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Stochastic and Integral Geometry

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STOCHASTIC AND INTEGRAL GEOMETRY

The Sine Representation of Centrally Symmetric Convex Bodies

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Abstract—The problem of the sine representation for the support function of centrally symmetric convex bodies is studied. We describe a subclass of centrally symmetric convex bodies which is dense in the class of centrally symmetric convex bodies. Also, we obtain an inversion formula for the sine-transform.

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1. INTRODUCTION

The cosine representation of the support function of centrally symmetric convex bodies plays a fundamental role in the integral geometry and in a number of related areas (see [2], [8] – [11], [17], [21]). In this paper we study (in a dual sense) sine representation for the support function of centrally symmetric convex bodies.

Denote by $\mathbf{R}^n$ $(n \geq 2)$ the n-dimensional Euclidean space. Let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ centered at the origin, and let λ_k be the spherical Lebesgue measure on $\mathbf{S}^k$ $(\lambda_k(\mathbf{S}^k) = \sigma_k)$. Denote by $\mathbf{S}_\omega \subset \mathbf{S}^{n-1}$ the great $(n-2)$-dimensional sphere with pole at $\omega \in \mathbf{S}^{n-1}$. The class of convex bodies (nonempty compact convex sets) **B** that are symmetric with respect to the origin in $\mathbf{R}^n$ (the so-called *centered* bodies) we denote by $\mathcal{B}_o^n$, and the class of centrally symmetric convex bodies in $\mathbf{R}^n$ by $\mathcal{B}^n$.

The most useful analytic description of a convex body is its support function (see [16]). The support function $H : \mathbf{R}^n \to (-\infty, \infty]$ of a convex body **B** is defined as follows:

$$H(\mathbf{B}, x) = H(x) = \sup_{y \in \mathbf{B}} \langle y, x \rangle, \quad x \in \mathbf{R}^n.$$

Here and below $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbf{R}^n$. The support function of **B** is positively homogeneous and convex. Below, we consider the support function $H(\cdot)$ of a convex body as a function defined on the unit sphere $\mathbf{S}^{n-1}$ (because of the positive homogeneity of $H(\cdot)$).

It is well known (see [16]) that a convex body **B** is uniquely determined by its support function, and **B** is k-smooth if its support function H is k times continuously differentiable function on $\mathbf{S}^{n-1}$. By $\mathcal{C}_c^k$ we denote the class of even, k times continuously differentiable functions defined on $\mathbf{S}^{n-1}$.

It is known (see [9], [8], [21]) that *the support function $H(\cdot)$ of an origin symmetric convex body $\boldsymbol{B} \in \mathcal{B}_o^n$ which is a limit in the Hausdorff metric of zonotopes (a finite sum of line segments) has the following representation:*

$$H(\xi) = \int_{\mathbf{S}^{n-1}} | \langle \xi, \Omega \rangle | \, m(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{1.1}$$

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with an even measure m.

The following question arises naturally. Does the support function of any centered convex body have a cosine representation?

It is known (see [9], [8], [17], [21]) that the *support function $H(\cdot)$ of a sufficiently smooth origin symmetric convex body $\boldsymbol{B} \in \mathcal{B}^n_o$ has the following representation:*

$$H(\xi) = \int_{\mathbf{S}^{n-1}} | \langle \xi, \Omega \rangle | \, h(\Omega) \, \lambda_{n-1}(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{1.2}$$

with an even continuous function $h(\cdot)$ (not necessarily positive) defined on $\mathbf{S}^{n-1}$.

Note that the function h in (1.2) is unique. Also, a body $\mathbf{B}$ whose support function has an integral representation of the form (1.1) with a signed even measure m is a centered *generalized zonoid*. If m is a measure on $\mathbf{S}^{n-1}$, then the centered convex body $\mathbf{B}$ is a zonoid.

It follows from (1.2) that the class of generalized zonoids is dense in $\mathcal{B}^n$. The right-hand side of (1.2) is called the cosine-transform of h.

