \). Then we have
\[
x_k = r_1^{(k)} y_1 + \ldots + r_{\nu}^{(k)} y_{\nu}, \quad k = 1, 2, \ldots, l,
\]
for some rational numbers \( r_i^{(k)} \). Let \( p \) be the least common multiple of the denominators of \( r_i^{(k)} \). Then we get
\[
x_k = \frac{n_1^{(k)} y_1 + n_2^{(k)} y_2 + \ldots + n_{\nu}^{(k)} y_{\nu}}{p},
\]
where
for some \( n_i^{(k)} \in \mathbb{Z} \). Denote
\[
\tau = \max_{i,k} |n_i^{(k)}|,
\]
and
\[
A_m = \left\{ y = \frac{n_1 y_1 + \ldots + n_\nu y_\nu}{p} : n_i \in \mathbb{Z}, \right. \\
\left. |n_i| \leq m\tau, i = 1, 2, \ldots, \nu - 1, |n_\nu| \leq \nu m\tau + 1 \right\}.
\]

**Lemma 1.** If (2.1) is an arbitrary sequence with \( \nu \geq 2 \), then for any interval \( I \subset (-1, 1) \) with \( |I| \leq y_\nu/p \) we have
\[
\#(A_m \cap I) \sim \gamma m^{\nu-1} |I| \quad \text{as} \quad m \to \infty,
\]
where \( \gamma = (2\tau)^{\nu-1} p/y_\nu \) is a constant depended on \( X \).

**Proof.** It is easy to observe that \( A_m \cap I = \left\{ y = \frac{n_1 y_1 + \ldots + n_\nu y_\nu}{p} : \right. \\
n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu} \in \frac{p}{y_\nu} \cdot I + \mathbb{Z} \cap [-(\nu m\tau + 1), (\nu m\tau + 1)], \\
|n_i| \leq m\tau, i = 1, 2, \ldots, \nu - 1 \left\} \right.
\]
On the other hand if \( y \in A_m \cap I \), then, by (2.4) we have
\[
|n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu}| \leq \nu m\tau.
\]
Using also the relation \( |I| \leq y_\nu/p \), we conclude
\[
A_m \cap I = \left\{ y = \frac{n_1 y_1 + \ldots + n_\nu y_\nu}{p} : \\
n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu} \in \frac{p}{y_\nu} \cdot I + \mathbb{Z}, \\
|n_i| \leq m\tau, i = 1, 2, \ldots, \nu - 1 \right\}
\]
Since \( y_1, \ldots, y_\nu \) are independent, the number
\[
\theta = y_{\nu-1}/y_\nu
\]
is irrational. Denoting
\[
E_m = \left\{ n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-2} \frac{y_{\nu-2}}{y_\nu} : |n_i| \leq m\tau, i = 1, 2, \ldots, \nu - 2 \right\}
\]
from (2.6) we get
\[
P_\frac{y_\nu}{y_\nu} \cdot (A_m \cap I) = \left( \{ n_{\nu-1} \theta : |n_{\nu-1}| \leq m\tau \} \cup E_m \right) \cap \left( \frac{p}{y_\nu} \cdot I + \mathbb{Z} \right).
\]
It is well known that \( \theta n + t, n = 1, 2, \ldots (n = -1, -2, \ldots) \), is a uniformly distributed sequence. This implies
\[
\#\left( \{ n_{\nu-1} \theta : |n_{\nu-1}| \leq m\tau \} + t \right) \cap \left( \frac{p}{y_\nu} \cdot I + \mathbb{Z} \right) \to \frac{p|I|}{y_\nu}, \quad \text{as} \quad m \to \infty,
\]

\[
\frac{n_{\nu-1} \theta + t}{2m\tau} \to \frac{p|I|}{y_\nu}, \quad \text{as} \quad m \to \infty,
\]
\[
\#(A_m \cap I) \sim \gamma m^{\nu-1} |I| \quad \text{as} \quad m \to \infty,
\]
where \( \gamma = (2\tau)^{\nu-1} p/y_\nu \) is a constant depended on \( X \).
for any \( t \in \mathbb{R} \) and the convergence is uniformly. Since \( y_1, \ldots, y_{\nu - 1} \) are independent from (2.7) we obtain
\[
|E_m| = (2m\tau + 1)^{\nu - 2}.
\]
Finally, using (2.8) and (2.9), we get
\[
\#(A_m \cap I) = \left( \frac{p}{y_\nu} \cdot (A_m \cap I) \right) \sim 2m\tau \frac{|I|}{y_\nu} |E_m| \sim (2m\tau)^{\nu - 1} \frac{|I|}{y_\nu}.
\]

**Lemma 2.** For any set (2.1) we have
\[
(2.10) \quad A_m \cap (-x_l, 0) + X \subset A_{m+1} \cap (-x_l, x_l), \quad m = 1, 2, \ldots,
\]
where \( A_m \) is defined in (2.4).

