L_2-ESTIMATES FOR CONVERGENCE RATE OF POLYNOMIAL-PERIODIC APPROXIMATIONS BY TRANSLATES

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Izvestiya Natsionalnoi Akademii Nauk Armenii. Matematika,
Vol. 36, No. 3, 2001

The paper considers the problem of approximation by translates of a fixed periodic function in the framework of L_2-convergence. Exact formulae for the principal terms of asymptotic expansions of error are obtained. The paper proposes fast algorithms which are especially efficient while approximating by even order splines and odd order "modified splines".

INTRODUCTION

The problem of approximation by translates of a fixed periodic function has been studied in numerous papers (see, for instance, [1] – [3]). In [4], [5] an approach for approximations of general form was developed, based on minimization of certain functionals. In particular, estimates for periodic biorthogonal expansions and interpolations by translates on a finite interval were obtained.

In the case, where the approximand function defined on a finite interval has no smooth periodic continuation, the problem of acceleration of the rate of convergence arises. For trigonometric approximation (that corresponds to translates of Dirichlet kernel) the acceleration can be reached by a method due to A. N. Krylov, based on application of Bernoulli polynomials (see [6] – [8] and section 1.3).

In the present paper we consider L_2–convergence of polynomial-periodic approximations of

The research described in this publication was made possible in part by AWARD # CS 037-01(BGP 7420) of the National Foundation for Sience and Advanced Technologies (NFSAT) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

sufficiently smooth on [-1,1] functions by translates of fixed periodic functions and obtain an exact formula for the principal term of the corresponding asymptotic expansions. The suggested approach is particular efficient for approximation by translates of even order splines and odd order "modified splines".

§1. BASIC FORMULAE

Our study is based on [4] – [8]. In this section we list some results from these papers we will need below.

1.1. Results from [4] and [5]. Consider the orthonormal system $\left\{\frac{1}{\sqrt{2}}e^{i\pi nx}\right\}_{n=-\infty}^{\infty}$ in the space $L_2(-1,1)$ with norm $||\cdot||$. Let Z be the set of integers, and the sequence $\{\theta_n\}$, $\theta_n \in \mathbf{C}$ be such that the series

$$\omega_n^2 = \sum_{s\in Z} |\theta_{n+s(2N+1)}|^2 \neq 0, \quad N \geq 1, \quad n = 0, \pm 1, \ldots, \pm N$$

converge. Then the finite system

$$\{\varphi_n\} = \{\varphi_n^N\}, \quad \varphi_n(x) = \frac{1}{\sqrt{2}\omega_n} \sum_{s\in Z} \theta_{n+s(2N+1)} e^{i\pi(n+s(2N+1))x}, \quad n = 0, \pm 1, \ldots, \pm N \tag{1}$$

is orthonormal in $L_2(-1,1)$. Consider the corresponding orthogonal expansion of a function $f \in L_2(-1,1)$:

$$S_N(f) = \sum_{n=-N}^{N} (f, \varphi_n)\varphi_n. \tag{2}$$

Lemma 1. Let $f \in L_2(-1,1)$ and $f_n = \frac{1}{2}(f, e^{i\pi nx})$ be its Fourier coefficients. The following estimate holds

$$||f - S_N(f)|| \leq 4 \left(\sum_{|n|>N} |f_n|^2 \right)^{1/2} + 2 \left(\sum_{n=-N}^{N} |f_n|^2 \left(1 - \frac{|\theta_n|^2}{\omega_n^2}\right)\right)^{1/2}. \tag{3}$$

The next result is an immediate consequence of (3).

Theorem A. The condition

$$\lim_{N\to\infty} \frac{|\theta_n|^2}{\omega_n^2} = 1, \quad n = const$$

is necessary and sufficient for the $||\cdot||$-convergence $S_N(f) \to f$ $(N \to \infty)$ for any $f \in L_2(-1,1)$.

The estimate (3) characterizes the rate of convergence $S_N(f) \to f$ in terms of Fourier coefficients $\{f_n\}$. The expansion (2) can be represented by the translates of the periodic function

$$b(x) = \sum_{n\in Z} \theta_n e^{i\pi nx}.$$

The following formulae are effectively realized by computer integration and the Fourier fast transformation (FFT):

$$\varphi_n(x) = \frac{1}{\omega_n(2N+1)} \sum_{s=-N}^{N} e^{\frac{2i\pi ns}{2N+1}} b\left(x - \frac{2s}{2N+1}\right),$$

$$S_N(f) = \frac{1}{(2N+1)^2} \sum_{s=-N}^{N} c_s b\left(x - \frac{2s}{2N+1}\right),$$

where

$$c_s = \sum_{r=-N}^{N} \mu_{s-r}\widetilde{f}_r, \quad \mu_k = \sum_{n=-N}^{N} e^{\frac{2i\pi nk}{2N+1}}/\omega_n^2, \quad \widetilde{f}_s = \int_{-1}^{1} f(t)\overline{b\left(t - \frac{2s}{2N+1}\right)}\, dt.$$

Now let the sequence $\{\theta_n\} = \left\{\theta\left(\frac{n}{2N+1}\right)\right\}$ be represented by Fourier transform

$$\theta(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\rho(x)\, dx. \tag{4}$$

Assuming that $\rho(x) \in L_2 \cap C_{loc}$ and $\sum_k |\rho((2N+1)x + 2k)| < \infty$, the function $b(x)$ can be represented by means of translations of $\rho(x)$:

$$b(x) = (N + 1/2) \sum_{r\in Z} \rho((2N+1)x - 2r).$$

1.2. Bernoulli polynomials and approximation of functions of one variable ([6] – [8]).

We put

$$A_k = f^{(k)}(1) - f^{(k)}(-1), \quad f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}\, dx.$$

The following result is well known

Lemma 2. For any $f(x) \in C^{q+1}[-1,1]$, $q \geq -1$,

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q} \frac{A_k}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^{q+1}} \int_{-1}^{1} f^{(q+1)}(t)e^{-i\pi nt}\, dt, \quad n \neq 0. \tag{5}$$

Using Lemma 2, the function f can be represented in the form

$$f(x) = \sum_{k=0}^{q} A_k B_k(x) + F(x), \tag{6}$$

where $F(x)$ is some 2-periodic and q times continuously differentiable function. The Bernoulli polynomials $B_k(x)$ have Fourier coefficients

$$B_{k,n} = \begin{cases} \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & \text{for } n \neq 0 \\ 0, & \text{for } n = 0, \end{cases}$$

and are determined by the recurrent relation

$$B_0(x) = \frac{x}{2}, \quad B_k(x) = \int B_{k-1}(x)\, dx, \quad x \in [-1, 1], \quad k = 1, 2, \ldots,$$

where the integration constant is determined from the condition

$$\int_{-1}^{1} B_k(t)\, dt = 0, \quad k = 1, 2, \cdots.$$

It follows from (6), that the Fourier coefficients of $F(x)$ are

$$F_n = f_n - \sum_{k=0}^{q} A_k B_{k,n}, \quad n = 0, \pm 1, \cdots, \pm N. \tag{7}$$

