BMO-ESTIMATION AND ALMOST EVERYWHERE EXPONENTIAL SUMMABILITY OF QUADRATIC PARTIAL SUMS OF DOUBLE FOURIER SERIES

U. GOGINAVA, L. GOGOLADZE AND G. KARAGULYAN

ABSTRACT. It is proved a BMO-estimation for quadratic partial sums of two-dimensional Fourier series from which it is derived an almost everywhere exponential summability of quadratic partial sums of double Fourier series.

1. INTRODUCTION

Let $\mathbb{T} := [-\pi, \pi) = \mathbb{R}/2\pi$ and $\mathbb{R} := (-\infty, \infty)$. We denote by $L_1(\mathbb{T})$ the class of all measurable functions f on $\mathbb{R}$ that are 2π-periodic and satisfy

$$\|f\|_1 := \int_{\mathbb{T}} |f| < \infty.$$

The Fourier series of the function $f \in L_1(\mathbb{T})$ with respect to the trigonometric system is the series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}, \tag{1}$$

where

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

are the Fourier coefficients of f.

Denote by $S_n(x, f)$ the partial sums of the Fourier series of f and let

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x, f)$$

be the $(C, 1)$ means of (1). Fejér [1] proved that $\sigma_n(f)$ converges to f uniformly for any 2π-periodic continuous function. Lebesgue in [15] established almost everywhere convergence of $(C, 1)$ means if $f \in L_1(\mathbb{T})$. The strong summability problem, i.e. the convergence of the strong means

$$\frac{1}{n+1} \sum_{k=0}^{n} |S_k(x, f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0, \tag{2}$$

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was first considered by Hardy and Littlewood in [11]. They showed that for any $f \in L_r(\mathbb{T})$ $(1 < r < \infty)$ the strong means tend to 0 a.e., if $n \to \infty$. The trigonometric Fourier series of $f \in L_1(\mathbb{T})$ is said to be (H, p)-summable at $x \in T$, if the values (2) converge to 0 as $n \to \infty$. The (H, p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [17] for $p = 2$, and later by Zygmund [26] for the general case $1 \le p < \infty$. K. I. Oskolkov in [19] proved the following

Theorem A. *Let $f \in L_1(\mathbb{T})$ and let Φ be a continuous positive convex function on $[0, +\infty)$ with $\Phi(0) = 0$ and*

$$\ln \Phi(t) = O(t/\ln\ln t) \qquad (t \to \infty). \tag{3}$$

Then for almost all x

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi(|S_k(x, f) - f(x)|) = 0. \tag{4}$$

It was noted in [19] that V. Totik announced the conjecture that (4) holds almost everywhere for any $f \in L_1(\mathbb{T})$, provided

$$\ln \Phi(t) = O(t) \quad (t \to \infty). \tag{5}$$

In [20] V.Rodin proved

Theorem B. *Let $f \in L_1(\mathbb{T})$. Then for any $A > 0$*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (\exp(A|S_k(x, f) - f(x)|) - 1) = 0$$

for a. e. $x \in \mathbb{T}$.

G. Karagulyan [12] proved that the following is true.

Theorem C. *Suppose that a continuous increasing function $\Phi : [0, \infty) \to [0, \infty), \Phi(0) = 0$, satisfies the condition*

$$\limsup_{t\to+\infty} \frac{\log \Phi(t)}{t} = \infty.$$

Then there exists a function $f \in L_1(\mathbb{T})$ for which the relation

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi(|S_k(x, f)|) = \infty$$

holds everywhere on $\mathbb{T}$.