W. Weil [20] showed that a local characterization of zonoids does not exist. Later it was shown that in even dimensions an equatorial characterization of zonoids exists (see [18], [11]), while in odd dimensions an equatorial characterization of zonoids does not exist (see [15]). In [4] was defined a subclass of zonoids admitting an equatorial characterization.

In the article, we consider a finite sum of $(n-2)$-dimensional centered balls and their limits. Let $b = (r, \Omega)$ be the $(n-2)$-dimensional centered ball in $\mathbf{R}^n$ with radius r, and let $\Omega \in \mathbf{S}^{n-1}$ be the unit vector normal to b. The support function of b has the form:

$$H(b, \xi) = r \, \sin(\widehat{\xi, \Omega}), \quad \xi \in \mathbf{S}^{n-1}. \tag{1.3}$$

Here and below by $(\widehat{\xi, \Omega})$ we denote the angle between two directions. Now we consider a finite sum (Minkowski sum) of $(n-2)$-dimensional centered balls in $\mathbf{R}^n$. The support function of P, which is the sum of $b_i = (r_i, \Omega_i)$, $i = \overline{1, m}$, has the form:

$$H(P, \xi) = \sum_{i=1}^{m} r_i \sin(\widehat{\xi, \Omega_i}) = \sum_{i=1}^{m} \frac{r_i}{2} [\sin(\widehat{\xi, \Omega_i}) + \sin(\widehat{\xi, -\Omega_i})], \quad \xi \in \mathbf{S}^{n-1}. \tag{1.4}$$

We define the class of convex bodies $\mathcal{D}$, the so-called diskoids, which are limits in the Hausdorff metric of finite sums of $(n-2)$-dimensional balls. For the support function of a centered diskoid the sum in (1.4) becomes into an integral, and we have

$$H(\xi) = \int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi, \Omega}) \, \upsilon(d\Omega), \quad \xi \in \mathbf{S}^{n-1}, \tag{1.5}$$

where υ is an even measure on $\mathbf{S}^{n-1}$.

The following question arises naturally. Does the support function of any centered convex body have a sine representation? In this article we prove the following theorem.

Theorem 1.1. *A centered convex body $\boldsymbol{B}$ is a diskoid if and only if the support function of $\boldsymbol{B}$ has representation* (1.5) *with an even measure υ on $\mathbf{S}^{n-1}$.*

Note that the class of diskoids is a subset of the class of zonoids because any diskoid is a zonoid. Also, there is a zonoid which is not a diskoid, for example a segment is not a diskoid. Thus we have: *the diskoids are nowhere dense in $\mathcal{B}^n$.*

Now we define the class of *generalized diskoids* (see also [21]). A centered convex body $\mathbf{B}$ is said to be a *generalized diskoid* if its support function H admits the following representation:

$$H(\xi) = \int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi, \Omega}) \, \upsilon(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{1.6}$$

with a signed even measure m. A generalized diskoid can also be defined as follows (see also [19]): a body $\mathbf{B}$ is a *generalized diskoid* if $\mathbf{B} + \mathbf{B}_1 = \mathbf{B}_2$, where $\mathbf{B}_1$, $\mathbf{B}_2$ are diskoids.

In this article, we prove the following theorem which states that the class of generalized diskoids is dense in $\mathcal{B}^n$.

Theorem 1.2. *The support function $H(\cdot)$ of a sufficiently smooth origin symmetric convex body $\boldsymbol{B} \in \mathcal{B}_o^n$ has the following representation:*

$$H(\xi) = \int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi, \Omega})\, h(\Omega)\, \lambda_{n-1}(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{1.7}$$

with an even continuous function $h(\cdot)$ (not necessarily positive) defined on $\mathbf{S}^{n-1}$. The function h is unique.

The right-hand side of (1.7) is called the sine-transform of h and is denoted by $(Qh)(\cdot)$. Theorem 2.1 below shows that the measure υ in (1.6) (the function h in (1.7)) is unique. Hence the transform $Q : \mathcal{C}_c^\infty \longrightarrow \mathcal{C}_c^\infty$ is injective. The function h in (1.7) is called the generating density of the body $\mathbf{B} \in \mathcal{B}_o^n$. Note, that the injectivity of the cosine-transform was proved by A. Aleksandrov [2].

