Available online at www.refaad.com
Gen.Lett. Math., 9(2) (2020), 74-79
Research Article

General Letters in Mathematics (GLM)

Journal Homepage: http://www.refaad.com/views/GLM/home.aspx
ISSN: 2519-9277 (Online) 2519-9269 (Print)

Chord length distribution and the distance between two random points in a convex body in $\mathbb{R}^n$

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Abstract

In this article for n-dimensional convex body D the relation between the chord length distribution function and the distribution function of the distance between two random points in D was found. Also the relation between their moments was found.

Keywords: Integral geometry, Chord length distribution, Random points, Convex body.
2010 MSC: 53C65, 53C60, 31A10.

1. Introduction

The integral geometric concepts such as the distribution of the chord length, the distribution of the distance between two random points in a convex body D and many others carry some information about D. In this article the relation between the chord length distribution function and the distribution function of the distance between two random points uniformly distributed in a convex body was found. By $\mathbb{R}^n$ we denote the n-dimensional Euclidean space (here we assume that $n > 2$, for case $n = 2$ see [1]). Let $D \subset \mathbb{R}^n$ be a compact convex set (convex body) with boundary ∂D. By $V_n(D)$ we denote the n-dimensional volume of D. By S^{n-1} we denote the unit sphere in $\mathbb{R}^n$ centered at the origin O and let $e_{O,\xi}$ be the hyperplane containing O and normal to $\xi \in S^{n-1}$. By $\mathbf{G}^n$ we denote the set of all lines in $\mathbb{R}^n$. We use the usual parametrization of a line $\gamma = (\omega, P)$: where $\omega \in S^{n-1}$ is the direction of γ and P is the intersection point of γ and $e_{O,\omega}$. By $[D]$ we denote the set of lines intersecting D. In $\mathbf{G}^n$ we consider the invariant measure (with respect to the group of Euclidean motions) $\mu(\cdot)$. It is known that the element $d\gamma$ of the measure, up to a constant, has the following form ([2], [4], [6])

$$d\gamma = d\omega dP, \tag{1.1}$$

here $d\omega$ and dP are elements of the Lebesgue measure on the sphere and the hyperplane, respectively. By $\chi(\gamma)$ we denote the length of a chord $D \cap \gamma$. We consider γ as a random line intersecting D with normalized measure $\dfrac{d\gamma}{\mu([D])}$. The distribution function $F_\chi(u)$ of $\chi(\gamma)$ is called the chord length distribution

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doi:10.31559/glm2020.9.2.3

Received 6 Jun 2020 : Revised : 12 Aug 2020 Accepted: 17 Oct 2020

function of D. Thus we have

$$F_\chi(u) = \frac{1}{\mu([D])}\int_{[D]} I_{\{\chi(\gamma)<u\}} d\gamma. \tag{1.2}$$

Now let $Q_1 = (x_1, ..., x_n), Q_2 = (y_1, ..., y_n)$ be random independent points uniformly distributed in D and denote by d distace between Q_1 and Q_2. By $F_d(t)$ we denote the distribution function of d. We have

$$F_d(t) = \frac{1}{V_n(D)^2}\int_{\{(Q_1,Q_2):|Q_1-Q_2|<t, Q_i\in D, i=1,2\}} dQ_1 dQ_2, \quad t\in \mathbf{R}^1, \tag{1.3}$$

here dQ_i $(i = 1, 2)$ is the usual Lebesgue's measure in $\mathbf{R}^n$.

2. Main results

Now we present the main results. The following theorem describes the relation between $F_\chi(t)$ and $F_d(t)$.

Theorem 2.1. *Let D be a convex body. We have the following relation between $F_\chi(t)$ and $F_d(t)$*

$$F_d(t) = \frac{V_{n-1}(\partial D)V_{n-2}(S^{n-2})}{(n-1)V_n(D)^2}\left(-\frac{t^{n+1}}{n+1} + \frac{(n-1)V_n(D)V_{n-1}(S^{n-1})t^n}{nV_{n-1}(\partial D)V_{n-2}(S^{n-2})} + \frac{t^n}{n}\int_0^t F_\chi(u)\,du - \frac{1}{n}\int_0^t u^n F_\chi(u)\,du\right). \tag{2.1}$$

Also in the article was found the following relation.