In fact, Rodin in [20] has obtained a BMO estimate for the partial sums of Fourier series and his theorem stated above is obtained from that estimate by using John-Nirenberg theorem. Recall the definition of BMO $[0, 1]$ space. It is the Banach space of functions $f \in L_1[0, 1]$ with the norm

$$\|f\|_{\mathrm{BMO}} = \mathfrak{R}(f) + \left| \int_0^1 f(t)dt \right|$$

where

$$\mathfrak{R}(f) = \sup_I (|f - f_I|)_I, f_I = \frac{1}{|I|}\int_I f(t)dt$$

and the supremum is taken over all intervals $I \subset [0,1]$ ([4], chap. 6). Let $\{\xi_n : n = 0,1,2,\ldots\}$ be an arbitrary sequence of numbers. Taking $\delta_k^n = [k/(n+1),(k+1)/(n+1)]$, we define

$$\text{BMO } [\xi_n] = \sup_{0\le n<\infty} \left\| \sum_{k=0}^{n} \xi_k \mathbb{I}_{\delta_k^n}(t) \right\|_{\text{BMO}}$$

where $\mathbb{I}_{\delta_k^n}(t)$ is the characteristic function of δ_k^n. Notice that the expressions

$$\text{BMO}\left[\widetilde{S}_n(x,f)\right], \quad \text{BMO}\left[S_n(x,f)\right], \quad f \in L_1(\mathbb{T}), x \in \mathbb{T} \tag{6}$$

define a sublinear operators, where $\widetilde{S}_n(x,f)$ is the conjugate partial sum. The following theorem is proved by Rodin in [20].

Theorem D. *The operators (6) are of weak type* $(1,1)$*, i.e. the inequalities*

$$|\{x \in \mathbb{T} : \text{BMO } [S_n(x,f)] > \lambda\}| \le \frac{c}{\lambda}\int_{\mathbb{T}} |f(t)|dt \tag{7}$$

and

$$|\{x \in \mathbb{T} : \text{BMO } [\widetilde{S}_n(x,f)] > \lambda\}| \le \frac{c}{\lambda}\int_{\mathbb{T}} |f(t)|dt \tag{8}$$

hold for any $f \in L_1(\mathbb{T})$.

In this paper we study the question of exponential summability of quadratic partial sums of double Fourier series. Let $f \in L_1(\mathbb{T}^2)$, be a function with Fourier series

$$\sum_{m,n=-\infty}^{\infty} \widehat{f}(m,n)\, e^{i(mx+ny)}, \tag{9}$$

where

$$\widehat{f}(m,n) = \frac{1}{4\pi^2}\iint_{\mathbb{T}^2} f(x,y)e^{-i(mx+ny)}dxdy$$

are the Fourier coefficients of the function f. The rectangular partial sums of (9) are defined as follows:

$$S_{MN}(x,y,f) = \sum_{m=-M}^{M}\sum_{n=-N}^{N} \widehat{f}(m,n)\, e^{i(mx+ny)}.$$

We denote by $L\log L\left(\mathbb{T}^2\right)$ the class of measurable functions f, with

$$\iint_{\mathbb{T}^2} |f| \log^+ |f| < \infty,$$

where $\log^+ u := \mathbb{I}_{(1,\infty)} \log u$. For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [18] has proved, that if $f \in L\log L\left(\mathbb{T}^2\right)$, then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(S_{kk}\left(x,y,f\right) - f\left(x,y\right)\right) = 0$$

for a. e. $(x,y) \in \mathbb{T}^2$. L. Zhizhiashvili [24] improved this result showing that class $L\log L\left(\mathbb{T}^2\right)$ can be replaced by $L_1\left(\mathbb{T}^2\right)$.

From a result of S. Konyagin [14] it follows that for every $\varepsilon > 0$ there exists a function $f \in L\log^{1-\varepsilon}\left(\mathbb{T}^2\right)$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|S_{kk}\left(x,y,f\right) - f\left(x,y\right)\right| \neq 0 \quad \text{for a. e. } (x,y) \in \mathbb{T}^2. \tag{10}$$

The main result of the present paper is the following.

Theorem 1. *If $f \in L\log L\left(\mathbb{T}^2\right)$, then*

$$\begin{aligned} &\left|\left\{(x,y) \in \mathbb{T}^2 : \mathrm{BMO}\left[S_{nn}(f,x,y)\right] > \lambda\right\}\right| \\ &\leq \frac{c}{\lambda}\left(1 + \iint\limits_{\mathbb{T}^2} |f| \log^+ |f)|\right) \end{aligned} \tag{11}$$

for any $\lambda > 0$, where c is an absolute positive constant.

