FORMULAS FOR DERIVATIVES OF SOLUTIONS OF $\overline{\partial}$-EQUATION IN THE BALL

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Abstract. Let $f(z) = \sum_{k=1}^{n} f_k(z)d\overline{z}_k$ be a closed $(0,1)$-form in the unit ball $\mathbf{B}^n \in \mathbb{C}^n$, and let u_α be the solution of the equation $\overline{\partial}u = f$, which has the minimal norm in the weighted space $L^2[(1-|z|^2)^\alpha dv]$.
Some explicit integral formulas for the derivatives of u_α are obtained. These formulas are used for estimations of the derivatives of $u_\alpha(z)$ in C^m-norm. Similar formulas and estimates are obtained also for the derivatives of the "canonical" solution having the minimal L^2-norm on the unit sphere.

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The Cauchy-Green formula

$$u(z) = P[u](z) + T[\overline{\partial}u](z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbf{B}^n} \frac{u(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int\limits_{\mathbf{B}^n} \frac{\partial u(\zeta)}{\partial\overline{\zeta}} \frac{d\overline{\zeta}\wedge d\zeta}{\zeta - z}. \quad (1)$$

and its weighted version are well known as one of the main tools in complex analysis. In the case of the unit disc $\mathbf{B}^n$, the weighted formula is written in the form

G.A. Barsegian and H.G.W. Begehr (eds.), Topics in Analysis and its Applications, 229–233.

$$u(z) = P_\alpha[u](z) + T_\alpha[\overline{\partial}u](z) = \frac{\alpha+1}{2\pi i}\int\limits_{\mathbf{B}^n} u(\zeta)\frac{(1-|\zeta|^2)^\alpha}{(1-\overline{\zeta}z)^{\alpha+2}}d\overline{\zeta}\wedge d\zeta -$$
$$-\frac{1}{2\pi i}\int\limits_{\mathbf{B}^n}\frac{\partial u(\zeta)}{\partial\overline{\zeta}}\left(\frac{1-|\zeta|^2}{1-\overline{\zeta}z}\right)^{\alpha+1}\frac{d\overline{\zeta}\wedge d\zeta}{\zeta - z}. \quad (2)$$

The first summand of the right-hand side of this formula singles out the "analytic" part of $u(z)$. In the complex one-dimensional case, this feature of the Cauchy-Green formula permits to find some explicit solutions of the $\overline{\partial}$-equation

$$\frac{\partial u(z)}{\partial \overline{z}} = f(z), \quad (3)$$

where f is a continuous function in $\overline{\mathbf{B}^n}$. Namely, we mean the functions $u_0(z) = T[f(\zeta)d\overline{\zeta}](z)$ and $u_\alpha(z) = T_\alpha[f(\zeta)d\overline{\zeta}](z)$.

One has to note that in case of holomorphic functions formula (2) was first given by M.M.Djrbashian [2, 3].

In several complex variables the natural similarity (3) is

$$\overline{\partial}u = f, \quad (4)$$

where f is a $\overline{\partial}$-closed $(0,1)$ form continuous in $\mathbf{B}^n$. As in the one-dimensional case, the Cauchy-Green formula is used for finding integral representations of solutions of (4).

The integral representation method developed by G.M.Henkin (see, eg. [1]) and others permits to find explicit expressions for solutions of the $\overline{\partial}$-equation in different types of domains. However, we shall deal with (4) in the unit ball $\mathbf{B}^n$ of the space $\mathbb{C}^n$.

The well-known multidimensional similarities of formulas (1) and (2) were found in [4]:

$$u(z) = P[u](z) + T[\overline{\partial}u](z) = \int\limits_{\partial\mathbf{B}^n} u(\zeta)| \wedge C_0(\zeta,z) - \int\limits_{\mathbf{B}^n}\overline{\partial}u(\zeta)\wedge C_0(\zeta,z), \quad (5)$$

$$u(z) = P_\alpha[u](z) + T_\alpha[\overline{\partial}u](z) =$$
$$= \frac{1}{nB(n,\alpha)}\int\limits_{\mathbf{B}^n} u(\zeta)\frac{(1-|\zeta|^2)^\alpha}{(1-\langle z,\zeta\rangle)^{n+1+\alpha}}d\lambda(\zeta) - \int\limits_{\mathbf{B}^n}\overline{\partial}u(\zeta)\wedge C_\alpha(\zeta,z). \quad (6)$$

