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On the Derivatives of Cauchy-Type Integrals in the Polydisk

A. I. Petrosyan*

Institute of Mathematics, National Academy of Sciences of the Republic of Armenia, Armenia

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Abstract—In the paper the formulas are provided for the derivatives of Cauchy-type integral $K[u]$ which are smooth on the skeleton of the polydisk of functions u. These formulas express the derivatives of the order m of $K[u]$ through the derivatives of lower order (Theorem 2.1). They are used for estimating the smoothness of the derivatives of the Cauchy-type integral in terms of Hölder order scale (Theorem 3.1).

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1. INTRODUCTION

1.1. In work [1] Joricke obtained the following result: if the function given on the skeleton of polydisk or non-singular Weyl polyhedron in the space $\mathbb{C}^n$ satisfies the Hölder condition of the order α, $\alpha \in (0,1)$, then the modulus of continuity of its Cauchy-type or Weyl-type integral is majorized by const $\cdot\, \delta^\alpha (\log 1/\delta)^{n-1}$. For the functions satisfying the Hölder condition of the order α, see, e.g., [2].

We present the corresponding example which implies that this result is exact, that is, it cannot be improved. We recall the definition of the non-singular Weyl polyhedron.

Suppose that $z = (z_1, \ldots, z_n)$ is a point of n-dimensional complex space $\mathbb{C}^n$. The domain D in $\mathbb{C}^n$ is referred to as the analytic polyhedron if there exist N functions $\chi_n(z)$, $n = 1, \ldots, N$, holomorphic in some neighborhood $U(\overline{D})$ of the closure D and such that

$$D = \{z \in U(\overline{D}):\ |\chi_\alpha(z)| < 1,\ \alpha = 1, \ldots, N\}.$$

It is clear that the polydisk is a special case of polyhedron if $N = n$ and $\chi_\alpha(z)| = z_\alpha$.

The analytic polyhedron is referred to as the Weyl polyhedron if $N \geqslant n$ and the intersection of any $k, 1 \leqslant k \leqslant n$, hypersurfaces $|\chi_{\alpha_i}(z)| = 1$, $i = 1, \ldots, k$, has the dimension not higher than $2n - k$. In this case, the collection of n-dimensional edges

$$\sigma_{\alpha_1 \ldots \alpha_n} = \{z:\ z \in \overline{D},\ |\chi_{\alpha_i}(z)| = 1,\ i = 1, \ldots, n\},$$

oriented naturally is called the skeleton of the polyhedron D and is denoted by $\Delta(D)$ (see, e.g., [3]):

$$\Delta(D) = \bigcup_{\alpha_1 < \cdots < \alpha_n} \sigma_{\alpha_1 \ldots \alpha_n}.$$

There is one more result proved in [1]: if the function defined on the skeleton is continuously continued to the pluriharmonic function in the polydisk, then the logarithmic co-multiplier does not appear.

There arises a natural question about the derivatives of the Cauchy-type integral if the derivatives of the function defined on the skeleton satisfy the Hölder condition. The current paper is devoted to this question.

*e-mail: apetrosyan@ysu.am

We obtained the formula expressing the derivatives of the Cauchy-type integral on the skeleton of the polydisk through the derivatives of lower order (Lemma 2.1). Due to this property of the formula and by the inductive reasoning, the Joricke theorem is extended to the case of derivatives (Theorem 2.1).

1.2. We will use the following denotations:

$U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n\colon\ |z_k| < 1,\ k = 1, \dots, n\}$ is the unit polydisk in $\mathbb{C}^n$;

$T^n = \{z \in \mathbb{C}^n\colon\ |z_k| = 1,\ k = 1, \dots, n\}$ is the skeleton of the polydisk U^n.

For the function u defined on the skeleton T^n through $K[u](z)$, we define its n-fold Cauchy-type integral

$$K[u](z) = \frac{1}{(2\pi i)^n} \int\limits_{T^n} \frac{u(\zeta)d\zeta}{\prod_{k=1}^{n}(\zeta_k - z_k)},$$

where $\zeta = (\zeta_1, \cdots, \zeta_n) \in T^n$, $z = (z_1, \dots, z_n) \in U^n$, and $d\zeta = d\zeta_1 \dots d\zeta_n$.

In the following, for the set X (as X we have either $\overline{U^n}$, or T^n), we denote

$C^m(X)$ is the set of functions which are m times continuously differentiable on X;

$C^{m,\alpha}(X)$ is the subset of those functions from $C^m(X)$ which have all derivatives of order m satisfying the Hölder condition with the exponent α.