The following theorem shows that the quadratic sums of two-dimensional Fourier series of a function $f \in L\log L\left(\mathbb{T}^2\right)$ are almost everywhere exponentially summable to the function f. It will be obtained from the previous theorem by using John-Nirenberg theorem.

Theorem 2. *Suppose that $f \in L\log L\left(\mathbb{T}^2\right)$. Then for any $A > 0$*

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^{m} \left(\exp\left(A\left|S_{nn}\left(x,y,f\right) - f\left(x,y\right)\right|\right) - 1\right) = 0$$

for a. e. $(x,y) \in \mathbb{T}^2$.

According to a Lemma of L. D. Gogoladze [9], this theorem can be formulated in more general settings.

Theorem 3. *Let $\psi : [0,\infty) \to [0,\infty)$ be a increasing function, satisfying the conditions*

$$\lim_{u\to 0} \psi\left(u\right) = \psi\left(0\right) = 0, \limsup_{u\to\infty} \frac{\log\psi\left(u\right)}{u} < \infty.$$

Then for any $f \in L\log L\left(\mathbb{T}^2\right)$ we have

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^{m} \psi\left(\left|S_{nn}\left(x,y,f\right) - f\left(x,y\right)\right|\right) = 0$$

almost everywhere on $\mathbb{T}^2$.

The results on Marcinkiewicz type strong summation for the Fourier series have been investigated in [2, 3, 10, 6, 7, 5, 8, 16, 23, 27, 28, 24]

2. Notations and lemmas

The relation $a \lesssim b$ bellow stands for $a \le c \cdot b$, where c is an absolute constant. The conjugate function of a given $f \in L_1(\mathbb{T}^2)$ is defined by

$$\tilde{f}(x) = \text{p.v.}\frac{1}{\pi}\int\limits_{\mathbb{T}} \frac{f(x+t)}{2\tan(t/2)}dt = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int\limits_{\varepsilon<|t|<\pi}\frac{f(x+t)}{2\tan(t/2)}dt$$

According to Kolmogorv's and Zygmund's inequalities (see [26], chap. 7), we have

$$|\{x\in\mathbb{T}: |\tilde{f}(x)|>\lambda\}| \lesssim \frac{\|f\|_{L_1(\mathbb{T})}}{\lambda}, \tag{12}$$

$$\int\limits_{\mathbb{T}} |\tilde{f}(x)|dx \lesssim 1 + \int\limits_{\mathbb{T}} |f(x)|\log^+|f(x)|dx. \tag{13}$$

It will be used two simple properties of BMO norm bellow. First one says, if $\xi_n = c$, $n = 1,2,\ldots$, then BMO $[\xi_n] = |c|$. The second one is, the bound

$$\text{BMO}\,[\xi_n] \le 3\sup_n|\xi_n|.$$

We shall consider the operators

$$U_n(x,f) = \text{p.v.}\frac{1}{\pi}\int\limits_{\mathbb{T}}\frac{\cos nt}{2\tan(t/2)}f(x+t)dt.$$

The following lemma is an immediate consequence of Theorem D.

Lemma 1. *The inequality*

$$|\{x\in\mathbb{T}: \text{BMO}\,[U_n(x,f)]>\lambda\}| \lesssim \frac{\|f\|_{L_1(\mathbb{T})}}{\lambda}$$

holds for any $f \in L_1(\mathbb{T})$.