**Proof.** Take an arbitrary point \( x \in A_m \cap (-x_l, 0) \). According to the definition of \( y_1, \ldots, y_\nu \) we will have
\[
x = n_1 y_1 + n_2 y_2 + \ldots + n_\nu y_\nu.
\]
Then suppose \( x_k \in X \) has representation (2.2). Since \( x \in (-x_l, 0) \) and \( 0 < x_k \leq x_l \) we get
\[
(2.11) \quad x + x_k \in (-x_l, x_l).
\]
On the other hand
\[
x + x_k = \left( n_1 + n_1^{(k)} \right) y_1 + \left( n_2 + n_2^{(k)} \right) y_2 + \ldots + \left( n_\nu + n_\nu^{(k)} \right) y_\nu,
\]
and by (2.4) (2.3) we have
\[
(2.12) \quad |n_i + n_i^{(k)}| \leq m\tau + \tau = (m + 1)\tau, \quad i = 1, 2, \ldots, \nu - 1,
\]
\[
|n_\nu + n_\nu^{(k)}| \leq \nu m\tau + 1 + \tau \leq \nu(m + 1)\tau.
\]
This means \( x + x_k \in A_{m+1} \). Combining (2.11) and (2.12) we get (2.10). \( \square \)

**Lemma 3.** For any numbers \( \delta > 0, \ 0 < \varepsilon < 1/3 \) and measure
\[
(2.13) \quad \mu = \sum_{k=1}^{l} m_k \delta_{x_k}, \quad m_k > 0, \ 0 < x_1 < x_2 < \ldots < x_l,
\]
there exists a real number \( \lambda, \) with \( 0 < \lambda \leq \delta \), such that
\[
(2.14) \quad S_{\mu} I_{\{t: \{t/\lambda\} > \varepsilon\}}(x) = \int_{\mathbb{T}} I_{\{t: \{t/\lambda\} > \varepsilon\}}(x + t) d\mu(t) > (1 - 3\varepsilon)|\mu|, \text{ as } \{x/\lambda\} < \varepsilon.
\]

**Proof.** Denote
\[
(2.15) \quad E_t = \{ \lambda > 0 : \{t/\lambda\} \in (\varepsilon, 1 - \varepsilon) \}, \quad t > 0.
\]
It is clear
\[
E_t = \bigcup_{k=0}^{\infty} \left( \frac{t}{k + 1 - \varepsilon}, \frac{t}{k + \varepsilon} \right).
\]
Hence if
\[
r = \min \left\{ \frac{\varepsilon x_1}{2(1 - \varepsilon)}, \delta \right\}
\]
and \( t \geq x_1 \), we obtain

\[
|E_t \cap [0,r]| > \sum_{k > t/r} \left( \frac{t}{k + \varepsilon} - \frac{t}{k + 1 - \varepsilon} \right)
\]

\[
= \sum_{k > t/r} \left( \frac{(1 - 2\varepsilon)t}{(k + \varepsilon)(k + 1 - \varepsilon)} \right) > (1 - 2\varepsilon)t \sum_{k > t/r} \frac{1}{(k + 1)^2}
\]

\[
> \frac{(1 - 2\varepsilon)t}{r} > \frac{(1 - 2\varepsilon)x_1 r}{x_1 + 2r} > \frac{(1 - 2\varepsilon)x_1 r}{x_1 + \varepsilon x_1/(1 - \varepsilon)}
\]

\[
= (1 - 2\varepsilon)(1 - \varepsilon)r > (1 - 3\varepsilon)r.
\]

Thus, denoting

\[
F = \{t > 0 : \{t\} \in (\varepsilon, 1 - \varepsilon)\},
\]

by (2.15) we have

\[
E_t = \{\lambda > 0 : t \in \lambda F\}
\]

and therefore, using (2.16), we get

\[
\int_0^r S_{\mu} I_{\lambda F}(0)d\lambda = \int_0^r \int_\mathbb{T} I_{\lambda F}(t)d\mu(t)d\lambda
\]

\[
= \int_\mathbb{T} \int_0^r I_{\lambda F}(t)d\mu(t) = \int_\mathbb{T} |E_t \cap [0,r]|d\mu(t)
\]

\[
= \sum_{i=1}^m m_i |E_{x_i} \cap [0,r]| \geq (1 - 3\varepsilon)r|\mu|.
\]

