ISSN 1068-3623, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2021, Vol. 56, No. 3, pp. 128–133. © Allerton Press, Inc., 2021.
Russian Text © The Author(s), 2021, published in Izvestiya Natsional'noi Akademii Nauk Armenii, Matematika, No. 3, pp. 3–9.

PROBABILITY THEORY

Conditional Moments for a d-Dimensional Convex Body

R. Aramyan[1*] and V. Mnatsakanyan[1]**

[1]*Russian-Armenian University, Yerevan, Armenia*

Received December 07, 2020; revised December 07, 2020; accepted March 01, 2021

Abstract—For a d-dimensional convex body we define new integral geometric concepts: conditional moments of the random chord length and conditional moments of the distance of two independent uniformly distributed points in the body. Also in this article the relations between the concepts are found.

MSC2010 numbers: 53C65, 53C60, 31A10.
DOI: 10.3103/S106836232103002X

Keywords: *integral geometry; convex body; geometric tomography; conditional moments; random chord length.*

1. INTRODUCTION

Geometric tomography (the term introduced by R. Gardner in [7]) is a field of mathematics engaged in extracting information about a geometric object from data on its sections or projections to reconstruct the geometric object. The reconstruction of convex domain using random sections makes it possible to simplify the calculation, since mathematical statistics methods can be used to estimate the geometric characteristics of random sections. The integral geometric concepts such as the distribution of the chord length, the distribution of the distance between two random points in a convex body and many others carry some information about the body. In this article for a d-dimensional convex body D we define two new integral geometric concepts: conditional moments of the chord length distribution of a convex body and conditional moments of the distribution of the distance of two random points in D. Also in this article we find the relation between the two concepts.

By $\mathbf{R}^d$ ($d \geqslant 2$) we denote the d-dimensional Euclidean space, by S^{d-1} the unit sphere in $\mathbf{R}^d$ centered at the origin. Let L_d be the Lebesgue measure on $\mathbf{R}^d$. For $\omega \in S^{d-1}$ by e_ω we denote the hyperplane containing the origin and orthogonal to ω. Let $\mathcal{N}$ be the set of nonnegative integers. Let $\mathbf{G}^d$ be the space of all lines in $\mathbf{R}^d$. We use the usual parametrization of a line $g = (\omega, P)$, where $\omega \in S^{d-1}$ is the direction of g and P is the intersection point of g and e_ω. By $[D]$ we denote the set of lines intersecting D. In $\mathbf{G}^d$ we consider the invariant measure (with respect to the group of Euclidean motions) $\mu(\cdot)$. It is known that the element dg of the measure, up to a constant, has the following form ([8, 1, 3])

$$dg = d\omega dP, \tag{1.1}$$

here $d\omega$ and dP are elements of the Lebesgue measure on S^{d-1} and the hyperplane, respectively.

Definition 1.1. *Let D be a compact convex set in $\mathbf{R}^d$ below we call D a convex body. We consider the random line g with normed invariant measure ($\frac{dg}{\mu([D])}$, here $\mu([D])$ is the invariant measure of lines intersecting $[D]$). For a random line g intersecting D by $X(g)$ we denote the length of the chord $D \cap g$. The conditional n-th moment of the distribution of the chords length (with respect to condition $X > u \geqslant 0$) we define as*:

$$I_{n,u} = \frac{1}{\mu([D])} \int_{X(g)>u} X(g)^n dg, \quad n = 1, 2, ... \tag{1.2}$$

*E-mail: rafikaramyan@yahoo.com
**E-mail: v.mnatsakanyan95@gmail.com

Lemma 2.2 (below) gives the explicit formula for $\mu([D])$. In the sequel by $F_X(t)$ we denote the distribution function of $X(g)$.

Definition 1.2. *For two independent uniformly distributed points Q_1, Q_2 in a convex domain D we denote the distance between the points by $r = |Q_1 - Q_2|$. The conditional n-th moment of the distribution of the distance (with respect to condition $r > u \geqslant 0$) we define as:*

$$J_{n,u} = \frac{1}{L_d(D)^2} \int\limits_{|Q_1-Q_2|>u} r^n dQ_1 dQ_2 \tag{1.3}$$

here $L_d(D)$ is the volume of D, dQ_i $(i = 1, 2)$ is the usual Lebesgue's measure in $\mathbf{R}^d$. Also, in the sequel by $F_r(u)$ we denote the distribution function of the distance of two uniformly distributed points Q_1, Q_2 in a convex body D.

In the following theorem we obtain relation between the conditional moments of the distribution of the distance of two random points in D and the conditional moments of the distribution of the chords length.