Proof. For the conjugate Dirichet kernel we have

$$\begin{aligned}\tilde{D}_n(t) &= \frac{\cos(t/2)-\cos(n+1/2)t}{2\sin(t/2)}\\ &= \frac{1}{2\tan(t/2)} + \frac{\sin nt}{2} - \frac{\cos nt}{2\tan(t/2)}\end{aligned} \tag{14}$$

and we get

$$\begin{aligned}\tilde{S}_n(x,f) &= \frac{1}{\pi}\int\limits_{\mathbb{T}}\tilde{D}_n(t)f(x+t)dt\\ &= \tilde{f}(x) + \frac{1}{2\pi}\int\limits_{\mathbb{T}} f(x+t)\sin nt dt - U_n(x,f).\end{aligned}$$

Thus, applying simple properties of BMO norm, we obtain

$$\mathrm{BMO}\,[U_n(x,f)] \le |\tilde{f}(x)| + \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|dt + \mathrm{BMO}\,\left[\tilde{S}_n(x,f)\right]$$

Applying the bound (12) and Theorem D, the last inequality completes the proof of lemma. □

We consider the square partial sums

(15)
$$S_{nn}(x,y,f) = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} \frac{\sin(n+1/2)\,t\sin(n+1/2)\,s}{4\sin(t/2)\sin(s/2)} f(x+t,y+s)\,dtds$$

and their modification, defined by

$$S^*_{nn}(x,y,f) = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} \frac{\sin nt\sin ns}{4\tan(t/2)\tan(s/2)} f(x+t,y+s)\,dtds.$$

Lemma 2. *If $f \in L\log L(\mathbb{T}^2)$, then*

$$\iint_{\mathbb{T}^2} \sup_n |S_{nn}(x,y,f) - S^*_{nn}(x,y,f)|\,dxdy \lesssim 1 + \iint_{\mathbb{T}^2} |f|\log^+|f|.$$

Proof. Substituting the expression for Dirichlet kernel

$$D_n(t) = \frac{\sin(n+1/2)t}{2\sin t/2} = \frac{\sin nt}{2\tan(t/2)} + \frac{\cos nt}{2}$$

in (15), we get

$$\begin{aligned}
S_{nn}(x,y,f) &\\
&= S^*_{nn}(x,y,f)\\
&+ \frac{1}{\pi^2}\iint_{\mathbb{T}^2} \frac{\sin nt\cdot\cos ns}{4\tan(t/2)} f(x+t,y+s)\,dtds\\
&+ \frac{1}{\pi^2}\iint_{\mathbb{T}^2} \frac{\cos nt\cdot\sin ns}{4\tan(s/2)} f(x+t,y+s)\,dtds\\
&+ \frac{1}{4\pi^2}\iint_{\mathbb{T}^2} \cos nt\cdot\cos ns\cdot f(x+t,y+s)\,dtds\\
&= S^{(1)}_{nn}(x,y,f) + S^{(2)}_{nn}(x,y,f) + S^{(3)}_{nn}(x,y,f).
\end{aligned}$$

It is clear, that

$$|S^{(3)}_{nn}(x,y,f)| \lesssim \|f\|_{L^1(\mathbb{T}^2)} \lesssim 1 + \iint_{\mathbb{T}^2} |f|\log^+|f|. \tag{16}$$

Everywhere below the notation

$$\text{p.v.} \iint_{\mathbb{T}^2} f(t,s)dtds$$

stands for either

$$\text{p.v.} \int_{\mathbb{T}} \left(\text{p.v.} \int_{\mathbb{T}} f(t,s)dt \right) ds, \text{ or } \text{p.v.} \int_{\mathbb{T}} \left(\text{p.v.} \int_{\mathbb{T}} f(t,s)ds \right) dt$$

and in each cases we have equality of these two iterated integrals. To observe that we will need just the fact that $f(x) \in L\log L(\mathbb{T})$ implies $\tilde{f}(x) \in L_1(\mathbb{T})$. Hence, making simple transformations and then changing the variables, we get