Observe that Theorem 1.2 can be proved by using the expansion of h in the spherical harmonics (see [3]). In this article, we prove the theorem by finding an inversion formula for transform (1.7).

Note that R. Schneider and R. Schuster [19], and S. Alesker [3] have proved several significant results for sums of similar convex bodies and spherical harmonics. Also, note that the Minkowski class $\mathcal{M}_{b,GL(n)}$, where b is a $(n-2)$-dimensional centered ball, coincides with the class of zonoids. In [13] was considered the sine-transform of isotropic measures and isoperimetric inequalities were established.

An inversion formula for the sine-transform. By R we denote the Radon transform on the sphere (the Funk's transform) defined by:

$$RF(\xi) = \frac{1}{\sigma_{n-2}} \int_{\mathbf{S}_\xi} F(\omega)\, \lambda_{n-2}(d\omega), \quad \xi \in \mathbf{S}^{n-1} \tag{1.8}$$

for $F \in \mathcal{C}_c^\infty$. For $n \geq 3$ an inversion formula for R was given by Helgason [14] (for $n = 3$ an inversion formula was obtained by Minkowski and Blashke (see [9])). In [5] was considered the generalized Radon transform on the sphere and was found an inversion formula (see also [6]). By Ξ we denote the transform $\Xi : \mathcal{C}_c^\infty \longrightarrow \mathcal{C}_c^\infty$ define by: $\Xi = ((n-1) + \Delta)$, where Δ is the Laplace-Beltrami operator on $\mathbf{S}^{n-1}$.

The next theorem contains an inversion formula for the sine-transform.

Theorem 1.3. *Let $H(\cdot)$ be the support function of a sufficiently smooth origin symmetric convex body $\boldsymbol{B} \in \mathcal{B}_o^n$. Then $h = Q^{-1}H = \frac{1}{(n-2)\sigma_{n-3}} \Xi R^{-2} H$ is the solution of integral equation* (1.7).

2. THE SINE REPRESENTATION FOR AN ORIGIN SYMMETRIC CONVEX BODY

Proof of Theorem 1.1. Necessity. Let $\mathbf{B}$ be a diskoid. Then there exists a sequence P_m of finite sums of $(n-2)$-dimensional balls, which converges to $\mathbf{B}$ in the Hausdorff metric. To each P_m corresponds an even measure υ_m with finite support on $\mathbf{S}^{n-1}$ through (1.4). The sequence υ_m is uniformly bounded in total variation norm because $\upsilon_m(\mathbf{S}^{n-1}) < C\mu([\mathbf{K}])$, where C is a constant, $\mathbf{K} \in \mathcal{B}_o^n$ is a convex body containing $\mathbf{B}$ and $\mu([\mathbf{K}])$ is the invariant measure of hyperplanes intersecting $\mathbf{K}$. Hence one can select a subsequence υ'_m, which weakly converges to an even measure υ on $\mathbf{S}^{n-1}$, and the support function $H(\mathbf{B}, \cdot)$ has the representation (1.5) with measure ν.

Sufficiency. Let the support function of $\mathbf{B}$ have the representation (1.5) with an even measure υ on $\mathbf{S}^{n-1}$. Then there exists a sequence of even measures υ_m with finite supports, which weakly converges to υ. To each υ_m corresponds P_m (a finite sum of $(n-2)$-dimensional balls) through (1.4). Then $H(P_m, \cdot)$ converges pointwise to $H(\mathbf{B}, \cdot)$. Also, it is known that pointwise convergence of a sequence of convex functions implies the uniform convergence on each compact. Thus, we have that $H(P_m, \cdot)$ converges uniformly to $H(\mathbf{B}, \cdot)$ on $\mathbf{S}^{n-1}$. Hence, P_m converges to $\mathbf{B}$ in the Hausdorff metric, and thus $\mathbf{B}$ is a diskoid. Theorem 1.1 is proved.

The next theorem shows that the measure υ in (1.5) is unique (see also [13]).