where

$$C_0(\zeta,z)=c_n\frac{(1-\langle\zeta,z\rangle)^{n-1}}{D^n(\zeta,z)}\left[\sum_{i=1}^{n}(-1)^{i-1}(\overline{\zeta}_i-\overline{z}_i)\underset{j\neq i}{\wedge}d\overline{\zeta}_j\right]\overset{n}{\underset{k=1}{\wedge}}d\zeta_k,$$

$$D(\zeta,z)=|1-\langle\zeta,z\rangle|^2-(1-|\zeta|^2)(1-|z|^2),\qquad c_n=(-1)^{\frac{n(n-1)}{2}}\frac{(n-1)!}{(2\pi i)^n},$$

$d\lambda(\zeta)$ is the normed element of the volume in the ball,

$$C_\alpha(\zeta,z)=\Psi_\alpha(\zeta,z)C_0(\zeta,z)$$

and

$$\Psi_\alpha(\zeta,z)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)}\left(\frac{1-|\zeta|^2}{1-\langle z,\zeta\rangle}\right)^{\alpha+1}\times$$
$$\times\left[\sum_{p=0}^{n-1}\binom{n-1}{p}\frac{(-1)^p}{\alpha+1+p}\left(\frac{(1-|\zeta|^2)(1-|z|^2)}{1-\langle z,\zeta\rangle}\right)^p\right].$$

One more remarkable property of formulas (5) and (6) is that the operators $P[u]$ and $P_\alpha[u]$ are the orthogonal projectors which map $L^2(d\sigma)$ and $L^2[(1-|z|^2)^\alpha d\lambda(z)]$ onto their subspaces of holomorphic functions. Therefore, the solutions of $\overline{\partial}$-equation (4), which are given by $u=T[f]$ and $u_\alpha=T_\alpha[f]$, have minimal norms in $L^2(d\sigma)$ and $L^2[(1-|z|^2)^\alpha d\lambda(z)]$ correspondingly.

In applications, it is significant to have more than simply a solution of the $\overline{\partial}$-equation. It is necessary to have a solution which is estimated in some norm, and the minimal solutions are of this type as they are written by an explicit formula.

We are aimed to obtain some explicit formulas for derivatives, using which we come to some estimates containing derivatives.

Below we assume that:

$C^m(\overline{\mathbf{B}^n})$ is the space of all functions $u(z)$ which are m times continuously differentiable in $\overline{\mathbf{B}^n}$;

$\|u\|_m$ is the norm in $C^m(\overline{\mathbf{B}^n})$;

$C^m_{(0,1)}(\overline{\mathbf{B}^n})$ is the space of all $(0,1)$ forms $f=\sum_{k=1}^n f_k(z)d\overline{z}_k$, the coefficients of which belong to $C^m(\overline{\mathbf{B}^n})$;

$\|f\|_m=\sum_{k=1}^n\|f_k\|_m$.

Further, by $L_j^{s_j}$ we denote the differential operator of the order $r_1+\cdots+r_j-s_j-1$, $1\le j\le n$, $0\le s_j\le r_j-1$:

$$L_j^{s_j}=\left(n+\sum_{i=1}^n z_i\frac{\partial}{\partial z_i}\right)\cdots\left(n+r_1+\cdots+r_j-s_j-2+\sum_{i=1}^n z_i\frac{\partial}{\partial z_i}\right)$$

here we assume that $L_j^{s_j}$ is the identical operator if for some j we have $s_j = r_1 + \cdots + r_j - 1$ (i.e. $r_1 = \cdots = r_{j-1} = 0$).