Theorem 1.1. *Let D be a convex domain and $u \geqslant 0$. For any $n \in \mathcal{N}$*

$$J_{n,u} = \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u}u^{n+d+1}}{(n+d+1)} - \frac{I_{1,u}u^{n+d}}{n+d} + \frac{I_{n+d+1,u}}{(n+d)\,(n+d+1)} \right). \tag{1.4}$$

The moments of the distribution of the chords length and the distribution of the distance between two independent uniformly distributed point in a convex domain was considered in [8] and [6].

For the planar case $d = 2$ (1.4) was proved in [4].

For the distribution function of the distance between two independent uniformly distributed points in a convex body in $\mathbf{R}^d$ we have

Theorem 1.2.

$$F_r(u) = 1 - J_{0,u} = 1 - \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u}u^{d+1}}{d+1} - \frac{I_{1,u}u^{d}}{d} + \frac{I_{d+1,u}}{d(d+1)} \right). \tag{1.5}$$

2. PRELIMINARY RESULTS

To prove Theorem 1.1 we need to prove the following lemmas. Let $D \subset \mathbf{R}^d$ be a convex body.

Lemma 2.1. *For the invariant measure of the lines intersecting D we have*

$$\mu([D]) = \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{2\,(d-1)}. \tag{2.1}$$

Proof. By definition we have

$$\mu([D]) = \int\limits_{[D]} dg = \int\limits_{[D]} d\omega dP = \frac{1}{2} \int\limits_{S^{d-1}} d\omega \int\limits_{D_\omega} dP = \frac{1}{2} \int\limits_{S^{d-1}} L_{d-1}(D_\omega)\, d\omega, \tag{2.2}$$

where D_ω is the orthogonal projection of D onto hyperplane e_ω. For $\xi \in S^{d-1}$ we denote by $s(\xi)$ the point on ∂D the outer normal of which is ξ. In [6] (see also [8]) was proved that

$$L_{d-1}(D_\omega) = \frac{1}{2} \int\limits_{\partial D} |\cos\widehat{(\omega,\xi)}| ds_\xi, \tag{2.3}$$

where ds_ξ is the element of $(d-1)$-dimensional Lebesgue's measure on ∂D and $\widehat{(\omega,\xi)}$ is the angle between two directions ω and ξ. Substituting (2.3) into (2.2) and using the Fubini's theorem we obtain

$$\mu([D]) = \frac{1}{4} \int\limits_{S^{d-1}} \int\limits_{\partial D} |\cos\widehat{(\omega,\xi)}| ds_\xi d\omega = \frac{1}{4} \int\limits_{\partial D} \int\limits_{S^{d-1}} |\cos\widehat{(\omega,\xi)}| d\omega ds_\xi. \tag{2.4}$$

For any $\xi \in S^{d-1}$ we have (see [2])

$$\int_{S^{d-1}} |\cos\widehat{(\omega,\xi)}| d\omega = \frac{2L_{d-2}\left(S^{d-2}\right)}{d-1}. \tag{2.5}$$

Finally substituting (2.5) into (2.4) we obtain

$$\mu\left([D]\right) = \frac{L_{d-2}\left(S^{d-2}\right)}{2\left(d-1\right)} \int_{\partial D} ds_{\xi} = \frac{L_{d-1}\left(\partial D\right) L_{d-2}\left(S^{d-2}\right)}{2\left(d-1\right)} \tag{2.6}$$

Lemma 2.1 is proved.

Now we consider a pair of points (Q_1, Q_2) in $\mathbf{R}^d$. There are two equivalent representations of (Q_1, Q_2).

1. A pair of points Q_1, Q_2 can be determined by the usual cartesian coordinates.
2. A pair of points Q_1, Q_2 can be determined by the line $g = (\omega, P)$ passing through the points and pair of two one dimensional coordinates (t_1, t_2) which determine Q_1 and Q_2 on the line g (for 3-dimensional case see [8]). Thus

$$(Q_1, Q_2) = (g, t_1, t_2) = (\omega, P, t_1, t_2). \tag{2.7}$$

Note that as a reference point on g one can take the point P on g.

Lemma 2.2. *The Jacobian of that transform* (2.7) *is*

$$dQ_1 dQ_2 = |t_1 - t_2|^{d-1} dt_1 dt_2 d\omega dP. \tag{2.8}$$

Proof. For a fixed Q_1 we represent Q_2 by polar coordinates with respect to Q_1. It is known that

$$dQ_2 = r^{d-1} dr d\omega \tag{2.9}$$

where $r = |Q_1 - Q_2|$ and ω is the direction of the vector $\overrightarrow{Q_1Q_2}$. For a fixed ω the point Q_1 can be represented by P and t_1. Thus

$$dQ_1 = dt_1 dP \tag{2.10}$$

and by multiplying (2.9) and (2.10) and taking into account that $r = |t_1 - t_2|$, we get

$$dQ_1 dQ_2 = |t_1 - t_2|^{d-1} dt_1 dt_2 d\omega dP. \tag{2.11}$$

Lemma 2.2 is proved.