Theorem 2.1. *If υ is an even signed measure on $\mathbf{S}^{n-1}$ with*

$$\int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi, \Omega})\, \upsilon(d\Omega) = 0 \tag{2.1}$$

for all $\xi \in \mathbf{S}^{n-1}$, then $\upsilon \equiv 0$.

Proof. We use expansions in spherical harmonics. Let Q_d be a spherical harmonic of order d. We multiply (2.1) by Q_d, integrate over $\mathbf{S}^{n-1}$, and use Fubini theorem to obtain

$$\int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, \upsilon(d\Omega) \right) Q_d(\xi)\lambda_{d-1}(d\xi)$$
$$= \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, Q_d(\xi)\lambda_{d-1}(d\xi) \right) \upsilon(d\Omega) = 0. \tag{2.2}$$

Next, the Funk-Hecke formula states that (see [12])

$$\int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, Q_d(\xi)\lambda_{d-1}(d\xi) = a_{n,d} Q_d(\Omega), \tag{2.3}$$

where $a_{n,d}$ is a coefficient depending only on d and n, and $a_{n,d} \neq 0$ if d is even. Thus, for all spherical harmonics of order d we have

$$\int_{\mathbf{S}^{n-1}} Q_d(\Omega)\, \upsilon(d\Omega) = 0. \tag{2.4}$$

Notice that if d is odd, then (2.4) is true because υ is an even measure. Using uniform approximation of continuous functions on $\mathbf{S}^{n-1}$, for every continuous g we obtain $\int_{\mathbf{S}^{n-1}} g(\Omega)\, \upsilon(d\Omega) = 0$. Taking into account known results of integration theory we can conclude that this is possible only if $\upsilon \equiv 0$. Theorem 2.1 is proved.

3. AN INVERSION FORMULA FOR THE SINE-TRANSFORM

Let $\mathbf{B} \in \mathcal{B}_o^n$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature at every point of the boundary $\partial\mathbf{B}$. Let $s(\omega)$ be the point on $\partial\mathbf{B}$, the outer normal of which is ω. Further, let $R_i(\omega)$ be the i-th principal radii of curvature ($i = 1, ..., n-1$) of ∂B at $s(\omega)$. $k_1(\omega) \cdots k_{n-1}(\omega) > 0$, where $k_1(\omega), ..., k_{n-1}(\omega)$ signify the principal curvatures of $\partial\mathbf{B}$ at $s(\omega)$.

The concept of a flag in $\mathbf{R}^n$, which naturally emerges in combinatorial integral geometry, will be of importance below. A detailed account of the concept in $\mathbf{R}^3$ can be found in [1]. Here we consider the so-called *directed* flags (below just a flag). A flag is a pair $\{g, e\}$, where g is a directed line containing the origin O and e is an oriented hyperplane (a hyperplane with specified positive normal direction) containing g. There are two equivalent representations of flags: (ω, φ) or (ξ, Φ), where $\omega \in \mathbf{S}^{n-1}$ is the normal of e and φ is the *planar* direction in $\mathbf{S}_\omega$ that coincides with the direction of g, while $\xi \in \mathbf{S}^{n-1}$ is the spatial direction of g and Φ is the *planar* direction in $\mathbf{S}_\xi$ that coincides with the normal of e.

Let $\xi \in \mathbf{S}^{n-1}$ and $\Phi \in \mathbf{S}_\xi$. By $B(\Phi)$ we denote the projection of $\mathbf{B}$ onto the hyperplane with normal Φ containing the origin O. Then for $(n-1)$-dimensional volume of $B(\Phi)$ we have

$$V_{n-1}(B(\Phi)) = \frac{1}{2} \int_{\mathbf{S}^{n-1}} |\langle \Phi, \omega \rangle| \prod_{i=1}^{n-1} R_i(\omega)\, \lambda_{n-1}(d\omega). \tag{3.1}$$