This implies

\[
S_{\mu} I_{\lambda F}(0) > (1 - 3\varepsilon)|\mu|
\]

for some \( 0 < \lambda \leq r \leq \delta \). From (2.18) it follows that

\[
S_{\mu} I_{\lambda F + x}(x) > (1 - 3\varepsilon)|\mu|, \quad x \in \mathbb{R}.
\]

It is clear

\[
\bigcup_{x : \{x/\lambda\} < \varepsilon} (\lambda F + x) = \{t : \{t/\lambda\} > \varepsilon\}.
\]

Thus, using (2.19) and (2.20), for any \( x \), \( \{x/\lambda\} < \varepsilon \), we obtain

\[
S_{\mu} I_{\{t/\lambda\} > \varepsilon}(x) \geq S_{\mu} I_{\lambda F + x}(x) > (1 - 3\varepsilon)|\mu|.
\]

This implies (2.14) and lemma is proved.

\[\Box\]

**Lemma 4.** For any measure (2.13) and number \( 0 < \varepsilon < 1/3 \) there exist finite sets \( E, G \subset (-x_1, x_1) \) such that

\[
E \cap G = \varnothing, \quad \#E > \varepsilon \#G \frac{4}{\varepsilon^2},
\]

\[
S_{\mu} I_{E}(x) > (1 - 3\varepsilon)|\mu|, \quad x \in E.
\]

**Proof.** Denote

\[
U_\varepsilon = \{t \in (-x_1, 0) : \{t/\lambda\} < \varepsilon\}, \quad V_\varepsilon = \{t \in (-x_1, x_1) : \{t/\lambda\} > \varepsilon\}
\]

and

\[
\sum_{x_i \in V_\varepsilon} m_i |E_{x_i} \cap [0,r]| \geq (1 - 3\varepsilon)r|\mu|.
\]

This implies (2.14) and lemma is proved.
It is clear $|U_\lambda| \to \varepsilon x_I$ and $|V_\lambda| \to 2(1-\varepsilon)x_I$ as $\lambda \to 0$. On the other hand, by Lemma 3, for $\lambda$ small enough we have (2.14). So we can fix $\lambda$ satisfying (2.14) and the conditions

\begin{equation}
0 < \lambda < x_1, 
|V_\lambda| < 2x_I, 
|U_\lambda| > \frac{\varepsilon x_I}{2}.
\end{equation}

Denote

\begin{equation}
E_m = A_m \cap U_\lambda, 
G_m = A_{m+1} \cap V_\lambda.
\end{equation}

Since the sets $U_\lambda$ and $V_\lambda$ are finite union of intervals in $(-1,1)$, according to Lemma 1 we have

\[ \#E_m \sim \gamma m^{-1} |U_\lambda|, \quad \#G_m \sim \gamma m^{-1} |V_\lambda| \]

as $m \to \infty$. Hence for an integer $m$ large enough, denoting

\[ E = E_m, \quad G = G_m \]

and taking into account (2.24) we will have

\begin{equation}
\# E > \varepsilon \# G.
\end{equation}

Besides, since $U_\lambda \cap V_\lambda = \emptyset$ we have $E \cap G = \emptyset$ and so (2.21). To show (2.22) we take an arbitrary $x \in E$. Because of (2.23) and (2.25) we will have

\[ x \in A_m \cap (-x_I,0), \quad \{x/\lambda\} < \varepsilon. \]

From Lemma 2 we get $x + X \in A_{m+1} \cap (-x_I, x_I)$. Thus we get

\[ S_{\mu} I_G(x) = S_{\mu} I_{V_\lambda}(x) = S_{\mu} I_{\{t: \{\lambda t\} > \varepsilon\}}(x) \]

and therefore, since we have $\{x/\lambda\} < \varepsilon$, from Lemma 3 we obtain (2.22). \hfill \Box

For an arbitrary nonempty finite set $A \subset \mathbb{R} \setminus \{0\}$ we define

\[ (A) = \begin{cases} 
\min\{|x-y|: x,y \in A, x \neq y\}, & \text{if } \#A \geq 2, \\
|x|, & \text{if } A = \{x\}. 
\end{cases} \]

**Lemma 5.** Let $A_k \subset \mathbb{R} \setminus \{0\}$, $k = 1, 2, \ldots$, be a sequence of nonempty finite sets such that

\begin{equation}
\max A_{k+1} \leq \frac{1}{4} \cdot (A_k), \quad k = 1, 2, \ldots.
\end{equation}

Then the equality

\begin{equation}
x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n, \quad x_i, y_i \in A_i, \quad i = 1, 2, \ldots, n
\end{equation}

implies $x_i = y_i$, $i = 1, 2, \ldots, n$.