The following formula is called *polynomial-periodic approximation*:

$$f(x) \simeq f_N(x) = \sum_{n=-N}^{N} F_n e^{i\pi n x} + \sum_{k=0}^{q} A_k B_k(x). \tag{8}$$

If $f \in C^{q+2}[-1, 1]$, then $F_n = O(n^{-q-2})$ as $n \to \infty$. Hence, we can expect faster convergence of f_N to f as $N \to \infty$. To determine the numbers A_k $(k = 0, \cdots, q)$ we use (7). Replacing (7) by the following system with nonsingular Vandermonde matrix

$$f_{n_s} = \sum_{k=0}^{q} \widetilde{A}_k B_{k,n_s}, \quad s = 1, \ldots, q+1, \quad n_i \neq n_j \quad (i \neq j), \tag{9}$$

$$const\, N \leq |n_s| \leq N, \quad N \to \infty,$$

we get an error of order $O(N^{-q-2})$, $N \to \infty$. The solution of system (9) was studied in [7], [8], where it was shown, that the error $|A_k - \widetilde{A}_k|$ is of order $O(N^{-q+k-1})$ as $N \to \infty$.

1.3. Polynomial-periodic approximation. As applied to expansions by translates of certain function, this method can be described as follows. Let $f \in C^{q+1}[-1,1]$. Consider the following approximating formula, that corresponds to the expansion (6), and also *called polynomial-periodic approximation*

$$S_N^q(f) = S_N\left(f(x) - \sum_{k=0}^{q} A_k B_k(x)\right) + \sum_{k=0}^{q} A_k B_k(x). \tag{10}$$

Note that for the following choice

$$\theta_n = \begin{cases} 1, & \text{for } |n| \le N, \\ 0, & \text{for } |n| > N, \end{cases} \tag{11}$$

(10) coincides with (8). In this case $\{\varphi_n(x)\}_{n=-N}^{N}$ coincides with the truncated Fourier system $\left\{\frac{1}{\sqrt{2}}e^{i\pi nx}\right\}_{n=-N}^{N}$ (see Example 1 below).

We are interested in asymptotic L_2–estimates for the error $S_N^q(f) - f$ as $N \to \infty$. However, our analysis does not touch the question of approximate determination of the jumps $\{A_k\}$ by means of (f, φ_n) in the general case (see 1.1).

§2. ASYMPTOTIC L_2–ESTIMATES

Let $f(x)$ be a function defined on $[a,b]$, and let $\omega(\delta, f)$ be its modulus of continuity

$$\omega(\delta, f) = sup|f(x_1) - f(x_2)|, \quad x_1, x_2 \in [a,b], \quad |x_1 - x_2| \le \delta.$$

Let $\theta_n = \theta\left(\frac{n}{2N+1}\right)$. It is clear, that $\omega_n = \omega\left(\frac{n}{2N+1}\right)$. Assuming that $\omega(x) \ne 0$ for $x \in [-1/2, 1/2]$, we put

$$\Phi(x) = \sum_{s \in Z} \frac{(-1)^s \overline{\theta(x+s)}}{(x+s)^{q+2}}, \quad \widetilde{\Phi}(x) = \sum_{s \in Z}{}' \frac{(-1)^s \overline{\theta(x+s)}}{(x+s)^{q+2}}, \quad \Psi(x) = 1 - \frac{|\theta(x)|^2}{\omega^2(x)}.$$

Here and below $\sum_s'$ stands for the sum, that contains no term corresponding to $s = 0$. Observe that $\Psi(x) \ge 0$. From (1) and (2)

$$||f - S_N(f)||^2 = 2\sum_{n=-N}^{N} \sum_{r \in Z} \left| f_{n+r(2N+1)} - \frac{\theta_{n+r(2N+1)}}{\omega_n^2} \sum_{s \in Z} \overline{\theta_{n+s(2N+1)}} f_{n+s(2N+1)} \right|^2 =$$

$$= 2\sum_{n=-N}^{N} \sum_{r \in Z} |f_{n+r(2N+1)}|^2 - 2\sum_{n=-N}^{N} \frac{1}{\omega_n^2} \left| \sum_{r \in Z} \overline{\theta_{n+r(2N+1)}} f_{n+r(2N+1)} \right|^2. \tag{12}$$

After some easy calculation we obtain

$$||f - S_N(f)||^2 = 2\sum_{n=-N}^{N}\Psi\left(\frac{n}{2N+1}\right)|f_n|^2 - 2\sum_{n=-N}^{N}\frac{1}{\omega_n^2}\left|{\sum_{r\in Z}}'\overline{\theta_{n+r(2N+1)}}f_{n+r(2N+1)}\right|^2 +$$

$$+2\sum_{|n|>N}|f_n|^2 - 4\,\mathrm{Re}\,\left(\sum_{n=-N}^{N}\left(\frac{\theta_n\overline{f_n}}{\omega_n^2}{\sum_{r\in Z}}'\overline{\theta_{n+r(2N+1)}}f_{n+r(2N+1)}\right)\right). \tag{13}$$

Theorem 1. The following three conditions

1°. *The function* $\theta(x)$ *is piecewise continuous, and the series*

$$\sum_{s\in Z}|\theta(x+s)|^2 \neq 0$$

converge uniformly on $[-1/2, 1/2]$,

2°. *There exists a monotone on* $(-1/2, 0)$ *as well as on* $(0, 1/2)$ *and integrable on* $(-1/2, 1/2)$ *function* $\tau(x) \geq 0$, *such that* $\Psi(x)x^{-2q-4} \leq \tau(x)$,

3°. $f \in C^{q+2}[-1, 1]$, $\quad q \geq -1$,

imply

$$\lim_{N\to\infty}(2N+1)^{2q+3}||f - S_N^q(f)||^2 = \frac{|A_{q+1}|^2}{\pi^{2q+4}}\left(\frac{2^{2q+3}}{2q+3} + \frac{1}{2}\int_{-1/2}^{1/2}\left(\frac{1}{x^{2q+4}} - \frac{|\Phi(x)|^2}{\omega^2(x)}\right)\,dx\right). \tag{14}$$

Proof: We use the inequality

$$||f - S_N^q(f)|| = ||F - S_N(F)|| \leq ||F_1 - S_N(F_1)|| + ||F_2 - S_N(F_2)||, \tag{15}$$

where (see Lemma 2)

$$F(x) = F_1(x) + F_2(x), \quad F_1(x) = {\sum_{n\in Z}}' F_{1,n}e^{i\pi nx}, \quad F_{1,n} = \frac{(-1)^{n+1}}{2}\frac{A_{q+1}}{(i\pi n)^{q+2}}, \quad n \neq 0,$$

$$F_2(x) = {\sum_{n\in Z}}' F_{2,n}e^{i\pi nx}, \quad F_{2,n} = \frac{1}{2(i\pi n)^{q+2}}\int_{-1}^{1} f^{(q+2)}(x)e^{-i\pi nx}\,dx, \quad n \neq 0.$$

