ON AN EQUIVALENCY OF DIFFERENTIATION BASIS OF DYADIC RECTANGLES

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Abstract. The paper considers differentiation properties of rare basis of dyadic rectangles corresponding to an increasing sequence of integers \( \{ \nu_k \} \). We prove that the condition
\[
\sup_k (\nu_{k+1} - \nu_k) < \infty
\]
is necessary and sufficient for such basis to be equivalent to the full basis of dyadic rectangles.

1. Introduction

Let \( \mathcal{R} \) be the family of half-closed rectangles \([a, b) \times [c, d)\) in \( \mathbb{R}^2 \) and \( \mathcal{Q} \subset \mathcal{R} \) be the family of half-closed squares in \( \mathbb{R}^2 \). Then let \( \mathcal{R}^{\text{dyadic}} \) be the family of dyadic rectangles of the form
\[
\left[ \frac{i-1}{2^n}, \frac{i}{2^n} \right] \times \left[ \frac{j-1}{2^m}, \frac{j}{2^m} \right], \quad i, j, n, m \in \mathbb{Z},
\]
and \( \mathcal{Q}^{\text{dyadic}} \) be the family of dyadic squares \((n = m)\). We have \( \mathcal{R}^{\text{dyadic}} \subset \mathcal{R} \) and \( \mathcal{Q}^{\text{dyadic}} \subset \mathcal{Q} \). For a given rectangle \( R \in \mathcal{R} \) we denote by \( \text{len}(R) \) the length of the bigger side of \( R \).

Definition 1.1. A family of rectangles \( \mathcal{M} \subset \mathcal{R} \) is said to be a differentiation basis (or simply basis), if for any point \( x \in \mathbb{R}^2 \) there exists a sequence of rectangles \( R_k \in \mathcal{M} \) such that \( x \in R_k \), \( k = 1, 2, \ldots \) and \( \text{len}(R_k) \to 0 \) as \( k \to \infty \).

Let \( \mathcal{M} \subset \mathcal{R} \) be a differentiation basis. For any function \( f \in L^1(\mathbb{R}^2) \) we define
\[
\delta_M(x, f) = \limsup_{\text{len}(R) \to 0; \ x \in R \in \mathcal{M}} \left| \frac{1}{|R|} \int_R f(t) \, dt - f(x) \right|.
\]
The integral of a function \( f \in L^1(\mathbb{R}^2) \) is said to be differentiable at a point \( x \in \mathbb{R}^2 \) with respect to the basis \( \mathcal{M} \), if \( \delta_M(x, f) = 0 \). Consider classes of functions
\[
\mathcal{F}(\mathcal{M}) = \{ f \in L(\mathbb{R}^2) : \delta_M(x, f) = 0 \text{ almost everywhere} \},
\]
\[
\mathcal{F}^+(\mathcal{M}) = \{ f \in L(\mathbb{R}^2) : f(x) \geq 0, \delta_M(x, f) = 0 \text{ almost everywhere} \}.
\]
Note that \( \mathcal{F}(\mathcal{M}) \) is the family of functions having almost everywhere differentiable integrals with respect to the basis \( \mathcal{M} \).

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Let $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a convex function. Denote by $\Phi(L)(\mathbb{R}^2)$ the class of measurable functions $f$ defined on $\mathbb{R}^2$ such that $\Phi(|f|) \in L^1(\mathbb{R}^2)$. If $\Phi$ satisfies the $\Delta_2$-condition $\Phi(2x) \leq k\Phi(x)$, then $\Phi(L)$ turns to be an Orlicz space with the norm

$$
\|f\|_\Phi = \inf \left\{ \epsilon > 0 : \int_{\mathbb{R}^2} \Phi\left(\frac{|f|}{\epsilon}\right) \leq 1 \right\}.
$$

The following classical theorems determine the optimal Orlicz space, which functions have a.e. differentiable integrals with respect to the entire family of rectangles $\mathcal{R}$ is the space

$$
L(1 + \log L)(\mathbb{R}^2) \subset L^1(\mathbb{R}^2),
$$
corresponding to the case $\Phi(t) = t(1 + \log t)$ ([1]).