Theorem 2.2. *Let D be a convex body. The following relation between the k-th moment of the distance between two random points and the moments of the chord length distribution of D is valid*

$$Ed^k = \frac{V_{n-1}(\partial D)V_{n-2}(S^{n-2})E\chi^{n+k+1}}{(n-1)(n+k)(n+k+1)V_n(D)^2}. \tag{2.2}$$

3. Preliminary results

In this section we need to prove the following lemmas.

Lemma 3.1. *For the invariant measure of the lines intersecting D we have*

$$\mu([D]) = \frac{V_{n-1}(\partial D)V_{n-2}(S^{n-2})}{2(n-1)}. \tag{3.1}$$

Proof of lemma 3.1. By definition we have

$$\mu([D]) = \int_{[D]} d\gamma = \int_{[D]} d\omega dP = \frac{1}{2}\int_{S^{n-1}} d\omega \int_{D_\omega} dP = \frac{1}{2}\int_{S^{n-1}} V_{n-1}(D_\omega)\,d\omega \tag{3.2}$$

where D_ω is the orthogonal projection of D onto hyperplane $e_{O,\omega}$. In this article, we consider a convex body D with positive Gaussian curvature at every point of ∂D. For $\xi \in S^{n-1}$ we denote by $s(\xi)$ the point on ∂D the outer normal of which is ξ. It is known that (see [3], [5])

$$V_{n-1}(D_\omega) = \frac{1}{2}\int_{\partial D} |\cos\widehat{(\omega,\xi)}| ds(\xi), \tag{3.3}$$

here $ds(\xi)$ is the element of $n-1$-dimensional Lebesgue's measure on ∂D. Substituting (3.3) into (3.2) and using the Fubini's theorem we obtain

$$\mu([D]) = \frac{1}{4}\int_{S^{n-1}}\int_{\partial D} |\cos\widehat{(\omega,\xi)}| ds(\xi)\, d\omega = \frac{1}{4}\int_{\partial D}\int_{S^{n-1}} |\cos\widehat{(\omega,\xi)}| d\omega ds(\xi). \tag{3.4}$$

For any $\xi \in S^{n-1}$ we have (see [3])

$$\int_{S^{n-1}} |\cos\widehat{(\omega,\xi)}| d\omega = \frac{2V_{n-2}(S^{n-2})}{n-1} \tag{3.5}$$

thus

$$\mu([D]) = \frac{1}{4}\int_{\partial D}\int_{S^{n-1}} |\cos\widehat{(\omega,\xi)}| d\omega ds(\xi) = \frac{V_{n-2}(S^{n-2})}{2(n-1)}\int_{\partial D} ds(\xi) = \frac{V_{n-1}(\partial D)V_{n-2}(S^{n-2})}{2(n-1)}. \tag{3.6}$$

□

For a line γ intersecting a convex body D we have the following lemma.

Lemma 3.2. *Let $\chi(\gamma)$ be the length of the chord $D \cap \gamma$. We have*

$$\int_{[D]} \chi(\gamma)\, d\gamma = \frac{V_n(D)V_{n-1}(S^{n-1})}{2}. \tag{3.7}$$

Proof of lemma 3.2. By definition we have ($\gamma = (\omega, P)$)

$$\int_{[D]} \chi(\gamma)\, d\gamma = \frac{1}{2}\int_{S^{n-1}} d\omega \int_{D_\omega} \chi(\omega, P)\, dP. \tag{3.8}$$

For any $\omega \in S^{n-1}$ it is obvious that $\chi(\omega, P)\, dP$ is the element of n-dimensional volume of D, hence the integrating by dP over D_ω we get $V_n(D)$.

$$\int_{[D]} \chi(\gamma)\, d\gamma = \frac{1}{2}\int_{S^{n-1}} d\omega \int_{D_\omega} \chi(\omega, P)\, dP = \frac{V_n(D)}{2}\int_{S^{n-1}} d\omega = \frac{V_n(D)V_{n-1}(S^{n-1})}{2}. \tag{3.9}$$