$C^{m,\alpha}_{\log}(X)$ is the set of those functions from $C^m(X)$ which have the modulus of continuity of derivatives of order m majorized by const $\cdot\, \delta^\alpha \left(\log \frac{1}{\delta}\right)^{n-1}$.

1.3. Everywhere below we apply the differential operators $\frac{\partial}{\partial \zeta_k}$ and $\frac{\partial}{\partial \overline{\zeta}_k}$ to the functions from the class $C^m(T^n)$. According to the Whittney theorem (see, e.g., [4]), these functions may be continued to the neighborhood of the skeleton T^n with the preserved class of smoothness. The mentioned differential operations are applied exactly to the continuations, and we will denote these continuations by the same letter as the original function.

2. FORMULAS FOR DERIVATIVES

Let us start from the preliminary lemma.

Lemma 2.1. *Suppose that $g \in C^1(T^n)$, $1 \leqslant k \leqslant n$ Then,*

$$K\left[\frac{\partial g(\zeta)}{\partial \zeta_k}\right](z) = K\left[\overline{\zeta}_k^2 \frac{\partial g(\zeta)}{\partial \overline{\zeta}_k}\right](z) + \frac{\partial}{\partial z_k} K\left[g\right](z). \tag{2.1}$$

Proof. At a fixed $z \in \mathbb{D}^n$ we consider the following form

$$\omega = \frac{(-1)^{k-1} g(\zeta)}{\prod_{j=1}^{n}(\zeta_j - z_j)} \underset{i \neq k}{\wedge} d\zeta_i.$$

We have

$$0 = \int\limits_{T^n} d_\zeta \omega = \int\limits_{T^n} \frac{\partial}{\partial \zeta_k} \frac{g(\zeta)}{\prod_{j=1}^{n}(\zeta_j - z_j)} d\zeta + \sum_{j=1}^{n} \int\limits_{T^n} \frac{\partial}{\partial \overline{\zeta}_j} \frac{(-1)^{k-1} g(\zeta)}{\prod_{j=1}^{n}(\zeta_j - z_j)} d\overline{\zeta}_j \wedge \left(\underset{i \neq k}{\wedge} d\zeta_i\right). \tag{2.2}$$

Here, the first equality follows from the Stokes formula. In the following, since on T^n we have $\zeta_j \overline{\zeta}_j = 1,\ j = 1, \dots, n$, $\zeta_j d\overline{\zeta}_j = -\overline{\zeta}_j d\zeta_j$. Therefore,

$$(-1)^{k-1} d\overline{\zeta}_j \wedge \left(\underset{i \neq k}{\wedge} d\zeta_i\right) = \begin{cases} -\overline{\zeta}_k^2 d\zeta, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases} \tag{2.3}$$

With account for (2.3) from (2.2) we obtain

$$\int\limits_{T^n} \frac{\partial g(\zeta)}{\partial \zeta_k} \frac{d\zeta}{\prod_{i=1}^{n}(\zeta_i - z_i)} - \int\limits_{T^n} \frac{g(\zeta)d\zeta}{(\zeta_k - z_k)^2 \prod_{i \neq k}(\zeta_i - z_i)} - \int\limits_{T^n} \frac{\partial g(\zeta)}{\partial \overline{\zeta}_k} \frac{\overline{\zeta}_k^2 d\zeta}{\prod_{i=1}^{n}(\zeta_i - z_i)} = 0. \tag{2.4}$$

Using the fact that

$$\frac{1}{(\zeta_k - z_k)^2} = \frac{\partial}{\partial z_k}\frac{1}{\zeta_k - z_k},$$

from (2.4) we derive

$$\int\limits_{T^n} \frac{\partial g(\zeta)}{\partial \zeta_k}\frac{d\zeta}{\prod_{i=1}^n(\zeta_i - z_i)} = \int\limits_{T^n} \frac{\partial g(\zeta)}{\partial \overline{\zeta}_k}\frac{\overline{\zeta}_k^2 d\zeta}{\prod_{i=1}^n(\zeta_i - z_i)} + \frac{\partial}{\partial z_k}\int\limits_{T^n} \frac{g(\zeta)d\zeta}{\prod_{i=1}^n(\zeta_i - z_i)}.$$

In our denotations this is exactly (2.1). In the following theorem we give the formula expressing the mth-order derivatives of the function $K[f]$ through the derivatives of lower order. □