By $\mathbf{D}(O, 1)$ we denote the n-dimensional ball of radius 1 centered at the origin. Now we write (3.1) for the Minkowski sum $\mathbf{B} + \varepsilon\, \mathbf{D}(O, 1)$ ($\varepsilon > 0$). Using the classical Steiner formula for volume we obtain

$$V_{n-1}(B(\Phi) + \varepsilon \mathbf{D}(O,1)) = \sum_{i=0}^{n-1} \varepsilon^i \binom{n-1}{i} W_i(B(\Phi)) = \frac{1}{2} \int_{\mathbf{S}^{n-1}} |\langle \Phi, \omega \rangle| \prod_{i=1}^{n-1} (R_i(\omega) + \varepsilon) \lambda_{n-1}(d\omega). \tag{3.2}$$

Here $W_i(B(\Phi))$ is the i-th quermassintegral of $B(\Phi)$ (see [16]). Comparing the orders of ε of both sides of (3.2) we get the following formula

$$(n-1) W_{n-2}(B(\Phi)) = \frac{1}{2} \int_{\mathbf{S}^{n-1}} |\langle \Phi, \omega \rangle| \sum_{i=1}^{n-1} R_i(\omega)\, \lambda_{n-1}(d\omega). \tag{3.3}$$

Next, using the spherical coordinates $\omega = (\nu, \varphi)$, where $\nu = (\widehat{\xi, \omega})$ is the polar angle measured from ξ (the zenith direction) and $\varphi \in \mathbf{S}_\xi$, and applying the spherical cosine rule, we find

$$| \langle \omega, \Phi \rangle | = \sin \nu \mid \cos(\widehat{\varphi, \Phi}) \mid . \tag{3.4}$$

Now integrating (3.3) with respect Φ over $\mathbf{S}_\xi$ and using (3.4) we obtain

$$\int\limits_{\mathbf{S}^{n-2}} (n-1) W_{n-2}(B(\Phi))\, \lambda_{n-2}(d\Phi) = \frac{1}{2} \int\limits_{\mathbf{S}^{n-1}} \sin(\widehat{\xi, \omega}) \sum_{i=1}^{n-1} R_i(\omega)\, \lambda_{n-1}(d\omega) \int\limits_{\mathbf{S}_\xi} \mid \cos(\widehat{\varphi, \Phi}) \mid \ \lambda_{n-2}(d\Phi).$$

Finally, for any $\varphi \in \mathbf{S}^{n-2}$ we have (see [4])

$$\int_{\mathbf{S}^{n-2}} \mid \cos(\widehat{\varphi, \Phi}) \mid \ \lambda_{n-2}(d\Phi) = \frac{2\sigma_{n-3}}{n-2}.$$

Thus, we have proved the following theorem.

Theorem 3.1. *Let $\boldsymbol{B} \in \mathcal{B}_o^n$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature. Then for $\xi \in \mathbf{S}^{n-1}$ we have*

$$\int_{\mathbf{S}_\xi} (n-1) W_{n-2}(B(\Phi))\, \lambda_{n-2}(d\Phi) = \frac{\sigma_{n-3}}{(n-2)} \int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi, \omega}) \sum_{i=1}^{n-1} R_i(\omega)\, \lambda_{n-1}(d\omega), \tag{3.5}$$

where $W_{n-2}(B(\Phi))$ is the $(n-2)$-th quermassintegral of the projection of $\boldsymbol{B}$ onto the hyperplane orthogonal to $\Phi \in \mathbf{S}_\xi$.

It is known (see [16]) that in $(n-1)$-dimensional space for $W_{n-2}(B(\Phi))$ we have

$$(n-1) W_{n-2}(B(\Phi)) = \int_{\mathbf{S}^{n-2}} H(B(\Phi), u)\, \lambda_{n-2}(du) = \int_{\mathbf{S}_\Phi} H(\mathbf{B}, u)\, \lambda_{n-2}(du), \tag{3.6}$$

where $H(B(\Phi), \cdot)$ is the support function of $B(\Phi)$ which is the restriction of the support function of $\mathbf{B}$ onto $\mathbf{S}_\Phi$. Let $\mathbf{B} \in \mathcal{B}_o^n$ be a convex body. In [7] it was shown that