It is well-known (see, for instance, [10]), that $|f_n| \leq const\,\omega(|n|^{-1}, f)$, hence

$$|F_{2,n}| \leq const\,\omega(|n|^{-1}, f^{(q+2)})|n|^{-q-2}. \tag{16}$$

According to Lemma 1 ($t_n = \frac{n}{2N+1}$),

$$||F_2 - S_N(F_2)|| \le 4 \left(\sum_{|n|>N} |F_{2,n}|^2 \right)^{1/2} + 2 \left(\sum_{n=-N}^{N}{}' |F_{2,n}|^2 \Psi(t_n) \right)^{1/2} \tag{17}$$

The first term on the right-hand side of (17) we estimate by means of (16):

$$\sum_{|n|>N} |F_{2,n}|^2 \le const \sum_{n=N+1}^{\infty} \frac{\omega^2(n^{-1}, f^{(q+2)})}{n^{2q+4}} \le \frac{const\, \omega^2(N^{-1}, f^{(q+2)})}{(2N+1)^{2q+3}}. \tag{18}$$

For the second term we have ($[x]$ stands for the integer part of x)

$$\sum_{n=-N}^{N}{}' |F_{2,n}|^2 \Psi(t_n) \le \frac{const}{(2N+1)^{2q+4}} \sum_{n=-[\sqrt{N}]}^{[\sqrt{N}]}{}' \frac{\Psi(t_n)}{t_n^{2q+4}} \omega^2(|n|^{-1}, f^{(q+2)}) +$$

$$+ \frac{const}{(2N+1)^{2q+4}} \sum_{[\sqrt{N}]<|n|\le N} \frac{\Psi(t_n)}{t_n^{2q+4}} \omega^2(|n|^{-1}, f^{(q+2)}). \tag{19}$$

Clearly, the assumptions of Theorem 1 imply integrability of $\Psi(x)x^{-2q-4}$ on $[-1/2, 1/2]$ and the existence of the limit (see, for instance, [11])

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}{}' \Psi(t_n) t_n^{-2q-4} = \int_{-1/2}^{1/2} \Psi(x) x^{-2q-4}\, dx.$$

Therefore, the second summand in (19) can be estimated by

$$const \frac{\omega^2([\sqrt{N}]^{-1}, f^{(q+2)})}{(2N+1)^{2q+3}} \int_{-1/2}^{1/2} \Psi(x) x^{-2q-4}\, dx.$$

Likewise, the first summand is majorized by

$$\frac{const}{(2N+1)^{2q+3}} \int_{-\frac{[\sqrt{N}]}{2N+1}}^{\frac{[\sqrt{N}]}{2N+1}} \tau(x)\, dx,$$

which tends to zero as $N \to \infty$, because $\tau(x)$ is integrable. Thus,

$$\lim_{N\to\infty} (2N+1)^{2q+3} ||F_2 - S_N(F_2)||^2 = 0,$$

and therefore,

$$\lim_{N\to\infty}(2N+1)^{2q+3}||f-S_N^q(f)||^2 = \lim_{N\to\infty}(2N+1)^{2q+3}||F_1-S_N(F_1)||^2.$$

On the other hand, by (12)

$$(2N+1)^{2q+3}||F_1-S_N(F_1)||^2 = \frac{|A_{q+1}|^2}{2\pi^{2q+4}}\frac{1}{(2N+1)}\left[\sum_{n=-N}^{N}{}'\sum_{r\neq 0}(t_n+r)^{-2q-4}+\right.$$

$$\left.+\sum_{n=-N}^{N}{}'\left(t_n^{-2q-4}-\frac{|\Phi(t_n)|^2}{\omega^2(t_n)}\right)\right]+O((2N+1)^{-1}),\quad N\to\infty. \tag{20}$$

Now we show that

$$x^{-2q-4}-\frac{|\Phi(x)|^2}{\omega^2(x)}\in L_1(-1/2,1/2),$$

and that for $N\to\infty$ the sums in (20) can be replaced by the corresponding integrals. We have

$$x^{-2q-4}-\frac{|\Phi(x)|^2}{\omega^2(x)} = \frac{\Psi(x)}{x^{2q+4}}-\frac{|\widetilde{\Phi}(x)|^2}{\omega^2(x)}-2\ \mathrm{Re}\left(\frac{\theta(x)\overline{\widetilde{\Phi}(x)}}{x^{q+2}\omega^2(x)}\right). \tag{21}$$

By assumptions 1° and 2° the functions $|\widetilde{\Phi}(x)|^2\omega^{-2}(x)$ and $\Psi(x)x^{-2q-4}$ are integrable on the segment $[-1/2,1/2]$. The third term on the right-hand side of (21) can be majorized by

$$const\frac{|\widetilde{\Phi}(x)x^{-q-2}|}{\omega^2(x)}\leq const(\Psi(x)x^{-2q-4})^{1/2}\leq const(\tau(x))^{1/2}.$$

To complete the proof, it remains to observe, that assumption 2° implies integrability of $\sqrt{\tau(x)}$ on $[-1/2,1/2]$. Theorem 1 is proved.

Given α $(0<\alpha\leq 1)$ we denote by Λ_α the class of functions f satisfying $\omega(\delta,f)\leq C\delta^\alpha$, where C is a constant not depending on δ. It is well-known (see, for instance, [10]), that $|f_n|\leq const|n|^{-\alpha}$.

Theorem 2. The following four conditions:

1°. *The function $\theta(x)$ is piecewise continuous and the series $\sum_{s\in Z}|\theta(x+s)|^2\neq 0$ converges uniformly on $[-1/2,1/2]$,*

2°. $\lim_{x\to 0}x^{-2q-3}\Psi(x)=B$,

3°. *There exists a monotone on each of the intervals* $(-1/2, 0)$ *and* $(0, 1/2)$ *and integrable on* $(-1/2, 1/2)$ *function* $\Omega(x) \geq 0$, *such that*

$$|(\Psi(x)x^{-2q-3} - B)x^{-1}| \leq \Omega(x), \tag{22}$$

$$\lim_{\varepsilon \to +0} (\ln\,\varepsilon)^{-1} \int_{\varepsilon < |x| \leq 1/2} \Omega(x)dx = 0, \tag{23}$$

4°. $f^{(q+2)} \in \Lambda_\alpha$, $q \geq -1$

imply

$$\lim_{N\to\infty} \frac{(2N+1)^{2q+3}}{\ln(2N+1)} ||f - S_N^q(f)||^2 = \frac{|A_{q+1}|^2}{\pi^{2q+4}} B. \tag{24}$$