Theorem 2.1. *Suppose that $u \in C^m(T^n)$ and the multiindex $r = (r_1, \dots, r_n)$ satisfies the condition $r_1 + \dots + r_n = m$. Then, the following formula is true:*

$$\begin{aligned} &\frac{\partial^{r_1+\dots+r_n}}{\partial z_1^{r_1}\dots\partial z_n^{r_n}}K[u](z) \\ &= \sum_{j=1}^{n}\sum_{k_j=1}^{r_j} \frac{\partial^{r_1+\dots+r_j-k_j}}{\partial z_1^{r_1}\dots\partial z_{j-1}^{r_{j-1}}\partial z_j^{r_j-k_j}}K\left[\overline{\zeta}_j^2\frac{\partial^{k_j+r_{j+1}+\dots+r_n}u(\zeta)}{\partial\overline{\zeta}_j\partial\zeta_j^{k_j-1}\partial\zeta_{j+1}^{r_{j+1}}\dots\partial\zeta_n^{r_n}}\right](z) \\ &\qquad - K\left[\frac{\partial^{r_1+\dots+r_n}u(\zeta)}{\partial\zeta_1^{r_1}\dots\partial\zeta_n^{r_n}}\right](z). \end{aligned} \tag{2.5}$$

Of course, in formula (2.5) the differential operator under the sign of double sum is considered to be the identical operator in the case when its order $r_1 + \dots + r_j - k_j$ is equal to zero.

Proof. We take $k = 1$ in Lemma 2.1 and

$$g(z) = \frac{\partial^{r_1-1+r_2+\dots+r_n}u(z)}{\partial z_1^{r_1-1}\partial\zeta_2^{r_2}\cdots\partial z_n^{r_n}}$$

and have

$$\begin{aligned} K\left[\frac{\partial^{r_1+\dots+r_n}u(\zeta)}{\partial\zeta_1^{r_1}\dots\partial\zeta_n^{r_n}}\right](z) &= K\left[\overline{\zeta}_1^2\frac{\partial^{r_1+\dots+r_n}u(\zeta)}{\partial\overline{\zeta}_1\partial\zeta_1^{r_1-1}\partial\zeta_2^{r_2}\dots\partial\zeta_n^{r_n}}\right](z) \\ &\quad + \frac{\partial}{\partial z_1}K\left[\frac{\partial^{r_1-1+r_2+\dots+r_n}u(\zeta)}{\partial\zeta_1^{r_1-1}\partial\zeta_2^{r_2}\cdots\partial\zeta_n^{r_n}}\right](z). \end{aligned} \tag{2.6}$$

We focus on the fact that the integrand in the second term in the right-hand side of (2.6) has the order of differentiation with respect to ζ_1 lower by one than the right-hand side, that is, formula (2.6) has the recurrent character. We successively apply this formula r_1 times and obtain

$$\begin{aligned} K\left[\frac{\partial^{r_1+\dots+r_n}u(\zeta)}{\partial\zeta_1^{r_1}\dots\partial\zeta_n^{r_n}}\right](z) &= K\left[\overline{\zeta}_1^2\frac{\partial^{r_1+\dots+r_n}u(\zeta)}{\partial\overline{\zeta}_1\partial\zeta_1^{r_1-1}\partial\zeta_2^{r_2}\dots\partial\zeta_n^{r_n}}\right](z) \\ + \frac{\partial}{\partial z_1}\left\{K\left[\overline{\zeta}_1^2\frac{\partial^{r_1-1+r_2+\dots+r_n}u(\zeta)}{\partial\overline{\zeta}_1\partial\zeta_1^{r_1-2}\partial\zeta_2^{r_2}\dots\partial\zeta_n^{r_n}}\right](z)\right. &+ \left.\frac{\partial}{\partial z_1}K\left[\frac{\partial^{r_1-2+r_2+\dots+r_n}u(\zeta)}{\partial\zeta_1^{r_1-2}\partial\zeta_2^{r_2}\dots\partial\zeta_n^{r_n}}\right](z)\right\} = \cdots \\ = \sum_{k_1=1}^{r_1}\frac{\partial^{r_1-k_1}}{\partial z_1^{r_1-k_1}}K\left[\overline{\zeta}_1^2\frac{\partial^{k_1+r_2+\dots+r_n}u(\zeta)}{\partial\overline{\zeta}_1\partial\zeta_1^{k_1-1}\partial\zeta_2^{r_2}\dots\partial\zeta_n^{r_n}}\right](z) &+ \frac{\partial^{r_1}}{\partial z_1^{r_1}}K\left[\frac{\partial^{r_2+\dots+r_n}u(\zeta)}{\partial\zeta_2^{r_2}\dots\partial\zeta_n^{r_n}}\right](z). \end{aligned} \tag{2.7}$$

Note that, in the last term in the right-hand side of Eq. (2.7), the derivatives with respect to ζ_1 vanish. To eliminate other derivatives, we apply formula (2.1), as above, successively in the variables $\zeta_2, \dots, \zeta_n$, each time choosing k and the function $g(z)$ in a corresponding manner. This leads to (2.5). □

Note 2.1. *The multipliers $\overline{\zeta}_j^2$ in the Cauchy-type integrals $K[\overline{\zeta}_j^2\dots]$ in the right-hand side of* (2.5) *arise by purely technical reasons and do no harm in applications* (*in Theorem* 3.1).