**Theorem A** (Jessen-Marcinkiewicz-Zygmund, [4]). $L(1 + \log L)(\mathbb{R}^2) \subset \mathcal{F}(\mathcal{R})$.

**Theorem B** (Saks, [7]). If

$$
\Phi(t) = o(t \log t) \text{ as } t \to \infty,
$$
then $\Phi(L)(\mathbb{R}^2) \not\subset \mathcal{F}(\mathcal{R})$. Moreover, there exists a positive function $f \in \Phi(L)(\mathbb{R}^2)$ such that $\delta_{\mathcal{R}}(x, f) = \infty$ everywhere.

Such theorems are valid also for the basis $\mathcal{R}_{\text{dyadic}}$. The first one trivially follows from embedding $L(1 + \log L)(\mathbb{R}^2) \subset \mathcal{F}(\mathcal{R}) \subset \mathcal{F}(\mathcal{R}_{\text{dyadic}})$. The second can be deduced from the following relation

$$
\mathcal{F}^+(\mathcal{R}_{\text{dyadic}}) = \mathcal{F}^+(\mathcal{R}),
$$
due to Zerekidze [9] (see also [10, 11]).

Let $\Delta = \{\nu_k : k = 1, 2, \ldots\}$ be an increasing sequence of positive integers. This sequence generates the rare basis $\mathcal{R}_{\Delta}^{\text{dyadic}}$ of dyadic rectangles of the form (1) with $n, m \in \Delta$. This kind of bases first considered in the papers [8], [2], [3]. Stokolos [8] proved that the analogous of Saks theorem holds for any basis $\mathcal{R}_{\Delta}^{\text{dyadic}}$ with an arbitrary $\Delta$ sequence. That means $L(1 + \log L)(\mathbb{R}^2)$ is again the largest Orlicz space containing in $\mathcal{F}(\mathcal{R}_{\Delta}^{\text{dyadic}})$. G. A. Karagulyan [5] proved some theorems, establishing an equivalency of some convergence conditions for multiple martingale sequences, those in particular imply some results of the papers [8], [2], [3].

In this paper we prove

**Theorem.** Let $\Delta = \{\nu_k\} \subset \mathbb{N}$ be an increasing sequence of positive integers. Then the condition

$$
(2) \quad \sup_{k \in \mathbb{N}} (\nu_{k+1} - \nu_k) < \infty
$$
is necessary and sufficient for the equality

$$
\mathcal{F}(\mathcal{R}_{\Delta}^{\text{dyadic}}) = \mathcal{F}(\mathcal{R}_{\text{dyadic}}).
$$
2. Some definitions and key functions

Denote by $\overline{E}$ and $\overset{\circ}{E}$ the closure and the interior of a set $E \subset \mathbb{R}^2$ respectively, $\mathbb{I}_E$ denotes the indicator function of $E$. A set $E \subset \mathbb{R}^2$ is said to be simple, if it can be written as a union of squares of the form

$$\left[\frac{i - 1}{2^n}, \frac{i}{2^n}\right] \times \left[\frac{j - 1}{2^n}, \frac{j}{2^n}\right].$$

If $n$ is the minimal integer with this relation, then we write $\text{wd}(E) = 2^{-n}$. Note that if $E$ is a dyadic rectangle, then $\text{wd}(E)$ coincides with the length of the smaller side of $E$. If $E$ is square, then $\text{len}(E) = \text{wd}(E)$. Denote

(3) $E_{ij}(n) = \bigcup_{k=0}^{n-1} \left[\frac{i}{2^k} + \frac{1}{2^{k+1}}, \frac{i}{2^k}\right] \times \left[\frac{j}{2^k} + \frac{1}{2^{k+1}}, \frac{j}{2^k}\right],$