Proof: We use the inequality (15), where the second term on the right-hand side we estimate by means of Lemma 1. The first term on the right-hand side of (3) with $f_n = F_{2,n}$ can be estimated as done in the proof of Theorem 1 (see (18)). The second term in (3) with $f_n = F_{2,n}$ we estimate as follows

$$\sum_{n=-N}^{N}{}' |F_{2,n}|^2 \Psi(t_n) \leq \frac{const}{(2N+1)^{2q+3}} \sum_{n=-N}^{N}{}' |n|^{2q+3} |F_{2,n}|^2 \leq$$

$$\leq \frac{const}{(2N+1)^{2q+3}} \sum_{n=-N}^{N}{}' \frac{\omega^2(|n|^{-1}, f^{(q+2)})}{|n|} \leq \frac{const}{(2N+1)^{2q+3}} \sum_{n=-\infty}^{\infty}{}' \frac{1}{|n|^{1+2\alpha}}.$$

Thus,

$$\lim_{N\to\infty} \frac{(2N+1)^{2q+3}}{\ln(2N+1)} ||F_2 - S_N(F_2)||^2 = 0,$$

and therefore,

$$\lim_{N\to\infty} \frac{(2N+1)^{2q+3}}{\ln(2N+1)} ||f - S_N^q(f)||^2 = \lim_{N\to\infty} \frac{(2N+1)^{2q+3}}{\ln(2N+1)} ||F_1 - S_N(F_1)||^2.$$

According to (13),

$$||F_1 - S_N(F_1)||^2 = 2 \sum_{n=-N}^{N}{}' \Psi(t_n) |F_{1,n}|^2 - 2 \sum_{n=-N}^{N} \frac{1}{\omega_n^2} \left| \sum_{r \in Z}{}' \overline{\theta_{n+r(2N+1)}} F_{1,n+r(2N+1)} \right|^2 +$$

$$+2 \sum_{|n|>N} |F_{1,n}|^2 - 4\,\mathrm{Re} \left(\sum_{n=-N}^{N}{}' \left(\frac{\theta_n \overline{F_{1,n}}}{\omega_n^2} \sum_{r \in Z}{}' \overline{\theta_{n+r(2N+1)}} F_{1,n+r(2N+1)} \right) \right). \tag{25}$$

The second summand in (25) can be majorized by the third term, since

$$const \sum_{n=-N}^{N}{}' \Psi(t_n) \sum_{r\in Z}{}' |F_{1,n+r(2N+1)}|^2 \leq const \sum_{|n|>N} |F_{1,n}|^2.$$

It is easy to show, that the third summand in (25) is of order $O((2N+1)^{-2q-3})$, as $N \to \infty$. The majorant of the fourth summand in (25) can be estimated by the expression

$$const \sum_{n=-N}^{N}{}' |F_{1,n}|\Psi^{1/2}(t_n) \left(\sum_{r\in Z0}{}' |F_{1,n+r(2N+1)}|^2 \right)^{1/2} \leq$$

$$\leq \frac{const}{(2N+1)^{q+1.5}} \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \left(\sum_{r\in Z}{}' \frac{1}{(n+r(2N+1))^{2q+4}} \right)^{1/2} \leq \frac{const}{(2N+1)^{2q+3.5}} \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \leq \frac{const}{(2N+1)^{2q+3}}.$$

Thus, we have

$$\lim_{N\to\infty} \frac{(2N+1)^{2q+3}}{\ln(2N+1)} ||f - S_N^q(f)||^2 = 2 \lim_{N\to\infty} \frac{(2N+1)^{2q+3}}{\ln(2N+1)} \sum_{n=-N}^{N}{}' |F_{1,n}|^2 \Psi(t_n) =$$

$$= \frac{|A_{q+1}|^2}{\pi^{2q+4}} \lim_{N\to\infty} \frac{1}{2\ln(2N+1)} \sum_{n=-N}^{N}{}' \frac{\Psi(t_n)}{t_n^{2q+3}} \frac{1}{n}. \tag{26}$$

To complete the proof, it remains to show, that the last limit on the right-hand side of (26) equals B. We have

$$\lim_{N\to\infty} \left(\frac{1}{2\ln(2N+1)} \sum_{n=-N}^{N}{}' \frac{\Psi(t_n)}{t_n^{2q+3}} \frac{1}{n} - B \right) = \lim_{N\to\infty} \frac{1}{2\ln(2N+1)} \sum_{n=1}^{N} \left(\frac{\Psi(t_n)}{t_n^{2q+3}} - B \right) \frac{1}{n} +$$

$$+ \lim_{N\to\infty} \frac{1}{2\ln(2N+1)} \sum_{n=1}^{N} \left(\frac{\Psi(-t_n)}{t_n^{2q+3}} - B \right) \frac{1}{n}. \tag{27}$$

First we show, that the first limit on the right-hand side of (27) is equal to zero. Setting $\lambda(x) = \Psi(x)x^{-2q-3} - B$, we obtain

$$\frac{1}{2\ln(2N+1)} \sum_{n=1}^{N} \frac{\lambda(t_n)}{n} = \frac{1}{2\ln(2N+1)} \sum_{n=1}^{[\sqrt{N}]} \frac{\lambda(t_n)}{n} + \frac{1}{2\ln(2N+1)} \sum_{n=[\sqrt{N}]+1}^{N} \frac{\lambda(t_n)}{n} \leq$$

$$\leq const\,\omega\left(\frac{[\sqrt{N}]}{2N+1},\lambda\right)+\frac{1}{2(2N+1)\ln(2N+1)}\sum_{n=[\sqrt{N}]+1}^{N}\Omega(t_n)\leq$$

$$\leq const\,\omega\left(\frac{[\sqrt{N}]}{2N+1},\lambda\right)+\frac{const}{\ln(2N+1)}\int_{\frac{[\sqrt{N}]}{2N+1}}^{1/2}\Omega(x)\,dx.$$

Likewise it can be proved, that the second limit on the right-hand side of (27) also is equal to zero. Theorem 2 is proved.