(17)

$$\begin{aligned}
&S_{nn}^{(1)}(x,y,f) \\
&= \text{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin n(t+s)}{2\tan(t/2)} f(x+t, y+s)\, dsdt \\
&+ \text{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin n(t-s)}{2\tan(t/2)} f(x+t, y+s)\, dsdt \\
&= \text{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin nu \cdot f(x+v, y+u-v)}{2\tan(v/2)} dvdu \quad (u = t+s,\ v = t) \\
&+ \text{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin nu \cdot f(x+v, y+v-u)}{2\tan(v/2)} dvdu \quad (u = t-s,\ v = t) \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} \sin nu \left(\text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+u-v)}{2\tan(v/2)} dv \right) du \\
&+ \frac{1}{2\pi} \int_{\mathbb{T}} \sin nu \left(\text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+v-u)}{2\tan(v/2)} dv \right) du.
\end{aligned}$$

Observe, that the functions

$$F_1(x,y,u) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+u-v)}{2\tan(v/2)} dv$$

$$F_2(x,y,u) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+v-u)}{2\tan(v/2)} dv$$

are defined for almost all triples (x, y, u). Moreover, using the inequality (13), one can deduce

$$\iiint\limits_{\mathbb{T}^3} |F_i(x,y,u)|dxdydu \lesssim 1 + \iint\limits_{\mathbb{T}^2} |f|\log^+|f|, \quad i = 1, 2. \tag{18}$$

On the other hand, from (17) it follows that

$$|S_{nn}^{(1)}(x,y,f)| \le \frac{1}{2\pi}\int\limits_{\mathbb{T}} |F_1(x,y,u)|du + \frac{1}{2\pi}\int\limits_{\mathbb{T}} |F_2(x,y,u)|du.$$

Combining this inequality with (18), we obtain

$$\iint\limits_{\mathbb{T}^2} \sup_n |S_{nn}^{(1)}(x,y,f)|dxdy \lesssim 1 + \iint\limits_{\mathbb{T}^2} |f|\log^+|f| \tag{19}$$

Similarly we can get the same bound for $S_{nn}^{(2)}(x, y, f)$, which together with (16) completes the proof of lemma. □

3. Proof of Theorems

Proof of Theorem 1. From Lemma 2 we obtain

$$|S_{nn}(x,y,f) - S_{nn}^*(x,y,f)| \le \phi(x,y), \quad n = 1, 2, \ldots,$$

where the function $\phi(x, y) \ge 0$ satisfies the bound

$$\iint\limits_{\mathbb{T}^2} \phi(x,y)dxdy \lesssim 1 + \iint\limits_{\mathbb{T}^2} |f|\log^+|f|.$$

Thus we get

$$\mathrm{BMO}\,[S_{nn}(x,y,f)] \le \mathrm{BMO}\,[S_{nn}^*(x,y,f)] + 3\phi(x,y).$$

Hence, the theorem will be proved, if we obtain BMO weak $(1,1)$ estimate for modified partial sums. We have

$$\begin{aligned}
S_{nn}^*(x,y,f) &= \frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} \frac{\cos n(t-s)\cdot f(x+t,y+s)}{4\tan(t/2)\tan(s/2)}dtds \\
&- \frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} \frac{\cos n(t+s)\cdot f(x+t,y+s)}{4\tan(t/2)\tan(s/2)}dtds \\
&= \frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} \frac{\cos nu\cdot f(x+u+v,y+v)}{4\tan((u+v)/2)\tan(v/2)}dudv \quad (u = t-s, v = s) \\
&- \frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} \frac{\cos nu\cdot f(x+u+v,y-v)}{4\tan((u+v)/2)\tan(v/2)}dudv \quad (u = t+s, v = -s) \\
&= I_n(x,y,f) - J_n(x,y,f).
\end{aligned}$$