In the sequel also, we use the following lemma. For a random line g intersecting a convex body $D \subset \mathbf{R}^d$ we have the following lemma.

Lemma 2.3. *Let $X(g)$ be the length of the chord $D \cap g$. We have*

$$\int_{[D]} X(g) dg = \frac{L_d\left(D\right) L_{d-1}\left(S^{d-1}\right)}{2} \tag{2.12}$$

Proof. By definition we have $(g = (\omega, P))$

$$\int_{[D]} X(g) dg = \frac{1}{2} \int_{S^{d-1}} d\omega \int_{D_\omega} X\left(\omega, P\right) dP. \tag{2.13}$$

For any $\omega \in S^{d-1}$ it is obvious that $X\left(\omega, P\right) dP$ is the element of d-dimensional volume of D, hence the integrating by dP over D_ω we get $L_d\left(D\right)$.

$$\int_{[D]} X\left(g\right) dg = \frac{1}{2} \int_{S^{d-1}} d\omega \int_{D_\omega} X\left(\omega, P\right) dP$$

$$= \frac{L_d\left(D\right)}{2} \int_{S^{d-1}} d\omega = \frac{L_d\left(D\right) L_{d-1}\left(S^{d-1}\right)}{2} \tag{2.14}$$

3. PROOF OF THEOREM 1.1

Let Q_1, Q_2 are two independent uniformly distributed points in a convex body D. For a random line g intersecting D we denote by $X(g) = |g \cap D|$ the length of the intersection. For $u \geqslant 0$ using (2.8), (2.1) and taking into account $r = |Q_1 - Q_2| = |t_1 - t_2|$ we have

$$J_{n,u} = \frac{1}{L_d(D)^2} \int\limits_{|P_1 - P_2| > u} |P_1 - P_2|^n \, dQ_1 dQ_2$$
$$= \frac{1}{L_d(D)^2} \int\limits_{X(g) > u} \int\limits_{|t_1 - t_2| > u} |t_1 - t_2|^{n+d-1} dt_1 dt_2 dg. \tag{3.1}$$

Consider the internal integral of (3.1). For two points t_1 and t_2 chosen at random, independently with uniform distribution in a segment of length $X > u$ we have.

$$\int\limits_{|t_1 - t_2| > u} |t_1 - t_2|^{n+d-1} dt_1 dt_2 = 2 \int\limits_0^{X-u} dt_1 \int\limits_{t_1+u}^{X} (t_2 - t_1)^{n+d-1} \, dt_2$$
$$= 2 \int\limits_0^{X-u} \frac{(X - t_1)^{n+d}}{n+d} - \frac{u^{n+d}}{n+d} dt_1 = 2 \left(\frac{u^{n+d+1}}{(n+d+1)} - \frac{X u^{n+d}}{n+d} + \frac{X^{n+d+1}}{(n+d)\,(n+d+1)} \right) \tag{3.2}$$

Substituting (3.2) into (3.1) we get

$$J_{n,u} = \frac{2}{L_d(D)^2} \int\limits_{X(g) > u} \left(\frac{u^{n+d+1}}{(n+d+1)} - \frac{X(g) u^{n+d}}{n+d} + \frac{X(g)^{n+d+1}}{(n+d)\,(n+d+1)} \right) dg$$
$$= \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u} u^{n+d+1}}{(n+d+1)} - \frac{I_{1,u} u^{n+d}}{n+d} + \frac{I_{n+d+1,u}}{(n+d)\,(n+d+1)} \right). \tag{3.3}$$

Theorem 1.1. is proved.

Not that for $u \geqslant Diam(D)$, both sides of (3.3) are 0.

Corollary 3.1. *For $u = 0$ and $d = 2$ from* (3.1) *for a convex domain D we get the following well known formula* (see [8])

$$J_{n,0} = \frac{2L_1(\partial D)}{L_2(D)^2} \left(\frac{I_{n+3,0}}{(n+2)\,(n+3)} \right). \tag{3.4}$$

Corollary 3.2. *For $n = 0$ we get*

$$J_{0,u} = \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u} u^{d+1}}{d+1} - \frac{I_{1,u} u^{d}}{d} + \frac{I_{d+1,u}}{d(d+1)} \right). \tag{3.5}$$

Taking into account

$$J_{0,u} = \frac{1}{L_d(D)^2} \int\limits_{|Q_1 - Q_2| > u} dQ_1 dQ_2 = 1 - F_r(u) \tag{3.6}$$

we get the following theorem.

4. A REPRESENTATION FOR $I_{n,u}$

Now we are going to find a representations for $I_{0,u}, I_{1,u}, I_{d+1,u}$.