Theorem 3. The following three conditions

1°. *The function* $\theta(x)$ *is piecewise continuous and the series* $\sum_{s\in Z}|\theta(x+s)|^2\neq 0$ *converges uniformly on* $[-1/2,1/2]$*,*

2°. *For some* $\alpha<q+1.5$ *there exists the limit*

$$\lim_{x\to 0}\Psi(x)x^{-2\alpha}=C, \tag{28}$$

3°. $f\in C^{q+2}[-1,1]$, $q\geq -1$

imply

$$\lim_{N\to\infty}(2N+1)^{2\alpha}||f-S_N^q(f)||^2=\frac{C}{2\pi^{2q+2}}\sum_{n\in Z}{}'\,\frac{1}{n^{2q+2-2\alpha}}\left|\int_{-1}^{1}f^{(q+1)}(x)e^{-i\pi nx}\,dx\right|^2. \tag{29}$$

Proof: In view of (13),

$$||f-S_N^q(f)||^2=||F-S_N(F)||^2=2\sum_{n=-N}^{N}{}'\,\Psi(t_n)|F_n|^2-2\sum_{n=-N}^{N}\frac{1}{\omega_n^2}\left|\sum_{r\in Z}{}'\,\overline{\theta_{n+r(2N+1)}}F_{n+r(2N+1)}\right|^2+$$

$$+2\sum_{|n|>N}|F_n|^2-4\,\mathrm{Re}\,\left(\sum_{n=-N}^{N}{}'\left(\frac{\theta_n\overline{F_n}}{\omega_n^2}\sum_{r\in Z}{}'\,\overline{\theta_{n+r(2N+1)}}F_{n+r(2N+1)}\right)\right), \tag{30}$$

where (see Lemma 2)

$$F_n=\frac{\varepsilon_n}{2(i\pi n)^{q+1}},\quad \varepsilon_n=\int_{-1}^{1}f^{(q+1)}(x)e^{-i\pi nx}\,dx,\quad n\neq 0.$$

Likewise in Theorem 2, it can be proved, that the second and third terms on the right-hand side of (30) are of order $O((2N+1)^{-2q-3})$, as $N\to\infty$. The last term on the right-hand side of (30) is majorized by the expression

$$\frac{const}{(2N+1)^{q+2}}\sum_{n=-N}^{N}{}'\,|F_n|\Psi^{1/2}(t_n)\leq\frac{const}{(2N+1)^{q+1.5+\alpha}}.$$

Therefore

$$\lim_{N\to\infty}(2N+1)^{2\alpha}||f-S_N^q(f)||^2 = 2\lim_{N\to\infty}(2N+1)^{2\alpha}\sum_{n=-N}^{N}{}' |F_n|^2\Psi(t_n) =$$

$$= \frac{1}{2\pi^{2q+2}}\lim_{N\to\infty}\sum_{n=-N}^{N}{}' \Psi(t_n)t_n^{-2\alpha}\frac{|\varepsilon_n|^2}{n^{2q+2-2\alpha}}.$$

Now we show that

$$\lim_{N\to\infty}\sum_{n=-N}^{N}{}' \Psi(t_n)t_n^{-2\alpha}\frac{|\varepsilon_n|^2}{n^{2q+2-2\alpha}} = C\sum_{n\in Z}{}' \frac{|\varepsilon_n|^2}{n^{2q+2-2\alpha}}.$$

Denoting

$$\widetilde{\Psi}(x) = x^{-2\alpha}\Psi(x), \quad \widetilde{\Psi}(0) = C,$$

and taking into account that $\widetilde{\Psi}(x)$ is continuous near the point $x=0$, we obtain

$$\left| C\sum_{n\in Z}{}' \frac{|\varepsilon_n|^2}{n^{2q+2-2\alpha}} - \sum_{n=-N}^{N}{}' \widetilde{\Psi}(t_n)\frac{|\varepsilon_n|^2}{n^{2q+2-2\alpha}}\right| =$$

$$= \left|\sum_{n=-N}^{N}{}' \left(\widetilde{\Psi}(0)-\widetilde{\Psi}(t_n)\right)\frac{|\varepsilon_n|^2}{n^{2q+2-2\alpha}} + C\sum_{|n|>N}\frac{|\varepsilon_n|^2}{n^{2q+2-2\alpha}}\right| \leq$$

$$\leq \sum_{n=-[\sqrt{N}]}^{[\sqrt{N}]}{}' \frac{\omega\left(\frac{n}{2N+1},\widetilde{\Psi}\right)}{n^{2q+4-2\alpha}} + const\sum_{|n|>[\sqrt{N}]} n^{-2q-4+2\alpha} \leq$$

$$\leq \omega\left(\frac{[\sqrt{N}]}{2N+1},\widetilde{\Psi}\right)\sum_{n\in Z}{}' n^{-2q-4+2\alpha} + const\sum_{|n|>[\sqrt{N}]} n^{-2q-4+2\alpha}.$$

We have used the fact, that the assumption $f\in C^{(q+2)}$ implies $|\varepsilon_n|\leq const|n|^{-1}$. It remains to observe, that the series $\sum_{n\in Z}{}' n^{-2q-4+2\alpha}$ converges for $\alpha < q+1.5$. Theorem 3 is proved.

§3. SPECIAL CASES AND NUMERICAL RESULTS

In this section we consider examples, that illustrate the results of §2.

Example 1. The simplest case (see also (11)) corresponds to the classical Fourier system

$$\theta(x) = \begin{cases} 1, & \text{for} \quad |x|\leq 1/2, \\ 0, & \text{for} \quad |x| > 1/2. \end{cases}$$

All conditions of Theorem 1 are fulfilled, and on $[-1/2, 1/2]$ we have $\Phi(x) = x^{-q-2}$. According to (14),

$$\lim_{N\to\infty} (2N+1)^{q+1.5} ||f - S_N^q(f)|| = |A_{q+1}| a(q), \tag{31}$$

where

$$a(q) = \frac{2^{q+1}}{\pi^{q+2}} \sqrt{\frac{2}{2q+3}}.$$

Thus, we have an exact asymptotic estimate of L_2–error for the Fourier–Bernoulli method (see [7], [8]). The following table contains numerical values of $a(q)$ for various values of q.

q	-1	0	1	2	3	4	5
$a(q)$	0.4501	0.1654	0.0816	0.0439	0.0246	0.0142	0.0083

Table 1. Numerical values of the constant $a(q)$ for various values of q.

Example 2.

$$\theta(x) = \begin{cases} \cos^s \frac{\pi}{2}x, & \text{for} \quad |x| \le 1, \\ 0, & \text{for} \quad |x| > 1, \end{cases} \qquad s > 0.$$

We have

$$\omega^2(x) = \cos^{2s} \frac{\pi}{2}x + \sin^{2s} \frac{\pi}{2}x, \quad \Psi(x) = \frac{\sin^{2s} \frac{\pi}{2}x}{\cos^{2s} \frac{\pi}{2}x + \sin^{2s} \frac{\pi}{2}x}. \tag{32}$$

It is clear, that the condition 1° of Theorem 1 is fulfilled. We have

$$\Psi(x) x^{-2q-4} \le const \ x^{2s-2q-4}. \tag{33}$$

The function on the right-hand side of (33) is monotone on each of the intervals $(-1/2, 0)$ and $(0, 1/2)$ and integrable on $(-1/2, 1/2)$ for $s > q + 1.5$. Hence the condition 2° of Theorem 1 is fulfilled. Besides, for $0 < x < 1/2$,

$$\Phi(x) = \theta(x)x^{-q-2} - \theta(x-1)(x-1)^{-q-2} = x^{-q-2} \cos^s \frac{\pi}{2}x - (x-1)^{-q-2} \sin^s \frac{\pi}{2}x,$$

and for $-1/2 < x < 0$,

$$\Phi(x) = \theta(x)x^{-q-2} - \theta(x+1)(x+1)^{-q-2} = x^{-q-2} \cos^s \frac{\pi}{2}x - (x+1)^{-q-2} \sin^s \frac{\pi}{2}x.$$

