Complex Variables
Vol. 50, No. 12, 10 October 2005, 953–966

On weighted classes of harmonic functions in the unit ball of $\mathbf{R}^n$

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Communicated by H. Begehr

(*Received 22 August 2004; in final form 12 April 2005*)

This article gives the main representation theorems for harmonic functions in the spaces $b^p_\omega(B)$ on the unit ball B in $\mathbf{R}^n$. These spaces depend on a parameter function ω and are arbitrarily large. We receive the integral representation for the functions of $b^p_\omega(B)$ over the unit ball. The article also gives a representation connected with the natural isometry between $b^2_\omega(B)$ and the ordinary space L^2 on the unit sphere, which is explicitly given in the form of an integral operator along with its inversion.

Keywords: Weighted spaces; Harmonic functions; Integral representation; Isometry

AMS 2000 Mathematics Subject Classifications: 30H05; 46E15

1. Introduction

This article gives the main representation theorems for arbitrarily large harmonic $b^p_\omega(B)$ spaces in the unit ball in $\mathbf{R}^n$, which are similar to the analytic spaces investigated in [1].

Section 2 is devoted to some preliminary notation and construction of Djrbashian's ω-kernel [2] in the unit ball of $\mathbf{R}^n$. In section 3, we introduce the spaces $b^p_\omega(B)$ and prove some preliminary statements. Section 4 is devoted to the main integral representation of $b^p_\omega(B)$ over the unit ball (Theorem 1) and to the orthogonal projection from $L^2_\omega(B)$ to $b^2_\omega(B)$ (Theorem 2). Note that for the particular case

$$\omega(t) = \frac{nV(B)}{2}\int_t^1 \tau^{(n/2)-1}(1-\tau)^\alpha \,\mathrm{d}\tau$$

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Complex Variables
ISSN 0278-1077 print: ISSN 1563-5066 online © 2005 Taylor & Francis Group Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/02781070500156868

the representation of Theorem 1 was obtained in [3] and for $\alpha = 0$ one can find this representation also in [4]. Section 5 gives an integral representation of the considered spaces $b^2_\omega(B)$ over the unit sphere. This leads to an isometry between the $b^2_\omega(B)$ spaces and the ordinary L^2-space on the unit sphere, which has an explicit form of integral operator along with its inversion (Theorems 3 and 4).

In this article, we shall frequently use some basic statements of the n-dimensional theory of harmonic functions explicitly given in [4].

Note that for the one-dimensional case, i.e. in the unit disc, the basics of the theory of arbitrarily large classes A^p_ω of analytic functions were more exhaustively constructed in [5,6].

2. Construction of the $\mathbf{R}_\omega$-kernel

2.1. We start by giving the notations, which we use throughout the article.

$B = \{x \in \mathbf{R}^n\colon\ |x| < 1\}$ is the open unit ball in $\mathbf{R}^n$ and $S = \{x \in \mathbf{R}^n\colon\ |x| = 1\}$ is its boundary, i.e. S is the unit sphere in $\mathbf{R}^n$;
σ is the normalized surface-area measure on S, so that $\sigma(S) = 1$;
$\mathcal{H}_m(\mathbf{R}^n)$ is the set of all complex-valued homogeneous harmonic polynomials of degree m in $\mathbf{R}^n$;
$\mathcal{H}_m(S)$ is the set of all spherical harmonics of degree m, i.e. the restrictions of functions from $\mathcal{H}_m(\mathbf{R}^n)$ on the sphere S;
$P[f]$ denotes the Poisson integral of f:

$$P[f](x) = \int_S P(x,\zeta) f(\zeta)\,\mathrm{d}\sigma(\zeta), \quad \text{where } P(x,\zeta) = \frac{1-|x|^2}{|\zeta - x|^n}. \tag{1}$$

We associate with each complex function f on $[a,b]$ its total variation $\bigvee_a^b f$ defined by $\bigvee_a^b f = \sup\left\{\sum_{j=1}^N |f(t_j) - f(t_{j-1})|\right\}$, where the supremum is taken over all N and over all choices of $\{t_j\}$ such that $a = t_1 < t_2 < \cdots < t_N = b$.

Further, as in [4], by Ω we denote the class of functions $\omega(t)$ in $[0,1]$ such that $\omega(1) = \omega(1-0)$ and

(i) $0 < \bigvee_\delta^1 \omega < \infty$ for any $\delta \in [0,1)$;
(ii) $\Delta_k \equiv \Delta_k(\omega) = -\int_0^1 t^k \mathrm{d}\omega(t) \neq 0, \infty,\ k = 0,1,\ldots;$
(iii) $\liminf\limits_{k\to\infty} \sqrt[k]{|\Delta_k|} \geq 1.$

2.2. For a given $\omega \in \Omega$ we introduce the ω-kernel

$$R_\omega(x,y) = \sum_{k=0}^\infty \Delta_k^{-1} Z_k(x,y). \tag{2}$$

Lemma 1 *The series in the right side of* (2) *converges absolutely and uniformly on the set* $\{(x,y) \in \mathbf{R}^{2n}\colon\ |x||y| \leq q,\ 0 < q < 1\}$ *and particularly on* $K \times \overline{B}$, *where K is an arbitrary compact subset of B.*