The main result of this work the following

Theorem 1. *Let* $u \in C^m(\overline{\mathbf{B}^n})$, $f \in C^m_{(0,1)}(\overline{\mathbf{B}^n})$ *and* $r_1 + \cdots + r_n \leq m$. *Then the following formula is true for derivatives of the minimal solutions of the* $\overline{\partial}$ *-equation* $u_0 = T[f]$*:*

$$D_1^{r_1} \cdots D_n^{r_n} T[f](z) = \\ = \sum_{j=1}^{n} \sum_{s_j=0}^{r_j-1} L_j^{s_j} \left\{ \overline{z}_1^{r_1} \cdots \overline{z}_{j-1}^{r_{j-1}} \overline{z}_j^{r_j - s_j} \left\langle D_j^{s_j} D_{j+1}^{r_{j+1}} \cdots D_n^{r_n} f(z), z \right\rangle \right\} - \\ - \sum_{j=1}^{n} \sum_{s_j=0}^{r_j-1} L_j^{s_j} \left\{ T\left[\overline{\partial} \left(\overline{\zeta}_1^{r_1} \cdots \overline{\zeta}_{j-1}^{r_{j-1}} \overline{\zeta}_j^{r_j - s_j} \left\langle D_j^{s_j} D_{j+1}^{r_{j+1}} \cdots D_n^{r_n} f(\zeta), \zeta \right\rangle \right) \right](z) \right\} + \\ + T\left[D_1^{r_1} \cdots D_n^{r_n} f(\zeta) \right](z). \quad (7)$$

Note that the multipliers of the form $\overline{z}_1^{r_1} \cdots \overline{z}_{j-1}^{r_{j-1}} \overline{z}_j^{r_j - s_j}$ arose in this formula by a purely technical reasons and they are not significant in further evaluation.

A similar formula is true for the derivatives of the solutions $u_\alpha = T_\alpha[f]$. We omit this formula as it is more cumbersome.

One can observe that formula (7) expresses the derivatives of $T[f](z)$ of the order $r_1 + \cdots + r_n$ by the derivatives of functions of the type $T[\overline{\partial}(\ldots)](z)$, which are of lower orders. This permits to use some inductive argument which leads to different estimates for $D_1^{r_1} \cdots D_n^{r_n} T[f](z)$. Particularly, the following theorem is true.

Theorem 2. *If* $f \in C^m_{(0,1)}(\overline{\mathbf{B}^n})$, *then the following estimates are true for the minimal solutions of the* $\overline{\partial}$*-equation, i.e. for* $u = T[f]$ *and* $u_\alpha = T_\alpha[f]$*:*

$$\|T[f]\|_m \leq \gamma \|f\|_m, \quad \|T_\alpha[f]\|_m \leq \gamma_\alpha \|f\|_m.$$

Here γ and γ_α are some constants which are independent of f. In other words, the minimal solution operators $T\colon C^m_{(0,1)}(\overline{\mathbf{B}^n}) \to C^m(\overline{\mathbf{B}^n})$ and $T_\alpha\colon C^m_{(0,1)}(\overline{\mathbf{B}^n}) \to C^m(\overline{\mathbf{B}^n})$ are bounded.

One has to mention the works [5] and [6] containing some estimates for the derivatives of the solutions of the $\overline{\partial}$-equation in the polydisc.

References

[1] Henkin, G. M. and Leiterer, J. (1984) Theory of functions on complex manifolds, *Akademie-Verlag*, Berlin.

[2] Djrbashian, M. M. (1945) On canonical representation of functions meromorphic in the unit disc [in Russian], *DAN of Armenia*, **Vol. 3, no. 1**, pp. 3–9

[3] Djrbashian, M. M. (1948) On the representability problem of analytic functions [in Russian], *Soobsch. Inst. Math. and Mech. AN Armenii*, **Vol. 2**, pp. 3–40

[4] Charpentier, P. (1980) Formules explecites pour les solutions minimales de l'equation $\overline{\partial} u = f$ dans la boule et dans le polydisque de $\mathbb{C}^n$, *Ann. Inst. Fourier*, **Vol. 30, no. 4**, pp. 121–154.

[5] Landucci, M. (1977) Uniform bounds on derivatives for the $\overline{\partial}$- problem in the polydisc, *Proc. Symp. Pure Math.,* **Vol. 30**, pp. 177–180.

[6] Petrosyan, A. I. (1991) The estimate in C^m-norm of the minimal solutions of $\overline{\partial}$-equation in polydisc [in Russian], *Izv. NAN Armenii*, **Vol. 26, no. 2,** pp. 99–107.