Using a simple and an important identity

$$\frac{1}{\tan((u+v)/2)\tan(v/2)} = \frac{1}{\tan(u/2)\tan(v/2)} - \frac{1}{\tan(u/2)\tan((u+v)/2)} - 1, \tag{20}$$

we obtain

$$\begin{aligned}
&I_n(x,y,f)\\
&= \text{p.v.}\frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} \frac{\cos nu \cdot f(x+u+v, y+v)}{4\tan(u/2)\tan(v/2)} dudv\\
&- \text{p.v.}\frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} \frac{\cos nu \cdot f(x+u+v, y+v)}{4\tan(u/2)\tan((u+v)/2)} dudv\\
&- \frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} f(x+t, y+s)\, dtds\\
&= \text{p.v.}\frac{1}{2\pi}\int\limits_{\mathbb{T}} \frac{\cos nu}{2\tan(u/2)} \left(\text{p.v.}\frac{1}{\pi}\int\limits_{\mathbb{T}} \frac{f(x+u+v, y+v)}{2\tan(v/2)} dv\right) du\\
&- \text{p.v.}\frac{1}{2\pi}\int\limits_{\mathbb{T}} \frac{\cos nu}{2\tan(u/2)} \left(\text{p.v.}\frac{1}{\pi}\int\limits_{\mathbb{T}} \frac{f(x+u+v, y+v)}{2\tan((u+v)/2)} dv\right) du\\
&- \frac{1}{2\pi^2}\iint\limits_{\mathbb{T}^2} f(t,s)\, dtds\\
&= I_n^{(1)}(x,y,f) - I_n^{(2)}(x,y,f) - I^{(0)},
\end{aligned}$$

where

$$|I^{(0)}| = \frac{1}{2\pi^2}\left|\iint\limits_{\mathbb{T}^2} f(t,s)\, dtds\right| \lesssim 1 + \iint\limits_{\mathbb{T}^2} |f(x,y)|\log^+|f(x,y)|dxdy. \tag{21}$$

Observe that

$$I_n^{(1)}(x,y,f) = \frac{1}{2}\cdot U_n(x, A(\cdot, y))$$

where

$$A(x,y) = \text{p.v.}\frac{1}{\pi}\int\limits_{\mathbb{T}} \frac{f(x+v, y+v)}{2\tan(v/2)} dv.$$

Note that this function is defined for a.a. $(x,y) \in \mathbb{T}^2$ and by inequality (13) we have

$$\iint\limits_{\mathbb{T}^2} |A(x,y)|\, dxdy \lesssim 1 + \iint\limits_{\mathbb{T}^2} |f(x,y)|\log^+|f(x,y)|dxdy. \tag{22}$$

Hence, applying the Lemma 1, we conclude

$$|\{(x,y)\in\mathbb{T}^2 : \mathrm{BMO}\,[I_n^{(1)}(x,y,f)]>\lambda\}| \lesssim \frac{1}{\lambda}\left(1+\iint\limits_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|dxdy\right). \tag{23}$$

After the changing of variable $u+v\to\nu$ in the inner integral of the expression of $I_n^{(2)}(x,y,f)$ we get

$$I_n^{(2)}(x,y,f)=\text{p.v.}\frac{1}{2\pi}\int\limits_{\mathbb{T}}\frac{\cos nu}{2\tan(u/2)}\left(\text{p.v.}\frac{1}{\pi}\int\limits_{\mathbb{T}}\frac{f(x+\nu,y+\nu-u)}{2\tan(t/2)}d\nu\right)du,$$

and then analogously we can prove that

$$|\{(x,y)\in\mathbb{T}^2 : \mathrm{BMO}\,[I_n^{(2)}(x,y,f)]>\lambda\}| \lesssim \frac{1}{\lambda}\left(1+\iint\limits_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|dxdy\right). \tag{24}$$

Hence, using (21), (23) and (24), we obtain

$$|\{(x,y)\in\mathbb{T}^2 : \mathrm{BMO}\,[I_n(x,y,f)]>\lambda\}| \lesssim \frac{1}{\lambda}\left(1+\iint\limits_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|dxdy\right).$$

Using the absolutely same process we may get the analogous estimate for $J_n(x,y,f)$ and therefore for $S_{nn}^*(x,y,f)$. The theorem is proved. □

Let X be either $[0,1]$ or $\mathbb{T}^2$ and $L_M=L_M(X)$ is the Orlicz space of functions on X, generated by Young function M, i. e. M is convex continuous even function such that $M(0)=0$ and

$$\lim_{t\to 0+}\frac{M(t)}{t}=\lim_{t\to\infty}\frac{t}{M(t)}=0.$$

It is well known that L_M is a Banach space with respect to Luxemburg norm

$$\|f\|_{(M)}:=\inf\left\{\lambda:\ \lambda>0,\ \int\limits_X M\left(\frac{|f|}{\lambda}\right)\le 1\right\}<\infty.$$

We will need some basic properties of Orlicz spaces (see [13]).