Proof Let $x = r\zeta$, $y = \rho\eta$, where $\zeta, \eta \in S$. Taking into account that the function $Z_k(x, y)$ is homogeneous by both variables, we obtain

$$|Z_k(x, y)| = r^k \rho^k |Z_k(\zeta, \eta)| \leq r^k \rho^k d_k, \tag{3}$$

where d_k is the dimension of $\mathcal{H}_k(S)$. The property (i) of the parameter function ω implies

$$\limsup_{k\to\infty} \sqrt[k]{|\Delta_k|} \leq \limsup_{k\to\infty} \sqrt[k]{\int_0^1 t^k |\mathrm{d}\omega(t)|} \leq \limsup_{k\to\infty} \sqrt[k]{\bigvee\nolimits_0^1 \omega} = 1.$$

Along with (iii), this means that

$$\lim_{k\to\infty} \sqrt[k]{|\Delta_k|} = 1.$$

The desired convergence follows from (3) in view of the estimate $d_k \leq Ck^{n-2}$ from [4]. ■

3. The spaces $b^p_\omega(B)$

3.1. For a given $\omega \in \Omega$, we denote

$$\mathrm{d}\mu_\omega(x) = -\mathrm{d}\omega(r^2)\, \mathrm{d}\sigma(\zeta),$$

where $x = r\zeta$ is the polar form of $x \in B$ (i.e. $r = |x|$, $\zeta \in S$), and define $L^p_\omega(B)$ as the set of all $\mathrm{d}\mu_\omega$-measurable functions in B for which

$$\|u\|_{p,\omega} = \left\{ \int_B |u(x)|^p \, |\mathrm{d}\mu_\omega(x)| \right\}^{1/p} < +\infty, \quad 1 \leq p < \infty.$$

We introduce $b^p_\omega(B)$ as the harmonic subset of $L^p_\omega(B)$.

It turns out that for any fixed p the classes $b^p_\omega(B)$ can contain harmonic functions of - arbitrary growth near the boundary.

Proposition 1 *For any fixed $p \in [1, \infty)$ the sum $\bigcup_{\omega\in\Omega} b^p_\omega(B)$ coincides with the set of all functions harmonic in B.*

Proof Evidently, it is sufficient to show that Ω contains functions of any rate of decrease as $t \to 1 - 0$. Indeed, if $\omega \searrow$ in $[0, 1]$ then

$$\Delta_k = -\int_0^1 t^k \, \mathrm{d}\omega(t) \geq -\int_{1-\varepsilon}^1 t^k \, \mathrm{d}\omega(t) \geq (1 - \varepsilon)^k \bigvee_{1-\varepsilon}^1 \omega$$

for any $\varepsilon \in (0,1)$. Therefore, by (i)

$$\liminf_{k\to\infty} \sqrt[k]{|\Delta_k|} \geq (1-\varepsilon)\sqrt[k]{\bigvee_{1-\varepsilon}^{1} \omega} = 1-\varepsilon,$$

and the passage $\varepsilon \to 0$ gives (iii). ■

For any fixed $x \in B$, the mapping $u \mapsto u(x)$ is a linear functional over $b^p_\omega(B)$. The following proposition shows that this is a continuous functional.

PROPOSITION 2 *For any function $u \in b^p_\omega(B)$ and any point $x \in B$*

$$|u(x)| \leq \frac{2^{n/p}}{(1-|x|)^{(n-1)/p}} \left(\int_{(1+|x|)/2}^{1} |\mathrm{d}\omega(t^2)| \right)^{-1/p} \|u\|_{p,\omega}.$$

Proof The following estimates obviously are true for the Poisson's kernel (1):

$$P(x,\zeta) = \frac{1-|x|^2}{|\zeta - x|^n} \leq \frac{1+|x|}{(1-|x|)^{n-1}} \leq \frac{2}{(1-|x|)^{n-1}}. \tag{4}$$

Let $x \in B$ and $|x| < R < 1$. Using the subharmonicity of the function $|u(Rx)|^p$ in the neighbourhood of the ball $\overline{B}$ and (4), we get

$$|u(Rx)|^p \leq \int_S |u(R\zeta)|^p P(x,\zeta)\,\mathrm{d}\sigma(\zeta) \leq \frac{2}{(1-|x|)^{n-1}} \int_S |u(R\zeta)|^p \,\mathrm{d}\sigma(\zeta). \tag{5}$$

The integral means $M(R) = \int_S |u(R\zeta)|^p \,\mathrm{d}\sigma(\zeta)$ is nondecreasing in R. Hence

$$\begin{aligned} \int_R^1 |\mathrm{d}\omega(t^2)| \int_S |u(R\zeta)|^p \,\mathrm{d}\sigma(\zeta) &\leq \int_R^1 \left(\int_S |u(t\zeta)|^p \,\mathrm{d}\sigma(\zeta) \right) |\mathrm{d}\omega(t^2)| \\ &= \int_{R<|x|<1} |u(x)|^p |\mathrm{d}\mu_\omega(x)| \leq \|u\|^p_{p,\omega}. \end{aligned} \tag{6}$$