(4) $F_{ij}(n) = \left[\frac{i}{2^k} + \frac{1}{2^{k+1}}, \frac{i}{2^k}\right] \times \left[\frac{j}{2^k} + \frac{1}{2^{k+1}}, \frac{j}{2^k}\right] = \bigcap_{k=0}^{n-1} \left[\frac{i}{2^k} + \frac{1}{2^{k+1}}, \frac{i}{2^k}\right] \times \left[\frac{j}{2^k} + \frac{1}{2^{k+1}}, \frac{j}{2^k}\right] \subset E_{ij}(n),$  \text{ i, j = 0, 1,}$

and define the sets

(5) $E(n) = E_{00}(n) \cup E_{01}(n) \cup E_{10}(n) \cup E_{11}(n),$ 

(6) $F(n) = F_{00}(n) \cup F_{01}(n) \cup F_{10}(n) \cup F_{11}(n) \subset E(n),$

Introduce the functions

$u(x, n) = (n + 1)2^{n-2}(\mathbb{I}_{F_{00}(n)}(x) + \mathbb{I}_{F_{11}(n)}(x) - \mathbb{I}_{F_{10}(n)}(x) - \mathbb{I}_{F_{01}(n)}(x)), \quad n \in \mathbb{N},$

$v(x) = \mathbb{I}_{[0,1/2]\times[0,1/2]}(x) + \mathbb{I}_{[1/2,1]\times[1/2,1]}(x) - \mathbb{I}_{[0,1/2]\times[1/2,1]}(x) - \mathbb{I}_{[1/2,1]\times[0,1/2]}(x).$

Let $\omega \in \mathcal{Q}$ be an arbitrary square and $\phi_\omega$ be the linear transformation of $\mathbb{R}^2$ taking $\omega$ onto unit square $[0,1)^2 \subset \mathbb{R}^2$. For an arbitrary function $f(x)$ defined on $[0,1)^2$ and for a set $E \subset [0,1)^2$ we define

$$f_\omega(x) = f(\phi_\omega(x)), \quad E_\omega = (\phi_\omega)^{-1}(E) \subset \omega.$$

We have

(7) $\text{supp} (u_\omega(x, n)) = F_\omega(n),$

(8) $\text{supp} (v_\omega(x)) = \omega,$

(9) $|E_\omega(n)| = \frac{(n + 1)|\omega|}{2^n}, \quad |F_\omega(n)| = \frac{|\omega|}{4^{n-1}},$

(10) $\text{wd} (E_\omega(n)) = \text{wd} (F_\omega(n)) = \text{wd}(\omega) \cdot 2^{-n}.$

Simple calculations show that

(11) $\|u_\omega(x, n)\|_1 = |E_\omega(n)| = \frac{n + 1}{2^n} |\omega|,$

(12) $\|v_\omega(x)\|_1 = |\omega|.$
Then observe that, if $\omega \in Q^{dyadic}$ is a dyadic square, then for any point $x \in E_\omega(n)$ there exists a dyadic rectangle $R(x) \in R^{dyadic}$ with

$$
\frac{1}{|R(x)|} \left| \int_{R(x)} u_\omega(x, n) dx \right| = \frac{n + 1}{2}, \quad x \in R(x) \subset E_\omega(n),
$$

besides this rectangle coincides with the $(\phi_\omega)^{-1}$-image of one of the representation rectangles from (3). Similarly, if $\omega \in D$, then

$$
\frac{1}{|R(x)|} \left| \int_{R(x)} v_\omega(x) dx \right| = 1, \quad x \in R(x) \subset \omega,
$$

for some square $R(x)$ with $|R(x)| = |\omega|/4$. In this case $R(x)$ coincides with one of the four squares forming $\omega$.

### 3. Auxiliary Lemmas

The following simple lemma has been proved in [6].