**Proof.** Suppose to the contrary in (2.28) we have $x_i = y_i$, $i < k$, and $x_k \neq y_k$. Hence we get

\begin{equation}
x_k + \ldots + x_n = y_k + \ldots + y_n.
\end{equation}

From (2.27) and the relation

\[ \max A_i \leq \frac{1}{4} \cdot (A_{i-1}) \leq \frac{1}{2} \max A_{i-1} \]
it follows that
\begin{equation}
|x_k|, |y_i| \leq \max A_i \leq \frac{1}{2} \max A_{i-1} \leq \ldots \\
\leq \frac{1}{2^{i-k}} \max A_{k+1} \leq \frac{(A_k)}{2^{i-k+1}} \leq \frac{|x_k - y_k|}{2^{i-k+1}}
\end{equation}
for any \( i = k + 1, k + 2, \ldots, n \). Thus, using (2.29) and (2.30), we get
\begin{equation}
|x_k - y_k| \leq |x_{k+1}| + |y_{k+1}| + \ldots + |x_n| + |y_n|
< 2|x_k - y_k| \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = |x_k - y_k|
\end{equation}
which is a contradiction and so \( x_i = y_i \) for all \( i = 1, 2, \ldots, n \). \( \square \)

**Lemma 6.** Let \( \mu_n \) be a sequence of measures, satisfying the condition (1.1). Then for any numbers \( \Delta > 0 \) and \( 0 < \delta < 1 \) there exists a measurable set \( A \subset T, |A| > 0 \), such that
\begin{equation}
|\{ x \in T : \sup_{n \in \mathbb{N}} S_{\mu_n} f_A(x) > \delta \}| > \Delta \cdot |A|.
\end{equation}

**Proof.** It is easy to observe that can be supposed each \( \text{supp} \mu_n \) is a finite set and moreover
\begin{equation}
\mu_n = \sum_{i=0}^{l(n)} m_i \delta_{x_i}, \quad n = 1, 2, \ldots,
\end{equation}
where \( 0 = l(0) < l(1) < l(2) < \ldots \) are integers, \( 1 > x_i \setminus 0 \) and \( m_i > 0, i = 1, 2, \ldots \).
Applying Lemma 4 with \( \varepsilon = (1 - \delta)/3 \) we define finite sets \( E_n \) and \( G_n \) with
\begin{align}
E_n, G_n &\subseteq (-x_{l(n)}, x_{l(n)}), \quad E_n \cap G_n = \emptyset, \\
\#(E_n) &> \frac{(1 - \delta) \#(G_n)}{12}, \\
S_{\mu_n} h_{G_n}(x) &> \delta, \quad x \in E_n.
\end{align}
Clearly we can chose a sequence of integers \( n_k, k = 1, 2, \ldots, \) satisfying
\begin{equation}
\max(E_{n_k+1} \cap G_{n_k+1}) < \frac{(E_{n_k} \cap G_{n_k})}{4}, \quad k = 1, 2, \ldots.
\end{equation}
So the sequence of sets \( A_k = E_{n_k} \cup G_{n_k} \) satisfies the condition (2.27). Fix an integer
\begin{equation}
m > \frac{12 \Delta}{1 - \delta},
\end{equation}
and denote
\begin{align}
G &= G_{n_1} + G_{n_2} + \ldots + G_{n_m}, \\
F_k &= \sum_{i \neq k} G_{n_i} + E_{n_k}, \quad E = \cup_{i=1}^{n} F_i.
\end{align}
Notice that the sets \( F_k \) are mutually disjoint. Indeed, suppose to the contrary \( F_p \cap F_q \neq \emptyset, p \neq q \), and \( x \in F_p \cap F_q \). We then have
\begin{align}
x &= x_1 + \ldots + x_m = y_1 + \ldots + y_m, \quad \text{where} \\
x, y_i &\in A_i, \quad x_p \in E_{n_p}, y_p \in G_{n_p}.
\end{align}
Since $G_n \cap E_n = \emptyset$ (see (2.32)), we have $x_n \neq y_n$. On the other hand because $x_i, y_i \in A_i$ and the family $A_i$ satisfies the hypothesis of Lemma 5 we get $x_i = y_i$ for all $i = 1, 2, \ldots, m$. This is a contradiction and so $F_k$ are mutually disjoint. Similarly we can prove that any point $x \in G$ has unique representation

$$x = x_1 + \ldots + x_m, \quad x_i \in G_n, \; i = 1, 2, \ldots, m.$$  

This implies

$$\#G = \prod_{i=1}^{m} \#(G_n).$$

By the same argument, using (2.33), we get

$$\#F_k = \prod_{i \neq k} \#(G_n) \cdot \#(E_{n_k}) \geq \prod_{i \neq k} \#(G_n) \cdot \frac{(1 - \delta) \#G}{12} = \frac{(1 - \delta) \#G}{12}.$$  