By (5) and (6)

$$|u(Rx)|^p \leq \frac{2}{(1-|x|)^{n-1}} \left(\int_R^1 |\mathrm{d}\omega(t^2)| \right)^{-1} \|u\|^p_{p,\omega},$$

and the change of a variable $Rx \mapsto x$ gives

$$|u(x)| \leq \frac{2^{n/p}}{(R-|x|)^{(n-1)/p}} \left(\int_R^1 |\mathrm{d}\omega(t^2)| \right)^{-1/p} \|u\|_{p,\omega}.$$

Taking $R = (1+|x|)/2$ we come to our assertion. ■

For a multi-index $s = (s_1, \ldots, s_n)$ (where s_i are nonnegative integers), the partial differentiation operator D^s is usually defined as $D_1^{s_1} \ldots D_n^{s_n}$. Using this definition and the Cauchy inequalities for harmonic functions, we come to

COROLLARY 1 *For any multi-index s there is a constant $C = C(s)$ such that*

$$\left|D^s u(x)\right| \leq \frac{C}{(1-|x|)^{|s|+(n-1)/p}} \left(\int_{(3+|x|)/4}^{1} |\mathrm{d}\omega(t^2)|\right)^{-1/p} \|u\|_{p,\omega}.$$

Proof Applied for the ball $B(x) = \{y\colon\ |y - x| < (1-|x|)/2\}$, the Cauchy inequalities give

$$\left|D^s u(x)\right| \leq \frac{C_s M(x)}{(1-|x|)^{|s|}}, \tag{7}$$

where $M(x) = \max_{y \in B(x)} u(y)$. By Proposition 2, for any $y \in B(x)$

$$|u(y)| \leq \frac{2^{n/p}}{(1-|y|)^{(n-1)/p}} \left(\int_{(1+|y|)/2}^{1} |\mathrm{d}\omega(t^2)|\right)^{-1/p} \|u\|_{p,\omega}.$$

Besides, the inequalities $1 - |y| \geq (1-|x|)/2$ and $(1+|y|)/2 \leq (3+|x|)/4$ obviously follow from $y \in B(x)$, and taking the maximum over all $y \in B(x)$ we get

$$M(x) \leq \frac{2^{(2n-1)/p}}{(1-|x|)^{(n-1)/p}} \left(\int_{(3+|x|)/4}^{1} |\mathrm{d}\omega(t^2)|\right)^{-1/p} \|u\|_{p,\omega}.$$

Our statement follows from (7) and the last inequalities. ■

PROPOSITION 3 *For any $1 \leq p < \infty$, $b^p_\omega(B)$ is a closed subset of $L^p_\omega(B)$.*

Proof Suppose $\|u_j - u\|_{p,\omega} \to 0$ as $j \to \infty$, where u_j is a sequence of functions in $b^p_\omega(B)$ and $u \in L^p_\omega(B)$. We shall show that u is equivalent to some function harmonic on B, with respect to the measure μ_ω.

Let $K \in B$ be a compact. Proposition 2 implies that there exists a constant $C \equiv C(K, p, \omega)$ such that

$$\max_{x \in K} |u(x)| \leq C\|u\|_{p,\omega}$$

for any $u \in b^p_\omega(B)$. Hence $|u_j(x) - u_k(x)| \leq C\|u_j - u_k\|_{p,\omega}$ for any $x \in K$ and j, k. The sequence u_j is fundamental in $b^p_\omega(B)$, and hence u_j converges uniformly on compact subsets of B to a function v harmonic on B. Besides, $u_j \to u$ in L^p_ω. Therefore, by Riesz' theorem there exists a subsequence of u_j converging to u pointwise almost everywhere in B, with respect to μ_ω. Thus, $u = v$ almost everywhere in B, and $u \in b^p_\omega(B)$. ■

COROLLARY 2 *$b^p_\omega(B)$ is a Banach space.*

3.2. The next assertion states the continuity of ϱ-dilatation in $b^p_\omega(B)$.

Proposition 4 *Let $u \in b^p_\omega(B)$ and $u_\varrho(x) = u(\varrho x)$. Then $\|u_\varrho - u\|_{p,\omega} \to 0$ as $\varrho \to 1-0$.*

Proof For any $\delta \in (0,1)$

$$\|u_\varrho - u\|^p_{p,\omega} \le \int_{\delta B} |u(\varrho x) - u(x)|^p \, \mathrm{d}\mu_\omega(x) + 2^p \int_\delta^1 \left\{ \int_S (|u(\varrho r\zeta)|^p + |u(r\zeta)|^p) \mathrm{d}\sigma(\zeta) \right\} \mathrm{d}\omega(r^2) \tag{8}$$

since $(a+b)^p \le 2^p(a^p + b^p)$ $(a, b > 0)$. Further $m(\varrho) = \int_S |u(\varrho r\zeta)|^p \, \mathrm{d}\sigma(\zeta)$ is nondecreasing and $m(\varrho) \le m(1)$ since $|u(x)|^p$ is subharmonic. Hence by (8)

$$\|u_\varrho - u\|^p_{p,\omega} \le \int_{\delta B} |u(\varrho x) - u(x)|^p \, \mathrm{d}\mu_\omega(x) + 2^{p+1} \int_{B \setminus \delta B} |u(x)|^p \, \mathrm{d}\mu_\omega(x).$$

It remains to see that the right-hand side of this inequality can be made arbitrarily small by taking δ and then ϱ close enough to 1. ■

It is well known that any function harmonic in a domain containing $\overline{B}$ can be uniformly approximated on $\overline{B}$ by harmonic polynomials. Using this fact, one can prove the following corollary of Proposition 4.