**Lemma 1.** Let $Q \in Q^{dyadic}$ be an arbitrary dyadic square, a function $f(x) = f(x_1, x_2) \in L^1(\mathbb{R}^2)$ satisfies the condition $\text{supp } f(x) \subset Q$ and

$$
\int_{\mathbb{R}} f(x_1, t) dt = \int_{\mathbb{R}} f(t, x_2) dt = 0, \quad x_1, x_2 \in \mathbb{R}.
$$

Then for any dyadic rectangle $R \in R^{dyadic}$ satisfying $R \not\subset Q$ we have

$$
\int_{R} f(x) dx = 0.
$$

**Proof.** We suppose

$$Q = [a_1, b_1] \times [a_2, b_2], \quad R = [a_1, b_1] \times [a_2, b_2].$$

If $R \cap Q = \emptyset$, then (18) is trivial. Otherwise we will have either $[a_1, b_1] \subset [a_1, b_1]$ or $[a_2, b_2] \subset [a_2, b_2]$. In the first case, using (17), we get

$$
\int_{R} f(x) dx = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) dx_1 dx_2 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) dx_1 dx_2
$$

$$
= \int_{a_2}^{b_2} \left( \int_{\mathbb{R}} f(x_1, x_2) dx_1 \right) dx_2 = 0.
$$

The second case is proved similarly. $\square$

**Lemma 2.** Let $m$ be a positive integer and $Q$ be a dyadic square. Then for any simple set $E \subsetneq [0, 1)^2$, there exists a finite family $\Omega$ of dyadic squares
\( \omega \subset Q \) such that
\[
E_\omega \cap E_{\omega'} = \emptyset , \quad \omega \neq \omega',
\]
and
\[
\min_{\omega \in \Omega} \text{wd}(\omega) = \text{wd}(Q) \cdot (\text{wd}(E))^m,
\]
then
\[
\left| Q \setminus \bigcup_{\omega \in \Omega} E_\omega \right| = |Q| (1 - |E|)^m.
\]

**Proof.** Define a sequence of sets \( G_k, k = 1, 2, \ldots, m, \) with
\[
Q = G_1 \supset G_2 \supset \ldots \supset G_m,
\]
and finite families of dyadic squares \( \Omega_k \subset D, k = 1, 2, \ldots, m + 1, \) such that
\[
\text{wd}(\omega) = \text{wd}(Q) \cdot (\text{wd}(E))^{k-1}, \quad \omega \in \Omega_k, \quad k = 1, 2, \ldots, m + 1,
\]
and
\[
G_k = \bigcup_{\omega \in \Omega_k} \omega, \quad k = 1, 2, \ldots, m + 1,
\]
and
\[
G_k = G_{k-1} \setminus \bigcup_{\omega \in \Omega_{k-1}} E_\omega = \bigcup_{\omega \in \Omega_{k-1}} (\omega \setminus E_\omega), \quad k = 2, \ldots, m + 1.
\]

We do it by induction. For the first step of induction we take just \( G_1 = Q \) and let \( \Omega_1 \) consist of a single rectangle \( Q \). Suppose we have already chosen the sets \( G_k \) and the families \( \Omega_k \) for \( k = 1, 2, \ldots, p \), satisfying (22)-(25). Set
\[
G_{p+1} = G_p \setminus \bigcup_{\omega \in \Omega_p} E_\omega = \bigcup_{\omega \in \Omega_p} (\omega \setminus E_\omega).
\]

From the induction hypothesis of (23) it follows that
\[
\text{wd}(\omega \setminus E_\omega) = \text{wd}(\omega) \cdot \text{wd}(E) = \text{wd}(Q) \cdot (\text{wd}(E))^p.
\]

Hence we conclude that \( G_{p+1} \) is a union of dyadic squares with side lengths \( \text{wd}(Q) \cdot (\text{wd}(E))^p \) and we define the family \( \Omega_{p+1} \) as a collection of these squares. Thus we get \( G_{p+1} \) and \( \Omega_{p+1} \) satisfying the conditions (22)-(25) for \( k = p + 1 \), that completes the induction process. Applying (11), (24) and (25) we obtain
\[
|G_k| = |G_{k-1}| - \left| \bigcup_{\omega \in \Omega_{k-1}} E_\omega \right| = |G_{k-1}| - |E||G_{k-1}| = (1 - |E|)|G_{k-1}|
\]
and therefore
\[
|G_{m+1}| = (1 - |E|)^m |Q|.
\]