□

It is known that a pair of points Q_1, Q_2 in $\mathbb{R}^n$ can be represented by the line $\gamma = (\omega, P)$ passing through the points and pair of one dimensional coordinates (t_1, t_2) (see [6]). Thus

$$(Q_1, Q_2) = (\gamma, t_1, t_2) = (\omega, P, t_1, t_2). \tag{3.10}$$

Lemma 3.3. *The Jacobian of the transform* (3.10) *is*

$$dQ_1 dQ_2 = |t_1 - t_2|^{n-1} dt_1 dt_2 d\omega dP. \tag{3.11}$$

Proof of lemma 3.3. For a fixed Q_1 we represent Q_2 by polar coordinates with respect to Q_1. It is known that

$$dQ_2 = r^{n-1} dr d\omega \tag{3.12}$$

where $r = |Q_1 - Q_2|$ and ω is the direction of $Q_2 - Q_1$. For a fixed ω the point Q_1 can be represented by P and t_1. Thus

$$dQ_1 = dt_1 dP \tag{3.13}$$

and by multiplying (3.12) and (3.13) and taking into account that $r = |t_1 - t_2|$ we get

$$dQ_1 dQ_2 = |t_1 - t_2|^{n-1} dt_1 dt_2 d\omega dP. \tag{3.14}$$

□

4. Proof of theorem 2.1

By definition of the distribution function we have

$$F_d(t) = Pr\{d < t\} = \frac{1}{V_n(D)^2} \int_{d<t} dQ_1 dQ_2. \tag{4.1}$$

Using (3.11) from lemma 3.3 we get

$$F_d(t) = \frac{1}{V_n(D)^2} \int_{[D]} \int_{|t_1 - t_2| < t} |t_1 - t_2|^{n-1} dt_1 dt_2 d\gamma. \tag{4.2}$$

We represent the integral by the form

$$\int_{[D]} \int_{|t_1 - t_2| < t} |t_1 - t_2|^{n-1} dt_1 dt_2 d\gamma = \int_{[D]} I_{\{\chi(\gamma) \geqslant t\}} J_1(t) + I_{\{\chi(\gamma) < t\}} J_1(t) \, d\gamma \tag{4.3}$$

where

$$J_1(t) = \int_{|t_1 - t_2| < t} |t_1 - t_2|^{n-1} dt_1 dt_2. \tag{4.4}$$

After calculating J_1 we obtain that

$$J_1(t) = \frac{-2t^{n+1}}{n+1} + \frac{2t^n}{n} \chi(\gamma) \tag{4.5}$$

for $\chi(\gamma) \geqslant t$ and

$$J_1(t) = \frac{2(\chi(\gamma))^{n+1}}{n(n+1)} \tag{4.6}$$

for $\chi(\gamma) < t$.

$$F_d(t) = \frac{1}{V_n(D)^2} \int_{[D]} \left(I_{\{\chi(\gamma) \geqslant t\}} \left(\frac{-2t^{n+1}}{n+1} + \frac{2t^n}{n} \chi(\gamma) \right) + I_{\{\chi(\gamma) < t\}} \left(\frac{2(\chi(\gamma))^{n+1}}{n(n+1)} \right) \right) d\gamma. \tag{4.7}$$

Now we denote by J_2, J_3, J_4 the following integrals

$$J_2(t) = \int_{[D]} I_{\{\chi(\gamma)\geqslant t\}} d\gamma, \tag{4.8}$$

$$J_3(t) = \int_{[D]} I_{\{\chi(\gamma)\geqslant t\}} \chi(\gamma)\, d\gamma, \tag{4.9}$$

$$J_4(t) = \int_{[D]} I_{\{\chi(\gamma)< t\}} (\chi(\gamma))^{n+1}\, d\gamma. \tag{4.10}$$

Easy to notice (using (3.1) from lemma 3.1) that

$$J_2(t) = \frac{V_{n-1}(\partial D)\, V_{n-2}\left(S^{n-2}\right)}{2(n-1)} \left(1 - F_\chi(t)\right). \tag{4.11}$$

To calculate J_3 we change it the following way

$$J_3(t) = \int_{[D]} I_{\{\chi(\gamma)\geqslant t\}} \chi(\gamma)\, d\gamma = \int_{[D]} \left(1 - I_{\{\chi(\gamma)<t\}}\right) \chi(\gamma)\, d\gamma =$$

$$\frac{V_n(D)\, V_{n-1}\left(S^{n-1}\right)}{2} - \int_{[D]} I_{\{\chi(\gamma)<t\}} \chi(\gamma)\, d\gamma \tag{4.12}$$