Corollary 3 *Harmonic polynomials are dense in $b^p_\omega(B)$.*

4. Representation over the ball

4.1. Let $u(x)$ be a harmonic function in the unit ball of the space $\mathbf{R}^n$. The following homogeneous expansion is well known:

$$u(x) = \sum_{k=0}^\infty p_k(x), \tag{9}$$

where $p \in \mathcal{H}_m(\mathbf{R}^n)$ and the series (9) is absolutely and uniformly convergent on compact subsets of the ball.

Let $Z_m(\zeta, \eta)$ $(\zeta \in S,\ \eta \in S)$ be the zonal harmonic of degree m. Then $Z_m(\zeta, \eta) = Z_m(\eta, \zeta)$, $Z_m(\zeta, \cdot) \in \mathcal{H}_m(S)$ and the following representation is true:

$$p_m(\zeta) = \int_S p_m(\eta) Z_m(\zeta, \eta) \, \mathrm{d}\eta. \tag{10}$$

The theorem below gives the main representation formulas in $b^p_\omega(B)$ spaces.

Theorem 1 *Let $u \in b^p_\omega(B)$. Then*

$$u(x) = \int_B u(y) R_\omega(x, y) \, \mathrm{d}\mu_\omega(y), \quad x \in B. \tag{11}$$

Proof Let $p_k \in \mathcal{H}_m(S)$. Then

$$\int_B p_k(y) R_\omega(x,y)\,\mathrm{d}\mu_\omega(y) = \int_B p_k(y)\left(\sum_{m=0}^{\infty} \frac{Z_m(x,y)}{\Delta_m}\right)\mathrm{d}\mu_\omega(y)$$
$$= \sum_{m=0}^{\infty} \frac{1}{\Delta_m} \int_B p_k(y) Z_m(x,y)\,\mathrm{d}\mu_\omega(y). \tag{12}$$

If $\zeta, \eta \in S$ and $x = r\zeta,\ y = \rho\eta$, then by homogeneity of the functions $p_k(y)$ and $Z_m(x,y)$

$$\int_B p_k(y) Z_m(x,y)\,\mathrm{d}\mu_\omega(y) = -\int_B \rho^k p_k(\eta) r^m \rho^m Z_m(\zeta,\eta)\,\mathrm{d}\omega(\rho^2)\,\mathrm{d}\sigma(\eta)$$
$$= -r^m \int_0^1 \rho^{k+m}\,\mathrm{d}\omega(\rho^2) \int_S p_k(\eta) Z_m(\zeta,\eta)\,\mathrm{d}\sigma(\eta).$$

The last integral vanishes for $m \neq k$ by orthogonality and is equal to $p_k(\zeta)$ for $m = k$ in accordance to (10). Hence

$$\int_B p_k(y) Z_k(x,y)\,\mathrm{d}\mu_\omega(y) = -r^k p_k(\zeta) \int_0^1 \rho^{2k}\,\mathrm{d}\omega(\rho^2)$$
$$= -p_k(x) \int_0^1 t^k\,\mathrm{d}\omega(t) = \Delta_k p_k(x). \tag{13}$$

By (12) and (13),

$$\int_B p_k(y) R_\omega(x,y)\,\mathrm{d}\mu_\omega(y) = p_k(x). \tag{14}$$

Further, let $u_\varrho(x) = u(\varrho x)$ $(0 < \varrho < 1)$. By the uniform convergence of the expansion $u(\varrho x) = \sum p_k(\varrho x)$ in $\overline{B}$ and by (14)

$$u_\varrho(x) = \sum_{k=0}^{\infty} \varrho^k p_k(x) = \sum_{k=0}^{\infty} \varrho^k \int_B p_k(y) R_\omega(x,y)\,\mathrm{d}\mu_\omega(y)$$
$$= \int_B \left(\sum_{k=0}^{\infty} \varrho^k p_k(y)\right) R_\omega(x,y)\,\mathrm{d}\mu_\omega(y) = \int_B \left(\sum_{k=0}^{\infty} p_k(\varrho y)\right) R_\omega(x,y)\,\mathrm{d}\mu_\omega(y)$$
$$= \int_B u_\varrho(y) R_\omega(x,y)\,\mathrm{d}\mu_\omega(y).$$

By Proposition 4, the passage $\varrho \to 0$ leads to the desired assertion. ■

4.2. Consider the special case

$$\omega(t) = \frac{nV(B)}{2} \int_t^1 \tau^{(n/2)-1}(1-\tau)^\alpha \mathrm{d}t.$$