1) According to a theorem from ([13], chap. 2, theorem 9.5) we have

$$\|f\|_{(M)}\le 1\Rightarrow\int\limits_X M(|f|)\le\|f\|_{(M)}, \tag{25}$$

$$\|f\|_{(M)} \geq 1 \Rightarrow \int_X M(|f|) \geq \|f\|_{(M)}. \tag{26}$$

2) From this fact we may deduce, that

$$c_1\left(1+\int_X M(|f|)\right) \leq \|f\|_{(M)} \leq c_2\left(1+\int_X M(|f|)\right) \tag{27}$$

provided $\|f\|_{(M)} = 1$.

3) From the definition of norm $\|\cdot\|_{(M)}$ immediately follows that $|f(x)| \leq |g(x)|$ implies $\|f\|_{(M)} \leq \|g\|_{(M)}$. Besides, for any measurable set E we have

$$\|\mathbb{I}_E\|_{(M)} = o(1) \text{ as } |E| \to 0 \text{ ([13], (9.23)}.$$

4) If M satisfies Δ_2-condition, that is

$$M(2t) \leq cM(t), t > t_0,$$

and $X = \mathbb{T}^2$, then the set of two variable trigonometric polynomials on $\mathbb{T}^2$ is dense in L_M ([13], §10).

5) From (25) it follows that for any sequence of functions f_n the condition $\|f_n\|_{(M)} \to 0$ implies $\int_X M(|f_n|) \to 0$.

Proof of Theorem 2. We will deal with two M-functions

$$\Phi(t) = t\log^+ t,$$
$$\Psi(t) = \exp t - 1.$$

We consider two Orlicz spaces $L_\Phi = L_\Phi(\mathbb{T}^2)$ and $L_\Psi = L_\Psi(0,1)$. Combining (27) with Theorem 1, we may obtain

$$|\{(x,y) \in \mathbb{T}^2 : \mathrm{BMO}[S_{nn}(x,y,f)] > \lambda\}| \lesssim \frac{\|f\|_{(\Phi)}}{\lambda}. \tag{28}$$

Indeed, at first we deduce the case when $\|f\|_{(\Phi)} = 1$, then, using a linearity principle, we get the inequality in the general case.

The inequality

$$\|f\|_{(\Psi)} \lesssim \|f\|_{\mathrm{BMO}} \tag{29}$$

proved in [20]. It is an immediate consequence of the John-Nirenberg theorem. Denote

$$\mathcal{B}f(x,y) = \sup_{0\leq n<\infty} \left\| \sum_{k=0}^n S_{kk}(x,y,f)\mathbb{I}_{\delta_k^n}(t) \right\|_{(\Psi)}. \tag{30}$$

Notice, that by the definition we have

$$\mathrm{BMO}[S_{nn}(f,x,y)] = \sup_{0\leq n<\infty} \left\| \sum_{k=0}^n S_{kk}(x,y,f)\mathbb{I}_{\delta_k^n}(t) \right\|_{\mathrm{BMO}}.$$

So, taking into account (28) and (29) we obtain

$$|\{(x,y)\in\mathbb{T}^2:\ \mathcal{B}f(x,y)>\lambda\}\lesssim\frac{\|f\|_{(\Phi)}}{\lambda}. \tag{31}$$

On the other hand we have

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n}&(\exp A|S_{kk}(x,y,f)-f(x,y)|-1)\\ &=\frac{1}{n+1}\sum_{k=0}^{n}\Psi(A|S_{kk}(x,y,f)-f(x,y)|)\\ &=\int_0^1\Psi\left(A\sum_{k=0}^{n}|S_{kk}(x,y,f)-f(x,y)|\mathbb{I}_{\delta_k^n}(t)\right)dt.\end{aligned}$$