Combining this and (2.36) we conclude

$$\#E = \sum_{k=1}^{m} \#F_k > \frac{m(1 - \delta) \#G}{12} > \Delta \cdot \#G.$$  

To prove (2.31), we take an arbitrary $x \in E$. We have $x \in F_k$ for some $1 \leq k \leq m$ and so

$$x = x_1 + \ldots + x_m, \quad x_i \in G_n, \; i \neq k, \; x_k \in E_{n_k}.$$  

From (2.37) it follows that $G_{n_k} \subset G - \sum_{i \neq k} x_i$. Therefore, by (2.34), we get

$$S_{\mu_{n_k}} 1_G(x) = S_{\mu_{n_k}} 1_{G - \sum_{i \neq k} x_i}(x_k) \geq S_{\mu_{n_k}} 1_{E_{n_k}}(x_k) > \delta.$$  

Hence we have

$$\sup_k S_{\mu_{n_k}} 1_G(x) > \delta, \quad x \in E,$$

Finally we let $\varepsilon = (G \cup E)/2$ and denote

$$A = G + (-\varepsilon, \varepsilon), \quad B = E + (-\varepsilon, \varepsilon).$$  

It is clear that the intervals $t + (-\varepsilon, \varepsilon), \; t \in G \cup E$, are pairwise disjoint. Hence

$$|A| = 2\varepsilon \#G, \quad |B| = 2\varepsilon \#E,$$

and so, by (2.39) we conclude

$$|B| > \Delta |A|.$$  

Then for an arbitrary $x \in B$ we have $x = t + y$ where $t \in E$ and $|y| < \varepsilon$. Hence, using (2.40), we get

$$\sup_k S_{\mu_{n_k}} 1_A(x) \geq \sup_k S_{\mu_{n_k}} 1_{G+y}(x) = \sup_k S_{\mu_{n_k}} 1_G(t) > \delta, \quad x \in B.$$  

Collecting (2.41) and (2.42) we obtain (2.31). Lemma is proved.  

**Definition.** A sequence of linear operators

$$U_n : L^1(T) \rightarrow \{ \text{measurable functions on } T\}.$$  

is said to be strong sweeping out, if given $\varepsilon > 0$ there is a set $E$ with $mE < \varepsilon$ such that $\limsup_{n \rightarrow \infty} U_n 1_E(x) = 1$ and $\liminf_{n \rightarrow \infty} U_n 1_E(x) = 0$ a.e.

To prove the theorem we need to show that the sequence $S_{\mu_n}$ is strong sweeping out. The following theorem gives a sufficient condition for a sequence of operators to be strong sweeping out.
Theorem 3 ([5], §7, Theorem 6). If the sequence of positive translation invariant operators $U_n$ satisfies the conditions

- **a:** $U_n(I_T) \to 1$ as $n \to \infty$,
- **b:** for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a number $\delta = \delta(\varepsilon, n) > 0$, such that if $G \subset \mathbb{T}$ and $m(G) < \delta$ then
  \begin{equation}
  (2.43)
  m\{x \in \mathbb{T} : U_n(I_G(x)) > \varepsilon\} < \varepsilon,
  \end{equation}
- **c:** for any $0 < \delta < 1$ we have
  \[
  \sup_{G \subset \mathbb{T}, |G| > 0} \left| \left\{ x \in X : \sup_{n \in \mathbb{N}} U_n(I_G(x)) \geq \delta \right\} \right| |G| = \infty.
  \]
then it is strong sweeping out.

Observe, that each $S_{\mu_n}$ is positive translation invariant. The conditions (a) follows from (1.1). To show (b) we simply note
\[
\int_T S_{\mu_n} I_G(x)dx = \int_T \int_T I_G(x+t)dt dx = |\mu_n| \cdot |G|,
\]
and therefore, by Chebyshev inequality, we will have (2.43) provided $|G| < \delta = |\mu_n|/\varepsilon$. The condition (c) immediately follows from Lemma 6. Theorem is proved

References


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