Let $x = r\zeta$, where $r = |x|$, $\zeta \in S$. Using the expression of the volume element in polar coordinates (see, for instance, [7]), we get

$$\begin{aligned}\mathrm{d}\mu_\omega(x) &= -\mathrm{d}\omega(r^2)\mathrm{d}\sigma(\zeta) = -\omega'(r^2)\,2r\,\mathrm{d}r\,\mathrm{d}\sigma(\zeta)\\ &= nV(B)r^{n-1}(1-r^2)^\alpha \mathrm{d}r\,\mathrm{d}\sigma(\zeta) = (1-r^2)^\alpha\,\mathrm{d}V(x).\end{aligned}$$

Thus, in the considered case $b^p_\omega(B)$ consists of all harmonic functions u in B, which satisfy

$$\|u\|_{p,\alpha} = \left(\int_B |u(x)|^p(1-|x|^2)^\alpha \mathrm{d}V(x)\right)^{1/p} < \infty.$$

We denote this space by $b^p_\alpha(B)$. Further,

$$\begin{aligned}\Delta_m &= -\int_0^1 t^m\,\mathrm{d}\omega(t) = \frac{nV(B)}{2}\int_0^1 t^{n/2+m-1}(1-t)^\alpha\,\mathrm{d}t\\ &= \frac{nV(B)}{2}\frac{\Gamma(n/2+m)\Gamma(\alpha+1)}{\Gamma(n/2+m+\alpha+1)},\end{aligned}$$

and denoting the corresponding kernel by R_α, we have

$$R_\alpha(x,y) = \frac{2}{nV(B)}\sum_{m=0}^\infty \frac{\Gamma(n/2+m+\alpha+1)}{\Gamma(n/2+m)\Gamma(\alpha+1)} Z_m(x,y).$$

Thus, in the considered case, formula (11) takes the form

$$u(x) = \frac{2}{nV(B)}\int_B u(y)\left(\sum_{m=0}^\infty \frac{\Gamma(n/2+m+\alpha+1)}{\Gamma(n/2+m)\Gamma(\alpha+1)} Z_m(x,y)\right)(1-|y|^2)^\alpha\,\mathrm{d}V(y),$$

i.e. coincides with that of [3].

We suppose that for any $x \in B$ the Poisson kernel $P(x,y)$ is harmonically extended to $\overline{B}$ as follows:

$$P(x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{n/2}},$$

where $\cdot$ denotes the usual Euclidean inner product. To obtain an expression of R_α by means of the Poisson kernel P, we use some well-known facts from the theory of fractional integro-differentiation in the Riemann–Liouville sense. The primitive of $f \in L^1(0,1)$ of order $\alpha > 0$ is defined as

$$D^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,\mathrm{d}\tau.$$

The derivative of order α is defined to be

$$D^{\alpha} f(t) = \frac{\mathrm{d}^p}{\mathrm{d}t^p}\left\{ D^{-(p-\alpha)} f(t)\right\},$$

where the integer p is deduced by the inequalities $p-1<\alpha\le p$. Using the simple equality

$$D^{\alpha+1} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(\gamma-\alpha)} t^{\gamma-\alpha-1},$$

we find that

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{\Gamma(n/2+m+\alpha+1)}{\Gamma(n/2+m)} Z_m(x,y) &= \sum_{m=0}^{\infty} D^{\alpha+1}\big(t^{n/2+m+\alpha} Z_m(x,y)\big)\big|_{t=1} \\ = D^{\alpha+1}\left(\sum_{m=0}^{\infty} t^{n/2+m+\alpha} Z_m(x,y)\right)\Bigg|_{t=1} &= D^{\alpha+1}\left(\sum_{m=0}^{\infty} t^{n/2+\alpha} Z_m(tx,y)\right)\Bigg|_{t=1} \\ = D^{\alpha+1}\big(t^{n/2+\alpha} P(tx,y)\big)\big|_{t=1}.\end{aligned}$$

Thus

$$R_\alpha(x,y) = \frac{2}{n\Gamma(\alpha+1)V(B)} D^{\alpha+1}\big(t^{n/2+\alpha} P(tx,y)\big)\big|_{t=1}.$$

When α is a nonnegative integer, the operator $D^{\alpha+1}$ is obtained from the usual derivation, and this allows to calculate $R_\alpha(x,y)$ in an explicit form. Particularly, for $\alpha=0$ this calculation results in the formula

$$R_0(x,y) = \frac{2}{nV(B)}\frac{\mathrm{d}}{\mathrm{d}t}\big(t^{n/2}P(tx,y)\big)\Bigg|_{t=1} = \frac{nP(x,y)+2(\mathrm{d}/\mathrm{d}t)P(tx,y)\big|_{t=1}}{nV(B)},$$

which coincides with that of [4] in view of

$$2\frac{\mathrm{d}}{\mathrm{d}t}P(tx,y)\Bigg|_{t=1} = \frac{\mathrm{d}}{\mathrm{d}t}P(tx,ty)\Bigg|_{t=1}.$$

4.3. The right-hand side integral of (11) defines the orthogonal projection of $L^2_\omega(B)$ onto its subspace $b^2_\omega(B)$, i.e. the following assertion is true.