According to Theorem 1

$$\lim_{N\to\infty}(2N+1)^{q+1.5}||f-S_N^q(f)|| = |A_{q+1}|b_1(q,s), \tag{34}$$

where

$$b_1^2(q,s) = \frac{1}{\pi^{2q+4}}\left(\frac{2^{2q+3}}{2q+3} + \int_0^{1/2}\left(x^{-2q-4} - \frac{|\Phi(x)|^2}{\omega^2(x)}\right)dx\right).$$

The numerical values of the constant $b_1(q,s)$ for some values of s and q are presented on Fig. 1. Comparison with Table 1 shows, that for even values of q (lower diagrams, Fig. 1) the values of constant $b_1(q,s)$ are always greater, than the corresponding values of $a(q)$, and if s increases, then $b_1(q,s)$ approaches $a(q)$. Thus, for even values of q the classical choice of $\theta(x)$ in Example 1 leads to more effective interpolation, than that of in Example 2.

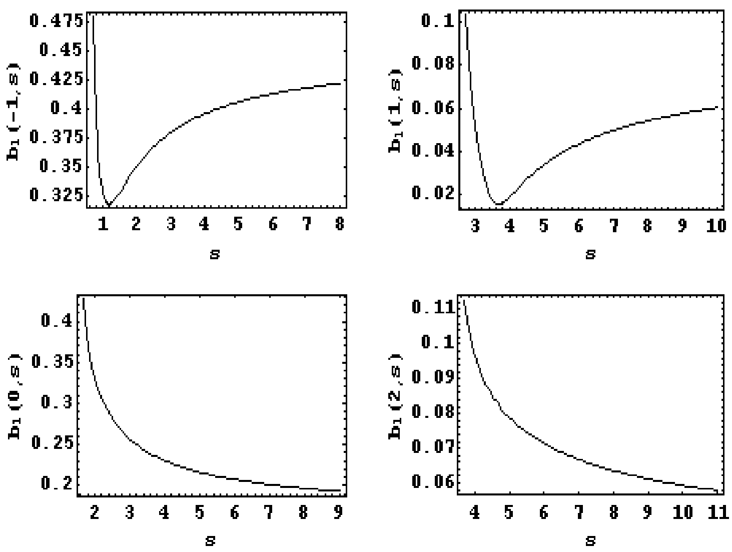

Fig. 1. Numerical values of the constant $b_1(q,s)$ for various values of s and q.

For odd values of q (upper diagrams, Fig. 1) the situation is different. The values of constant $b_1(q,s)$ first rapidly decrease (if s increases) up to some minimal value, and then increase up to the values of constant $a(q)$. Comparison with the results of Table 1 shows, that for greater values of s the values of $b_1(q,s)$ becomes less, than $a(q)$. Such behaviour of $b_1(q,s)$ with sharply expressed minimum allows to find an optimal value of the parameter s for fixed odd values of q. The optimal values of the parameter s determined by means of MATHEMATICA 3.0 are given in Table 2.

q	s	$b_1(q,s)$	$\frac{a(q)}{b_1(q,s)}$
-1	1.17501	0.318887	1.4
1	3.68683	0.015286	5.3
3	6.26922	0.001880	13.1
5	8.83021	0.000234	35.4

Table 2. Optimal values of the parameter s and constant $b_1(q,s)$.

The last column in Table 2 describes the efficiency of optimal interpolation relative to the classical case (see example 1). Note that the optimal value of the parameter s increases with q. For $s = q + 1.5$ from (32) we have (see also Theorem 2)

$$B = \lim_{x\to 0} \Psi(x)x^{-2q-3} = (\pi/2)^{2q+3}.$$

Thus, the conditions 1° and 2° of Theorem 2 are satisfied. Besides, we have

$$\left|\frac{\Psi(x)x^{-2q-3} - B}{x}\right| \leq const\ x,$$

and hence the condition 3° is satisfied. So, by Theorem 2

$$\lim_{N\to\infty} \frac{(2N+1)^{q+1.5}}{\sqrt{\ln(2N+1)}} ||f - S_N^q(f)|| = |A_{q+1}|b_2(q),$$

where

$$b_2(q) = \frac{1}{2^{q+1}}\sqrt{\frac{1}{2\pi}}.$$

The Table 3 contains numerical values of $b_2(q)$ for some values of q.

q	-1	0	1	2	3	4	5
$b_2(q)$	0.3989	0.1995	0.0997	0.0499	0.0249	0.0125	0.0062

Table 3. Numerical values of $b_2(q)$ for $-1 \leq q \leq 5$.

Finally, for $s < q+1.5$ we have (see also Theorem 3)

$$C = \lim_{x\to 0} \Psi(x)x^{-2s} = \left(\frac{\pi}{2}\right)^{2s}.$$

Therefore, for $\alpha = s$ all conditions of Theorem 3 are satisfied, and hence

$$\lim_{N\to\infty} (2N+1)^{2s}||f - S_N^q(f)||^2 = \pi^{2s-2q-2}2^{-2s-1} \sum_{n\in Z}{}' \frac{1}{n^{2q+2-2s}} \left| \int_{-1}^{1} f^{(q+1)}(x)e^{-i\pi nx}\, dx\right|^2. \tag{36}$$

So, for odd values of q and an optimal choice of the parameter s, we get more sharp algorithm as compared with the classical trigonometric approximation (Example 1).

Example 3. Consider the case, where

$$\frac{\theta_{n+r(2N+1)}}{\omega_n^2} = \left(\sum_{r\in Z}(n+r(2N+1))^{-2q-4}\right)^{-1/2} \frac{(-1)^{n+r}}{(n+r(2N+1))^{q+2}}, \tag{37}$$

for $n+r(2N+1) \neq 0$ and $\theta_0 = 1$, $\theta_{r(2N+1)} = 0$ for $r \neq 0$. By Theorem 1

$$||f - S_N^q(f)|| = o((2N+1)^{-q-1.5}), \quad N \to \infty.$$

Now we present a more exact estimate, displaying the principal term of $||f - S_N^q(f)||$.