(see (3.7) from lemma 3.2) then denote

$$J_5(t) = \int_{[D]} I_{\{\chi(\gamma)<t\}} \chi(\gamma)\, d\gamma \tag{4.13}$$

and calculate it instead. First we calculate the derivative of J_5 and then intergrate from 0 to t ($J_5(0) = 0$). Using the first mean value theorem for definite integrals we have

$$\frac{d}{dt} J_5(t) = \lim_{\Delta t \to 0} \frac{J_5(t+\Delta t) - J_5(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{[D]} I_{\{t\leqslant\chi(\gamma)<t+\Delta t\}} \chi(\gamma)\, d\gamma =$$

$$\lim_{\Delta t \to 0} \frac{V_{n-1}(\partial D)\, V_{n-2}\left(S^{n-2}\right) t \left(F_\chi(t+\Delta t) - F_\chi(t)\right)}{2(n-1)\,\Delta t} =$$

$$\frac{V_{n-1}(\partial D)\, V_{n-2}\left(S^{n-2}\right)}{2(n-1)} t f_\chi(t) \tag{4.14}$$

and after integrating that we get

$$J_5(t) = \frac{V_{n-1}(\partial D)\, V_{n-2}\left(S^{n-2}\right)}{2(n-1)} \left(t F_\chi(t) - \int_0^t F_\chi(u)\, du\right). \tag{4.15}$$

For J_3 we have

$$J_3(t) = \frac{V_n(D)\, V_{n-1}\left(S^{n-1}\right)}{2} -$$

$$\frac{V_{n-1}(\partial D)\, V_{n-2}\left(S^{n-2}\right)}{2(n-1)} \left(t F_\chi(t) - \int_0^t F_\chi(u)\, du\right). \tag{4.16}$$

Using the same technique we can calculate J_4 and get that

$$J_4(t) = \frac{V_{n-1}(\partial D)\, V_{n-2}\left(S^{n-2}\right)}{2(n-1)} \left(t^{n+1} F_\chi(t) - (n+1) \int_0^t u^n F_\chi(u) du\right). \tag{4.17}$$

After substituting (4.11), (4.16), (4.17) into (4.7) finally we obtain that

$$F_d(t) = \frac{V_{n-1}(\partial D) V_{n-2}(S^{n-2})}{(n-1) V_n(D)^2} \left(-\frac{t^{n+1}}{n+1} + \frac{(n-1) V_n(D) V_{n-1}(S^{n-1}) t^n}{n V_{n-1}(\partial D) V_{n-2}(S^{n-2})} + \frac{t^n}{n} \int_0^t F_\chi(u)\, du - \frac{1}{n} \int_0^t u^n F_\chi(u)\, du \right). \tag{4.18}$$

□

Differentiating the distribution function $F_d(t)$ we get the following expression for the density function $f_d(t)$ of d

$$f_d(t) = \frac{V_{n-1}(\partial D) V_{n-2}(S^{n-2})}{(n-1) V_n(D)^2} \left(-t^n + \frac{(n-1) V_n(D) V_{n-1}(S^{n-1}) t^{n-1}}{V_{n-1}(\partial D) V_{n-2}(S^{n-2})} + t^{n-1} \int_0^t F_\chi(u)\, du \right). \tag{4.19}$$

Note that in $\mathbb{R}^2$ formula (4.19) first was found in [1]. Now we are going to prove theorem 2.2.

5. Proof of theorem 2.2

We can prove that theorem by just putting the (4.19) in moments formula

$$Ed^k = \int_{-\infty}^{\infty} t^k f_d(t)\, dt \tag{5.1}$$

but we will do that by the following way

$$Ed^k = \frac{1}{V_n(D)^2} \int_{Q_1, Q_2 \in D} |Q_1 - Q_2|^k dQ_1 dQ_2 =$$
$$\frac{1}{V_n(D)^2} \int_{[D]} \int_0^{\chi(\gamma)} \int_0^{\chi(\gamma)} |t_1 - t_2|^{n+k-1} dt_1 dt_2 d\gamma =$$
$$\frac{2}{V_n(D)^2 (n+k)(n+k+1)} \int_{[D]} (\chi(\gamma))^{n+k+1} d\gamma =$$
$$\frac{V_{n-1}(\partial D) V_{n-2}(S^{n-2}) E\chi^{n+k+1}}{(n-1)(n+k)(n+k+1) V_n(D)^2}. \tag{5.2}$$

□

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