THEOREM 2 *The operator*

$$Q_\omega[u](x) = \int_B u(y) R_\omega(x,y)\,\mathrm{d}\mu_\omega(y), \quad u\in L^2_\omega(B), \ \ x\in B,$$

is the orthogonal projection of $L^2_\omega(B)$ *onto* $b^2_\omega(B)$.

Proof As $L^2_\omega(B) = b^2_\omega(B) \oplus (b^2_\omega(B))^\perp$, any $u \in L^2_\omega(B)$ is written in the form $u = u_1 + u_2$, where $u_1 \in b^2_\omega(B)$ and $u_2 \in (b^2_\omega(B))^\perp$. Hence $Q_\omega[u] = Q_\omega[u_1] + Q_\omega[u_2]$, where $Q_\omega[u_1] = u_1$ by Theorem 1. On the other hand,

$$Q_\omega[u_2](x) = \int_B u_2(y) R_\omega(x, y)\, \mathrm{d}\mu_\omega(y) = \langle u_2, R_\omega(x, \cdot)\rangle_\omega = 0,$$

where $\langle \cdot, \cdot \rangle_\omega$ is the inner product of $L^2_\omega(B)$, since due to Lemma 1 for a fixed $x \in B$ the function $R_\omega(x, y)$ is harmonic by y in a domain containing $\overline{B}$, and u_2 is orthogonal to $b^2_\omega(B)$. Thus $Q_\omega[u] = u_1$, i.e. Q_ω is the orthogonal projector $L^2_\omega(B) \mapsto b^2_\omega(B)$. ■

5. Representation over the sphere

5.1. We start by the following assertion proved in [6].

Proposition 5 *Let* $\tilde{\omega} \in \Omega$ *be continuously differentiable in* $[0, 1)$ *and such that* $\tilde{\omega}(t) \searrow$, $\tilde{\omega}(1) = 0$ *and* $\tilde{\omega}(0) = 1$. *Further, let* ω *be the Volterra square of* $\tilde{\omega}$, *i.e.*

$$\omega(t) = -\int_t^1 \tilde{\omega}\left(\frac{t}{\sigma}\right) \mathrm{d}\tilde{\omega}(\sigma), \quad 0 < t < 1. \tag{15}$$

Then $\omega \in \Omega$ *and*

$$\Delta_m(\omega) = [\Delta_m(\tilde{\omega})]^2, \quad m \geq 0. \tag{16}$$

Further, we denote the norm in $L^p(S)$ by $\| \cdot \|_p$ and prove

Proposition 6 *Let* $u \in b^2_\omega(B)$ *and* $u(x) = \sum p_k(x)$ *be its homogeneous expansion. Then*

$$\|u\|^2_{2,\omega} = \sum_{k=0}^{\infty} |\Delta_k(\omega)| \| p_k \|^2_2.$$

Proof For any $r \in (0, 1)$

$$\begin{aligned}
\int_{B(r)} |u(y)|^2 \, \mathrm{d}\mu_\omega(y) &= \int_0^r \mathrm{d}\omega(\rho^2) \left(\sum_{k=0}^{\infty} p_k(\rho\zeta) \sum_{s=0}^{\infty} \overline{p}_s(\rho\zeta) \right) \mathrm{d}\sigma(\zeta) \\
&= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \int_0^r \rho^{k+s} \, \mathrm{d}\omega(\rho^2) \int_S p_k(\zeta) \overline{p}_s(\zeta) \, \mathrm{d}\sigma(\zeta) \\
&= \sum_{k=0}^{\infty} \int_0^r \rho^{2k} \, \mathrm{d}\omega(\rho^2) \int_S | p_k(\zeta)|^2 \, \mathrm{d}\sigma(\zeta) \\
&= \sum_{k=0}^{\infty} \int_0^{r^2} t^k \, \mathrm{d}\omega(t) \int_S |p_k(\zeta)|^2 \, \mathrm{d}\sigma(\zeta).
\end{aligned}$$

Letting $r \to 1-0$ we get

$$\|u\|_{2,\omega}^2 = \int_B |u(y)|^2 \,\mathrm{d}\mu_\omega(y) = \sum_{k=0}^{\infty} |\Delta_k(\omega)| \|p_k\|_2^2. \qquad \blacksquare$$

On the basis that $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$, we prove

PROPOSITION 7 *Let* $f \in L^2(S)$ *and let* $f = \sum p_m$ *be its spherical harmonic expansion* (*i.e.* $p_m \in \mathcal{H}_m(S)$ *and the sum converges in* $L^2(S)$). *Then the following formulas are true for homogeneous harmonic polynomials* $p_m(x)$:

$$p_m(x) = \int_S f(\zeta) Z_m(x, \zeta) \,\mathrm{d}\sigma(\zeta).$$

Proof For any fixed $x = r\eta$ ($r \geq 0$, $\eta \in S$)

$$\begin{aligned} p_m(x) = r^m p_m(\eta) &= r^m \int_S p_m(\zeta) Z_m(\eta, \zeta) \,\mathrm{d}\sigma(\zeta) \\ &= r^m \int_S \left(\sum_{k=0}^{\infty} p_k(\zeta) \right) Z_m(\eta, \zeta) \,\mathrm{d}\sigma(\zeta) = r^m \int_S f(\zeta) Z_m(\eta, \zeta) \,\mathrm{d}\sigma(\zeta) \\ &= \int_S f(\zeta) Z_m(x, \zeta) \,\mathrm{d}\sigma(\zeta). \end{aligned}$$

where the third equality follows by the orthogonality of the spherical harmonics of different degrees. $\blacksquare$