Thus, according the property 5) of Orlicz spaces, to prove the theorem it is enough to prove that

$$\left\|\sum_{k=0}^{n}(S_{kk}(x,y,f)-f(x,y))\mathbb{I}_{\delta_k^n}(t)\right\|_{(\Psi)}\to 0, \tag{32}$$

almost everywhere on $\mathbb{T}^2$ as $n\to\infty$, for any $f\in L_\Phi$. It is easy to observe, that (32) holds if f is a real trigonometric polynomial in two variables. Indeed, if $P(x,y)$ is a polynomial of degree m, then we have

$$S_{kk}(x,y,P)-P(x,y)\equiv 0,\quad k\ge m.$$

Therefore, if $n\ge m$, then we get

$$\left|\sum_{k=0}^{n}(S_{kk}(x,y,P)-P(x,y))\mathbb{I}_{\delta_k^n}(t)\right|\le C\cdot\mathbb{I}_{[0,m/(n+1)]}(t),$$

where C is a constant, depending on P. Then, applying the property 3) of Orlicz spaces, we conclude that (32) holds if $f=P$. To prove the general case, we consider the set

$$\begin{aligned}G_\lambda=\{(x,y)\in\mathbb{T}^2:&\\ &\limsup_{n\to\infty}\left\|\sum_{k=0}^{n}(S_{kk}(x,y,f)-f(x,y))\mathbb{I}_{\delta_k^n}(t)\right\|_{(\Psi)}>\lambda\}.\end{aligned} \tag{33}$$

To complete the proof of theorem, it enough to prove that $|G_\lambda|=0$ if $\lambda>0$. It is easy to check that $\Phi(t)$ satisfies the Δ_2-condition. Therefore, according the property 4), we may chose a polynomial $P(x,y)$ such that $\|f-P\|_{(\Phi)}<\varepsilon$. Using the definition of (Φ)-norm, we get

$$\int\limits_{\mathbb{T}^2}\Phi\left(\left|\frac{f-P}{\varepsilon}\right|\right)<1.$$

From Chebishev's inequality, one can easily deduce

$$|\{(x,y)\in\mathbb{T}^2:\,|f(x,y)-P(x,y)|>\lambda\}|\le\frac{1}{\Phi(\lambda/\varepsilon)},\quad\lambda>0.$$

Thus, using (31) for any $\lambda>0$ we get

$$\begin{aligned}|G_\lambda|&=|\{(x,y)\in\mathbb{T}^2:\\&\limsup_{n\to\infty}\left\|\sum_{k=0}^{n}(S_{kk}(x,y,f-P)-f(x,y)+P(x,y))\mathbb{I}_{\delta_k^n}(dt)\right\|_{(\Psi)}>\lambda\}|\\&\le|\{\mathcal{B}(f-P)(x,y)+c|f(x,y)-P(x,y)|>\lambda\}|\\&\lesssim\frac{\|f-P\|_{(\Phi)}}{\lambda}+\frac{1}{\Phi(\lambda/\varepsilon)}\le\frac{\varepsilon}{\lambda}+\frac{1}{\Phi(\lambda/\varepsilon)}.\end{aligned}$$

Since $\varepsilon>0$ may be taken sufficiently small, we conclude $|G_\lambda|=0$ if $\lambda>0$. □

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U. Goginava, Department of Mathematics, Faculty of Exact and Natural Sciences, Iv. Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia

E-mail address: zazagoginava@gmail.com

L. Gogoladze, Department of Mathematics, Faculty of Exact and Natural Sciences, Iv. Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia

E-mail address: lgogoladze1@hotmail.com

G. Karagulyan, Institue of Mathematics of Armenian National Academy of Science, Bughramian Ave. 24/5, 375019, Yerevan, Armenia

E-mail address: g.karagulyan@yahoo.com