$$((n-1) + \Delta) H(\mathbf{B}, \cdot) = \sum_{i=1}^{n-1} R_i(\cdot), \tag{3.7}$$

where Δ is the Laplace-Beltrami operator on $\mathbf{S}^{n-1}$. Substituting (3.6) and (3.7) into (3.5) we get

$$\int_{\mathbf{S}_\xi} \left[\int_{\mathbf{S}_\Phi} H(u) \lambda_{n-2}(du) \right] \lambda_{n-2}(d\Phi) = \frac{\sigma_{n-3}}{(n-2)} \int_{\mathbf{S}^{n-1}} \sin(\widehat{\xi, \omega}) \left[((n-1) + \Delta) H(\xi) \right] \lambda_{n-1}(d\omega). \tag{3.8}$$

Thus, we have proved the following theorem.

Theorem 3.2. *Let $\boldsymbol{B} \in \mathcal{B}_o^n$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature. Then for the support function H of $\mathbf{B}$ we have*

$$R(RH)(\xi) = R^2 H(\xi) = \frac{1}{(n-2)\sigma_{n-3}} Q(\Xi H)(\xi) \quad \xi \in \mathbf{S}^{n-1}. \tag{3.9}$$

Theorem 1.3 is a consequence of Theorem 3.2.

4. EXAMPLE OF A CONVEX BODY WITH SIGNED GENERATING DENSITY

In this section, we give an example of a sufficiently smooth origin symmetric convex body $\mathbf{B} \in \mathcal{B}_o^3$ for which the solution of the equation takes also negative values. Let U be a ε neighborhood of a point $\Omega_0 \in \mathbf{S}^2$ and g be a sufficiently smooth even function defined on $\mathbf{S}^2$ such that $g(\Omega) \geq 1$ for $\Omega \in \mathbf{S}^2 \backslash \{U \cup \{-U\}\}$ and $g(\Omega) \leq -1$ for $\Omega \in \{U \cup \{-U\}\}$.

Consider the following function defined on $\mathbf{R}^3$

$$F(\xi) = |\xi| \int_{\mathbf{S}^2} \sin(\widehat{\overrightarrow{\xi}, \Omega})\, g(\Omega)\, \lambda_2(d\Omega), \quad \xi \in \mathbf{R}^3 \backslash O, \tag{4.1}$$

where $|\xi|$ is the norm of ξ and $\overrightarrow{\xi} = \xi/|\xi|$. Observe that F is positively homogeneous. Now we are going to show that for sufficiently small ε the function F is also convex.

For the first order partial derivative of F we have

$$\frac{\partial F(\xi)}{\partial \xi_1} = \int_{\mathbf{S}^2} \frac{\xi_1 - \Omega_1 \langle \xi, \Omega \rangle}{|\xi| \sin(\widehat{\overrightarrow{\xi}, \Omega})}\, g(\Omega)\, \lambda_2(d\Omega), \quad \xi \in \mathbf{R}^3 \backslash O, \tag{4.2}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

Next, for fixed $\xi \in \mathbf{S}^2$ and $\psi \in \mathbf{S}_\xi$ we choose ξ as the zenith direction and ψ as the azimuth reference. Then, for the second order derivative on direction ψ at $\xi = (0,0,1)$, we have

$$\frac{\partial^2 F(\xi)}{\partial^2 \xi_1} \,|_{\xi=(0,0,1)} = \int_{\mathbf{S}^2} \frac{\sin^2 \varphi}{\sin \nu}\, g(\Omega)\, \lambda_2(d\Omega), \tag{4.3}$$

where (ν, φ) are the usual spherical coordinates of Ω on $\mathbf{S}^2$ based on the choice of ξ as the North Pole and ψ as the reference direction on $\mathbf{S}_\xi$. It follows from (4.3) that for a sufficiently small ε for all $\xi \in \mathbf{S}^2$ we have $\frac{\partial^2 F(\xi)}{\partial^2 \xi_1} > 0$. Therefore, the function F is convex, and there is an origin symmetric convex body $\mathbf{B} \in \mathcal{B}_o^3$ for which F is the support function.

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