THEOREM 3 *The mapping* $f \mapsto R_{\tilde{\omega}}[f]$, *where*

$$R_{\tilde{\omega}}[f](x) = \int_S f(\zeta) R_{\tilde{\omega}}(x, \zeta) \,\mathrm{d}\sigma(\zeta),$$

is a linear isometry from $L^2(S)$ *to* $b^2_\omega(B)$.

Proof First, observe that the considered mapping is evidently linear. Next, suppose $f = \sum p_m$ as in Proposition 7. Then obviously

$$\begin{aligned} R_{\tilde{\omega}}[f](x) &= \int_S f(\zeta) R_{\tilde{\omega}}(x, \zeta) \,\mathrm{d}\sigma(\zeta) = \int_S f(\zeta) \sum_{m=0}^{\infty} \Delta_m^{-1}(\tilde{\omega}) Z_m(x, \zeta) \,\mathrm{d}\sigma(\zeta) \\ &= \sum_{m=0}^{\infty} \Delta_m^{-1}(\tilde{\omega}) \int_S f(\zeta) Z_m(x, \zeta) \,\mathrm{d}\sigma(\zeta) = \sum_{m=0}^{\infty} \Delta_m^{-1}(\tilde{\omega}) p_m(x). \end{aligned} \tag{17}$$

According to Proposition 6 and (16)

$$\left\| R_{\tilde{\omega}}[f] \right\|_{2,\omega}^2 = \sum \Delta_m(\omega) \| \Delta_m^{-1}(\tilde{\omega}) p_m \|_2^2 = \sum \|p_m\|_2^2 = \|f\|_2^2.$$

It remains to show that the range of values of the mapping $f \mapsto R_{\tilde{\omega}}[f]$ is the whole space $b^2_\omega(B)$. To prove this, suppose $u \in b^2_\omega(B)$ and $u(x) = \sum q_k(x)$. If $p_k(x) = \Delta_k(\tilde{\omega})q_k(x)$, then

$$\sum \|p_k\|_2^2 = \sum \|\Delta_k(\tilde{\omega})q_k\|_2^2 = \sum \Delta_k(\omega)\|q_k\|_2^2 = \|u\|^2_{2,\omega}$$

due to Proposition 6 and (16). Hence, the function $f = \sum p_k$ belongs to $L^2(S)$. As in (17), we obtain $R_{\tilde{\omega}}[f](x) = \sum \Delta_m^{-1}(\tilde{\omega})p_m(x)$, and therefore $R_{\tilde{\omega}}[f](x) = u(x)$. ∎

Further, we denote $h^p(B)$ the ordinary harmonic Hardy space, i.e. the class of functions u harmonic in B and such that

$$\|u\|_{h^p} = \sup_{0 \le r < 1} \|u_r\|_p < \infty.$$

Besides, we consider the operator

$$L_{\tilde{\omega}}[u](x) = -\int_0^1 u(tx)\, \mathrm{d}\tilde{\omega}(t).$$

THEOREM 4 *Let* $f \in L^2(S)$ *and* $u = R_{\tilde{\omega}}[f]$. *Then*

(a) $L_{\tilde{\omega}}[u] = P[f]$,
(b) *the mapping* $u \mapsto L_{\tilde{\omega}}[u]$ *is a linear isometry of* $b^2_{\tilde{\omega}}(B)$ *onto* $h^2(B)$.

Proof Let $f = \sum p_m$, and $u(x) = \sum q_m(x)$ be the homogeneous expansion of u in the unit ball. Then

$$q_m(x) = \Delta_m^{-1}(\tilde{\omega})p_m(x) \tag{18}$$

in accordance with (17). Further, it is known that $P[f]$ has the homogeneous expansion $P[f](x) = \sum p_k(x)$. Therefore, by (18) and Proposition 7

$$\begin{aligned} P[f](x) &= \sum_{k=0}^{\infty} \Delta_k(\tilde{\omega})q_k(x) = -\sum_{k=0}^{\infty} q_k(x) \int_0^1 t^k\, \mathrm{d}\tilde{\omega}(t) \\ &= -\sum_{k=0}^{\infty} \int_0^1 q_k(tx)\, \mathrm{d}\tilde{\omega}(t) = -\int_0^1 \left(\sum_{k=0}^{\infty} q_k(tx) \right) \mathrm{d}\tilde{\omega}(t) \\ &= -\int_0^1 u(tx)\, \mathrm{d}\tilde{\omega}(t) = L_{\tilde{\omega}}[u](x). \end{aligned}$$

This proves (a). For accomplishing the proof, it suffices to observe that the mapping $f \mapsto P[f]$ is a linear isometry of $L^2(S)$ onto $h^2(B)$, and consequently (b) follows from Theorem 3 and (a). ∎

Remark It is well known that for $f \in L^2(S)$ the function $P[f]$ has a nontangential limit $f(\zeta)$ at almost every point $\zeta \in S$. Thus, it is natural to identify f and $P[f]$ and to say that the operators L_ω and R_ω are mutually inverse.