3. HÖLDER ESTIMATES FOR DERIVATIVES

3.1. For some α, $0 < \alpha < 1$ we introduce the following denotations:

$A^m(U^n)$ denotes the space of all functions holomorphic in U^n which are m times continuously differentiable in $\overline{U}^n$;

$A^{m,\alpha}(U^n)$ denotes the subset of those functions from $A^m(U^n)$ whose mth-order derivatives belong to the α class of Hölder in $\overline{U}^n$;

$A^{m,\alpha}_{(\log)}(U^n)$ denotes the subset of those functions from $A^m(U^n)$, whose moduli of continuity of all mth-order derivatives are majorized by const $\cdot\, \delta^\alpha \left(\log\frac{1}{\delta}\right)^{n-1}$ in $\overline{U}^n$. In these denotations we omit m if $m = 0$. Note that $A^0(U^n) = A(U^n)$ is a usual polydisk algebra.

Theorem 3.1. *Suppose that* $u \in C^{m,\alpha}(T^n), 0 < \alpha < 1$. *Then*, $K[u] \in A^{m,\alpha}_{(\log)}(U^n)$.

Proof. We will carry out the proof by induction in m.

At $m = 0$ the statement of the theorem is the consequence of the Joricke theorem [1].

We make an inductive assumption: suppose that the statement of the theorem is valid for all orders of smoothness lower than m. We use identity (2.5) from Lemma 2.1. From the condition of the theorem it follows that the functions

$$\overline{\zeta}_j^2 \frac{\partial^{k_j + r_{j+1} + \cdots + r_n} u(\zeta)}{\partial \overline{\zeta}_j \partial \zeta_j^{k_j - 1} \partial \zeta_{j+1}^{r_{j+1}} \ldots \partial \zeta_n^{r_n}}$$

belong to the class $C^{r_1 + \cdots + r_j - k_j + \alpha}(T^n)$. Because $r_1 + \cdots + r_j - k_j < m$, by the inductive assumption,

$$K\left[\overline{\zeta}_j^2 \frac{\partial^{k_j + r_{j+1} + \cdots + r_n} u(\zeta)}{\partial \overline{\zeta}_j \partial \zeta_j^{k_j - 1} \partial \zeta_{j+1}^{r_{j+1}} \ldots \partial \zeta_n^{r_n}}\right](z) \in C_{(\log)}^{r_1 + \cdots + r_j - k_j + \alpha}(\overline{U^n}).$$

Therefore, the terms under the double sum sign in the right-hand side of (2.5) belong to $C^\alpha_{(\log)}(\overline{U^n})$, that is,

$$\frac{\partial^{r_1 + \cdots + r_j - k_j}}{\partial z_1^{r_1} \ldots \partial z_{j-1}^{r_{j-1}} \partial z_j^{r_j - k_j}} K\left[\overline{\zeta}_j^2 \frac{\partial^{k_j + r_{j+1} + \cdots + r_n} u(\zeta)}{\partial \overline{\zeta}_j \partial \zeta_j^{k_j - 1} \partial \zeta_{j+1}^{r_{j+1}} \ldots \partial \zeta_n^{r_n}}\right](z) \in C^\alpha_{(\log)}(\overline{U^n}). \tag{3.1}$$

Next, because by the condition $\dfrac{\partial^{r_1 + \cdots + r_n} u(\zeta)}{\partial \zeta_1^{r_1} \ldots \partial \zeta_n^{r_n}} \in C^\alpha(T^n)$, for the last term in (2.5) we have

$$K\left[\frac{\partial^{r_1 + \cdots + r_n} u(\zeta)}{\partial \zeta_1^{r_1} \ldots \partial \zeta_n^{r_n}}\right](z) \in C^\alpha_{(\log)}(\overline{U^n}). \tag{3.2}$$

From (3.1), (3.2), and (2.5) it follows that $\frac{\partial^{r_1 + \cdots + r_n}}{\partial z_1 \ldots \partial z_n} K[u](z) \in C^\alpha_{(\log)}(\overline{U^n})$, which means that $K[u] \in C^{m,\alpha}_{(\log)}(\overline{U^n})$. Since $K[u]$ is holomorphic in (U^n), this implies that $K[u] \in A^{m,\alpha}_{(\log)}(U^n)$. $\square$

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