Obviously the family of squares \( \Omega = \bigcup_{k=1}^{m+1} \Omega_k \) satisfies the hypothesis of lemma. Indeed, suppose \( \omega, \omega' \in \Omega \) are arbitrary squares. If \( \omega, \omega' \in \Omega_k \) for some \( k \), then according to (23) we have \( \omega \cap \omega' = \emptyset \) and so (19). If \( \omega \in \Omega_k, \omega' \in \Omega_{k'} \) and \( k < k' \), then
\[
E_{\omega'} \subset \omega' \subset G_{k'},
\]
\[
E_\omega \subset G_k \setminus G_{k+1} \Rightarrow E_\omega \cap G_{k'} = \emptyset.
\]
Thus we again get (19). The condition (20) immediately follows from (23), and (21) follows from (26) and from the relation

\[ \left| \bigcup_{\omega \in \Omega} E_\omega \right| = \left| \bigcup_{k=1}^{m+1} \bigcup_{\omega \in \Omega_k} E_\omega \right| = \left| \bigcup_{k=1}^{m+1} G_k \setminus G_{k+1} \right| = |Q \setminus G_{m+1}| = |Q|(1-(1-|E|)^m). \]

□

Lemma 3. Let \( L > 1 \) be a positive integer and let \( Q \in D \) be a dyadic square. Then there exist a function \( f \in L^\infty(\mathbb{R}^2) \) and numbers \( \alpha(L) \in \mathbb{N}, \beta(L) > 0 \), depended on \( L \) such that

(27) \( \text{supp } f \subset Q \),

(28) \( \|f\|_\infty \leq \beta(L) \),

(29) \( |\text{supp } f| \leq \frac{2|Q|}{\beta(L)} \),

(30) \( \text{wd}(\text{supp } f) \geq \text{wd}(Q) \cdot 2^{-\alpha(L)} \),

(31) \( \int_R f(x)dx = 0, \quad R \in \mathcal{R}^{\text{dyadic}}, \quad R \not\subset \hat{Q} \),

and for any point \( x \in Q \) there exists a rectangle \( R(x) \subset Q \) satisfying

(32) \( \text{wd}(R(x)) \geq \text{wd}(Q) \cdot 2^{-\alpha(L)} \),

(33) \( \frac{1}{|R(x)|} \left| \int_{R(x)} f(t)dt \right| \geq L \).

Proof. Let \( n = 2L \) and denote

(34) \( \alpha(L) = n(2^n + 1), \quad \beta(L) = (n+1)2^{n-2} \),

(35) \( m = m(L) = \left[ \frac{2^n(\ln(n+1) + (n-2)\ln 2)}{n+1} \right] + 1 < 2^n \).

Let \( E = E(n) \) be the set defined in (5). We have \( |E(n)| = (n+1)/2^n \) and \( \text{wd}(E(n)) = 2^{-n} \). Applying Lemma 2, we may find family \( \Omega \) of dyadic squares \( \omega \subset Q \) with properties (19)-(21). Set

(36) \( G = \bigcup_{\omega \in \Omega} E_\omega(n), \quad G_1 = Q \setminus G \).

According to (21), (34) and (35), we have

\[ |G_1| = (1-|E(n)|)^m |Q| = \left( 1 - \frac{n+1}{2^n} \right)^m |Q| < \frac{|Q|}{\beta(L)} \]

From (20) and (35) it follows that

\[ G_1 = \bigcup_{\omega \in \Omega_1} \omega, \]

where \( \Omega_1 \) is a family of squares with

(37) \( \min_{\omega \in \Omega_1} \text{wd}(\omega) = \min_{\omega \in \Omega} \text{wd}(\omega) = \text{wd}(Q) \cdot (\text{wd}(E(n)))^m \geq \text{wd}(Q) \cdot 2^{-n-2^m}. \)
Define

$$f(x) = \sum_{\omega \in \Omega} u_{\omega}(x, n) + \beta(L) \sum_{\omega \in \Omega_1} v_{\omega}(x) = g(x) + g_1(x).$$