1. For $I_{0,u}$ we have

$$I_{0,u} = \frac{2(d-1)}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})} \int\limits_{X(g)>u} dg$$
$$= P\left(X(g) > u\right) = \left(1 - P\left(X(g) \leqslant u\right)\right) = 1 - F_X\left(u\right). \tag{4.1}$$

Here $F_X\left(t\right)$ is the chord length distribution function.

2. For the derivative of $I_{1,u}$ we have

$$(I_{1,u})' = \left(\int\limits_{X(g)>u} X(g)\frac{dg}{\mu([D])}\right)' = -\lim_{\Delta u\to 0}\frac{\int_{u<X(g)<u+\Delta u} X(g)\frac{dg}{\mu([D])}}{\Delta u}$$
$$-\lim_{\Delta u\to 0}\frac{\mathrm{uP}\left(u < X(g) < u+\Delta u\right)}{\Delta u} = -\mathrm{uf}_X\left(u\right), \tag{4.2}$$

here $f_X\left(t\right)$ is the density function of the chord length distribution of $X(g)$.

It follows from Lemma 2.3 that

$$I_{1,0} = \frac{(d-1)L_d(D)L_{d-1}(S^{d-1})}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})}. \tag{4.3}$$

Integrating (4.2) and taking into account (4.3) we get

$$I_{1,u} = \frac{(d-1)L_d(D)L_{d-1}(S^{d-1})}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})} - \int\limits_0^u v f_x\left(v\right)\mathrm{dv}. \tag{4.4}$$

3. By the same way (see (4.2)) for the derivative of $I_{d+1,u}$ we have

$$(I_{d+1,u})' = \left(\int\limits_{X(g)>u} X(g)^{d+1}\frac{dg}{\mu([D])}\right)' = -u^{d+1}f_X\left(u\right). \tag{4.5}$$

It follows from (3.3) that

$$I_{d+1,0} = \frac{(d-1)d(d+1)L_d(D)^2}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})}. \tag{4.6}$$

Integrating (4.5) and taking into account (4.6) we get

$$I_{d+1,u} = \frac{(d-1)d(d+1)L_d(D)^2}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})} - \int\limits_0^u v^{d+1} f_X((v)\,\mathrm{dv}. \tag{4.7}$$

Finally substituting (4.7), (4.4), (4.1) into (3.5) we obtain the following relation between the distribution function of the distance of two uniformly distributed points of D and the chord length distribution function of D.

Theorem 4.1. *Let D be a convex body in $\mathbf{R}^d$. For $u \geqslant 0$*

$$F_r\left(u\right) = 1 - \frac{L_{d-1}\left(\partial D\right)L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\left(d-1\right)}\left(\frac{u^{d+1}}{d+1} - \frac{(d-1)L_d(D)L_{d-1}(S^{d-1})}{dL_{d-1}\left(\partial D\right)L_{d-2}(S^{d-2})}u^d\right.$$
$$\left. + \frac{(d-1)L_d(D)^2}{L_{d-1}\left(\partial D\right)L_{d-2}(S^{d-2})} - \frac{u^d}{2}\int\limits_0^u F_X(v)dv + \frac{1}{d}\int\limits_0^u v^d F_X(v)dv\right). \tag{4.8}$$

For the planar case $d = 2$ (4.8) was proved in [5] (see also [4]).

REFERENCES

1. R. V. Ambartzumian, “Combinatorial integral geometry, metric and zonoids,” Acta Appl. Math. **9**, 3–27 (1987). doi 10.1007/978-94-009-3921-9_2
2. R. H. Aramyan, “The sine representation of centrally symmetric convex bodies,” J. Contemp. Math. Anal. **53**, 363–368 (2018). doi 10.3103/S1068362318060079
3. R. Aramyan, “A flag representation for n-dimensional convex body,” J. Geom. Anal. **29**, 2998–3009 (2019). doi 10.1007/s12220-018-00102-1
4. R. Aramyan and V. Mnatsakanyan, “Conditional moments of the distribution of the distance of two random points in a convex domain in R^2,” Proc. Yerevan State Univ., Phys. Math. Sci. **54**, 3–8 (2020).
5. N. G. Aharonyan, “The distribution of the distance between two random points in a convex set,” Russ. J. Math. Res., Ser. A **1**, 4–8 (2015). doi 10.13187/rjmr.a.2015.1.4
6. W. Blaschke, *Kreis und Kugel*, 2nd ed. (W. de Gruyter, Berlin, 1956).
7. R. J. Gardner, *Geometric Tomography*, 2nd ed. (Cambridge Univ. Press, Cambridge, 2006).
8. L. A. Santalo, *Integral Geometry and Geometric Probability* (Addison-Wesley, Reading, MA, 1976).