Teorem 4. Let $f^{(q+2)} \in \Lambda_\alpha$, $\alpha > 1/2$, $q \geq -1$, and θ_n be a sequence from (37). Then

$$\lim_{N\to\infty} (2N+1)^{2q+4}||f - S_N^q(f)||^2 = \frac{\zeta(2q+4)}{\pi^{2q+4}} \left(|A_{q+1}|^2 + 2\int_{-1}^{1} |f^{(q+2)}(z)|^2 dz - \left|\int_{-1}^{1} f^{(q+2)}(z)\, dz\right|^2 \right), \tag{38}$$

where $\zeta(s) = \sum\limits_{r=1}^{\infty} r^{-s}$ is the Riemann function.

Proof: It is easy to check, that

$$\lim_{N\to\infty} (2N+1)^{2q+4}||F_1 - S_N(F_1)||^2 = \frac{|A_{q+1}|^2\zeta(2q+4)}{\pi^{2q+4}}.$$

Therefore,

$$\lim_{N\to\infty} (2N+1)^{2q+4}||f - S_N^q(f)||^2 = \lim_{N\to\infty} (2N+1)^{2q+4}||F_2 - S_N(F_2)||^2 + \frac{|A_{q+1}|^2\zeta(2q+4)}{\pi^{2q+4}}.$$

We use (13) with

$$f_n = F_{2,n} = \frac{\delta_n}{2(i\pi n)^{q+2}}, \quad \delta_n = \int_{-1}^{1} f^{(q+2)}(x)e^{-i\pi nx}\, dx, \quad n \neq 0.$$

Like in the proof of Theorem 2, the second term on the right-hand side of (13) for $f_n = F_{2,n}$ is majorized by the third term, for which we have the estimate

$$\sum_{|n|>N} |F_{2,n}|^2 \leq const \sum_{|n|>N} \frac{|\delta_n|^2}{n^{2q+4}} \leq const \sum_{|n|>N} n^{-2q-2\alpha-4} \leq \frac{const}{(2N+1)^{2q+3+2\alpha}}.$$

The last term on the right-hand side of (13) for $f_n = F_{2,n}$ is majorized by the expression

$$const \sum_{n=-N}^{N}{}' \left(\frac{|\delta_n|}{|n|^{q+2}} \Psi^{1/2}(t_n) \left(\sum_{r\in Z}{}' \frac{|\delta_{n+r(2N+1)}|^2}{(n+r(2N+1))^{2q+4}} \right)^{1/2} \right) \leq$$

$$\leq \frac{const}{(2N+1)^{q+2}} \sum_{n=-N}^{N}{}' |n|^{-\alpha} \left(\sum_{r\in Z}{}' (n+r(2N+1))^{-2q-4-2\alpha} \right)^{1/2} \leq$$

$$\leq \frac{const}{(2N+1)^{2q+4+\alpha}} \sum_{n=1}^{N} n^{-\alpha} \leq \frac{const}{(2N+1)^{2q+3+2\alpha}}.$$

Therefore,

$$\lim_{N\to\infty} (2N+1)^{2q+4} ||f - S_N^q(f)||^2 =$$

$$= \frac{1}{2\pi^{2q+4}} \lim_{N\to\infty} (2N+1)^{2q+4} \sum_{n=-N}^{N}{}' \frac{\Psi(t_n)}{t_n^{2q+4}} |\delta|^2 + \frac{|A_{q+1}|^2 \zeta(2q+4)}{\pi^{2q+4}}.$$

It remains to prove, that

$$\lim_{N\to\infty} \sum_{n=-N}^{N}{}' \frac{\Psi(t_n)}{t_n^{2q+4}} |\delta_n|^2 = 2\zeta(2q+4) \sum_{n\in Z}{}' |\delta_n|^2. \tag{39}$$

By the arguments of the proof of Theorem 3, the series on the right-hand side of (39) converges, because $|\delta_n|^2 \leq |n|^{-2\alpha}$, $\alpha > 1/2$. Hence,

$$\lim_{N\to\infty} (2N+1)^{2q+4} ||f - S_N^q(f)||^2 = \frac{\zeta(2q+4)}{\pi^{2q+4}} \sum_{n\in Z}{}' |\delta_n|^2 + \frac{|A_{q+1}|^2 \zeta(2q+4)}{\pi^{2q+4}}.$$

An application of Parseval equality completes the proof of Theorem 4.

Example 4. Consider the function

$$\theta(x) = \left(\frac{\sin \pi x}{\pi x}\right)^m, \tag{40}$$

where $m \geq 1$ is an integer. This is the well-known case of B–splines, for which $supp(\rho(x)) \in [-m, m]$. We have

$$\theta(x+s) = \frac{(-1)^{sm} \sin^m \pi x}{\pi^m (x+s)^m}, \quad \omega^2(x) = \frac{\sin^{2m} \pi x}{\pi^{2m}} \sum_{s\in Z} (x+s)^{-2m},$$

$$\Psi(x) = x^{2m} {\sum_{s\in Z}}' (x+s)^{-2m} \left(1 + x^{2m} {\sum_{s\in Z}}' (x+s)^{-2m}\right)^{-1}. \tag{41}$$

Besides,

$$\Psi(x) x^{-2q-4} \leq const\ x^{2m-2q-4}.$$

Thus, for $m \geq q+2$ all conditions of Theorem 1 are fulfilled. After some simplification, from (14) we obtain

$$\lim_{N\to\infty} (2N+1)^{q+1.5} ||f - S_N^q(f)|| = |A_{q+1}| c_1(q, m),$$

where

$$c_1^2(q,m) = \pi^{-2q-4} \left(\frac{2^{2q+3}}{2q+3} + \int_0^{1/2} \left(x^{-2q-4} - \frac{(\sum_{s\in Z} (-1)^{s(m+1)} (x+s)^{-m-q-2})^2}{\sum_{s\in Z} (x+s)^{-2m}} \right) dx \right).$$

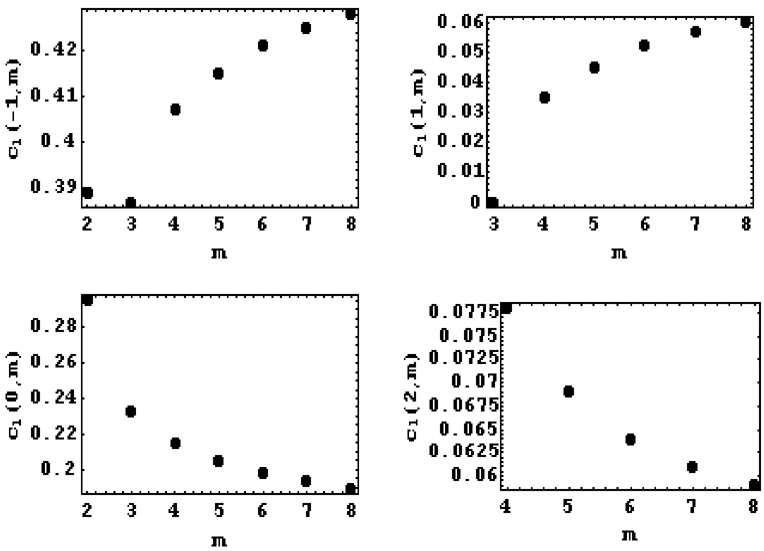

Fig. 2. The values of constant $c_1(q, m)$ for $-1 \leq q \leq 2$ and $2 \leq m \leq 8$.