5.2. In the special case mentioned in section 4, Theorems 3 and 4 take the following forms.

THEOREM 5 *Let* $u(x) \in b^2_\alpha(B)$ $(\alpha > -1)$. *Then the function*

$$\varphi(x) = \frac{\Gamma((n+\alpha+1)/2)}{\Gamma(n/2)\Gamma((\alpha+1)/2)} \int_0^1 u(tx) t^{(n/2)-1} (1-t)^{(\alpha-1)/2}\, \mathrm{d}t$$

belongs to $h^2(B)$ *and the following integral representation is true*:

$$u(x) = \int_S \varphi(\zeta) T_\alpha(x, \zeta)\, \mathrm{d}\sigma(\zeta),$$

where

$$T_\alpha(x,\zeta) = \sum_{k=0}^{\infty} \frac{\Gamma(n/2)\Gamma(k+(n+\alpha+1)/2)}{\Gamma((n+\alpha+1)/2)\Gamma(k+n/2)} Z_k(x,\zeta).$$

Proof In Theorems 3 and 4 we put

$$\tilde{\omega}(t) = \frac{\Gamma((n+\alpha+1)/2)}{\Gamma(n/2)\Gamma((\alpha+1)/2)} \int_t^1 \tau^{n/2-1} (1-\tau)^{(\alpha-1)/2}\, \mathrm{d}\tau,$$

where the coefficient before the integral is chosen to provide $\tilde{\omega}(0) = 1$ in Proposition 5. It is clear that $\tilde{\omega}(t)$ satisfies all the required conditions. Arguing as above, we see that the corresponding $R_{\tilde{\omega}}$ is equal to T_α. Hence, it remains to show that $b^2_\omega(B) = b^2_\alpha(B)$. The latter will be proved if we show that the Volterra square ω of $\tilde{\omega}$ satisfies the relation $\omega'(t) \asymp (1-t)^\alpha$. Indeed, denoting

$$c = \frac{\Gamma((n+\alpha+1)/2)}{\Gamma(n/2)\Gamma((\alpha+1)/2)},$$

we have

$$\begin{aligned}
\omega'(t) &= -\int_t^1 \tilde{\omega}'\Big(\frac{t}{\sigma}\Big) \tilde{\omega}'(\sigma)\, \mathrm{d}\sigma \\
&= -c^2 \int_t^1 \Big(\frac{t}{\sigma}\Big)^{(n/2)-1} \Big(1-\frac{t}{\sigma}\Big)^{(\alpha-1)/2} \sigma^{(n/2)-1} (1-\sigma)^{(\alpha-1)/2} \frac{\mathrm{d}\sigma}{\sigma} \\
&= -c^2 t^{(n/2)-1} \int_t^1 \Big(1-\frac{t}{\sigma}\Big)^{(\alpha-1)/2} (1-\sigma)^{(\alpha-1)/2} \frac{\mathrm{d}\sigma}{\sigma} \\
&= -c^2 t^{(n/2)-1} \int_0^1 \left(\frac{(1-t)(1-\tau)}{1-(1-t)\tau}\right)^{(\alpha-1)/2} \big((1-t)\tau\big)^{(\alpha-1)/2} \frac{(1-t)\mathrm{d}\tau}{1-(1-t)\tau} \\
&\asymp (1-t)^\alpha \int_0^1 (1-\tau)^{(\alpha-1)/2} \tau^{(\alpha-1)/2}\, \mathrm{d}\tau \asymp (1-t)^\alpha.
\end{aligned}$$

■

References

[1] Petrosyan, A.I., 2005, On A^p_ω spaces in the unit ball of $\mathbb{C}^n$. *Journal of Analysis and Applications*, **3**(1), 47–53.
[2] Djrbashian, M.M., 1975, Theory of factorization and boundary properties of functions meromorphic in the disc, In: *Proceedings of the ICM*, Vancouver, B.C., 1974, Vol. 2, pp. 197–202 (USA).
[3] Djrbashian, A.E. and Shamoian, F.A., 1988, *Topics in the Theory of A^p_α Spaces* (Leipzig: Teubner Verlag).
[4] Axler, Sheldon, Bourdon, Paul and Ramsey, Wade, 2001, *Harmonic Function Theory* (New York, Inc.: Springer-Verlag).
[5] Jerbashian, A.M., 2002, Weighted Classes of Regular Functions Area Integrable Over the Unit Disc, Preprint 2002-01, Institute of Mathematics, National Ac. of Sci. of Armenia, Yerevan. Available at: http://math.sci.am/People/ArmenJerbashianPubl.html (accessed 2 June 2005).
[6] Jerbashian, A.M., 2005, On the theory of weighted classes of area integrable regular functions. *Complex Variables*, **50**, 155–183.
[7] Rudin, Walter, 1987, *Real and Complex Analysis* (London: McGraw-Hill).