The Figures 1 and 2 are similar. As in Example 2, for even values of q (lower diagrams in Fig. 2) the values of constant $c_1(q,m)$ exceed than the corresponding values of $a(q)$, and $c_1(q,m)$ approaches $a(q)$ as m increases. For odd values of q (upper diagrams in Fig. 2) $c_1(q,m)$ is less than $a(q)$, and again approaches $a(q)$ as m increases.

It follows from (41), that (see Theorem 3)

$$C = \lim_{x\to 0} \Psi(x) x^{-2m} = \sum_{s\in Z}{}' s^{-2m} = 2\zeta(2m).$$

Hence, for $m \le q+1$ and $\alpha = m$ all conditions of Theorem 3 are fulfilled. By (29)

$$\lim_{N\to\infty} (2N+1)^{2m} ||f - S_N^q(f)||^2 = \frac{\zeta(2m)}{\pi^{2q+2}} \sum_{n\in Z}{}' \frac{1}{n^{2q+2-2m}} \left| \int_{-1}^{1} f^{(q+2)}(x) e^{-i\pi n x}\, dx \right|^2 .$$

It is easy to check, that the function (40) generates a sequence satisfying (37) for odd values of q. In this case the assertion of Theorem 4 remains valid for Example 4.

Example 5. Consider the function

$$\theta(x) = \left(\frac{\sin \pi x}{\pi x}\right)^m \cos \pi x, \tag{42}$$

where $m \ge 1$ is an integer. It is clear, that $supp\, \rho(x) \in [-m-1, m+1]$. For $m \ge q+2$ all conditions of Theorem 1 are fulfilled, and from (14) we obtain

$$\lim_{N\to\infty} (2N+1)^{q+1.5} ||f - S_N^q(f)|| = |A_{q+1}| d_1(q,m), \tag{43}$$

where

$$d_1^2(q,m) = \pi^{-2q-4} \left(\frac{2^{2q+3}}{2q+3} + \int_0^{1/2} \left(x^{-2q-4} - \frac{(\sum_{s\in Z} (-1)^{s(m+2)} (x+s)^{-m-q-2})^2}{\sum_{s\in Z} (x+s)^{-2m}} \right) dx \right).$$

The numerical values of $d_1(q,m)$ for various values of q and m are presented in Fig. 3.

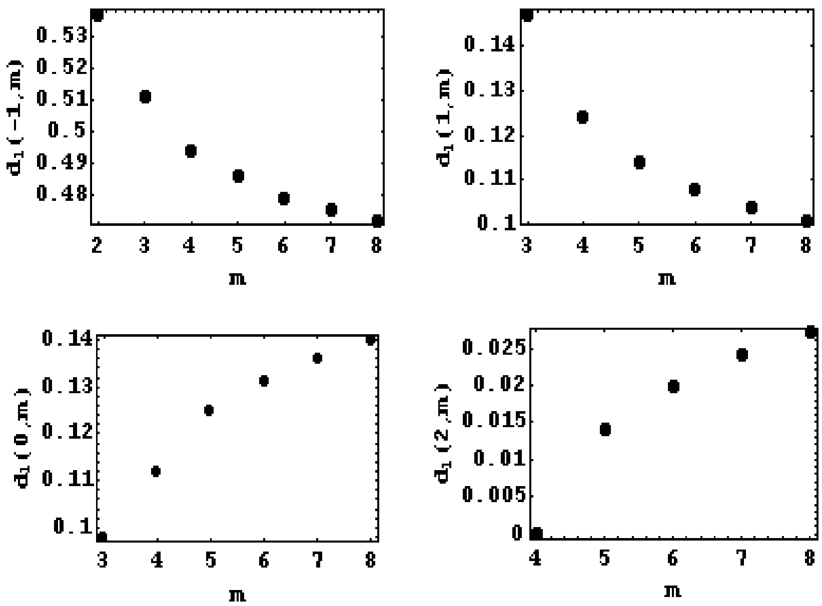

Fig. 3. The values of constant $d_1(q,m)$ for various values of q and m.

Finally, for $m < q+2$ all conditions of Theorem 3 are satisfied, and from (29) we obtain

$$\lim_{N\to\infty}(2N+1)^{2m}||f - S_N^q(f)||^2 = \frac{\zeta(2m)}{\pi^{2q+2}} \sum_{n\in Z}{}' \frac{1}{n^{2q+2-2m}} \left| \int_{-1}^{1} f^{(q+2)}(x) e^{-i\pi n x}\, dx \right|^2 . \tag{44}$$

For even values of q the condition (37) is fulfilled, and Theorem 4 can be applied.

In fact, by means of choice of the spline as in Example 4 (for odd $q \geq -1$) and "modified spline" as in Example 5 (for odd $q \geq 0$), we obtain fast algorithms of high accuracy based on functions with local supports.

R E F E R E N C E S

1. W. Gautschi, "Attenuation factors in practical Fourier analysis", Numer. Math., vol. 18, pp. 373 – 400, 1972.
2. F. Locher, "Interpolation on uniform meshes by the translates of a function and related attenuation factors", Math. Comp., vol. 37, no. 156, pp. 403 – 416, 1981.
3. F. J. Narcowich, J. D. Ward, "Wavelets associated with periodic basis functions", Appl. and Comput. Harmonic Analysis, vol. 3, pp. 40 – 56, 1996.
4. A. B. Nersessian, "Parametric approximation and some applications" [in Russian], DNAN Armenii, vol. 98, no. 1, pp. 23 – 30, 1998.
5. A. B. Nersessian, "A family of approximation formulas and some applications", Numerical Functional Analysis and Optimization, vol. 21, no. 1 – 2, 2000.
6. A. V. Kantorovich, V. I. Krylov, Approximate Methods of Analysis [in Russian], Gostekhizd, Moscow–Leningrad, 1950.

7. K. S. Eckhoff, “Accurate reconstructions of functions of finite regularity from truncated Fourier series expansions”, Math. Comp., vol. 64, no. 210, pp. 671 – 690, 1995.
8. K. S. Eckhoff, C. E. Wasberg, “On the numerical approximation of derivatives by a modified Fourier collocation method”, Thesis of C. E. Wasberg, Department of Mathematics, University of Bergen, Norway, 1996.
9. S. Wolfram, The MATHEMATICA Book, Third Edition, Wolfram Media, Cambridge University Press, 1996.
10. A. Zigmund, Trigonometric Series [in Russian], Mir, Moscow, 1965.
11. G.M.Fixtengolc, Course of differential and integral calculus [in Russian], vol.3, Nauka,1970.

16 October 2000

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