Clearly this function satisfies (27) and (28). Then, we have

$$\text{supp } g = \bigcup_{\omega \in \Omega} F_{\omega}(n) \subset G, \quad \text{supp } g_1 = G_1,$$

$$\text{supp } f = \text{supp } g \bigcup \text{supp } g_1.$$ 

This together with (9) and (36) implies

$$|\text{supp } f| = \bigcup_{\omega \in \Omega} |F_{\omega}(n)| + |G_1| = \frac{1}{(n+1)2^{n-2}} \sum_{\omega \in \Omega} |E_{\omega}(n)| + |G_1|$$

and therefore we get (29). Using (37), we obtain

$$\text{wd}(\text{supp } g) \geq \min_{\omega \in \Omega} \text{wd}(\omega) \cdot \text{wd}(F(n)) = \text{wd}(Q) \cdot 2^{-n(2^n+1)} = \text{wd}(Q) \cdot 2^{-\alpha(L)},$$

$$\text{wd}(\text{supp } g_1) \geq \min_{\omega \in \Omega_1} \text{wd}(\omega) \geq \text{wd}(Q) \cdot 2^{-n^2} > \text{wd}(Q) \cdot 2^{-\alpha(L)},$$

and therefore we get (30). The condition (31) follows from Lemma 1, since $f(x)$ satisfies the condition (17) according the definitions of functions $u_{\omega}(x, n)$ and $v_{\omega}(x)$. To prove (33) we take an arbitrary point $x \in Q$. We have either $x \in G$ or $x \in G_1$. In the first case we will have $x \in E_{\omega}(n)$ for some square $\omega \in \Omega$. By (13) there exists a dyadic rectangle $R = R(x)$, $x \in R \subset E_{\omega}(n)$, such that

$$\frac{1}{|R|} \left| \int_{R} f(t) dt \right| = \frac{1}{|R|} \left| \int_{R} u_{\omega}(t, n) dt \right| = \frac{n+1}{2} > L.$$ 

In the second case from (15) we obtain

$$\frac{1}{|R|} \left| \int_{R} f(t) dt \right| = \frac{\beta(L)}{|R|} \left| \int_{R} v_{\omega}(t) dt \right| \geq 2^n > L$$

for some square $R = R(x)$, $x \in R \subset \omega$. Obviously in any case $R(x)$ satisfies (32). Lemma is proved. \qed

Proof of Theorem. Necessity: Let $\Delta = \{\nu_k\}$ be a sequence with

$$\gamma = \sup_{k \in \mathbb{N}} (\nu_{k+1} - \nu_k) < \infty,$$

and suppose conversely, we have

$$\mathcal{F}(\mathcal{R}_\Delta^{\text{dyadic}}) \setminus \mathcal{F}(\mathcal{R}_\Delta^{\text{dyadic}}) \neq \emptyset.$$ 

That means there exists a function $f \in L^1(\mathbb{R}^2)$ such that

$$\delta_{\mathcal{R}_\Delta^{\text{dyadic}}}(x, f) = 0 \text{ a.e.,}$$

$$\delta_{\mathcal{R}_\Delta^{\text{dyadic}}}(x, f) > \alpha, \quad x \in E,$$
where $\alpha > 0$ and $|E| > 0$. According to (39) for any $x \in \mathbb{R}^2$ one can chose a number $\delta(x) > 0$ such that the conditions

$$x \in R \in \mathcal{R}_\Delta^{\text{dyadic}}, \quad \text{len}(R) < \delta(x),$$

imply

$$\left| \frac{1}{|R|} \int_R f - f(x) \right| < \frac{\alpha}{2}. \quad (41)$$

For some $\delta > 0$ the set $F = \{x \in E: \delta(x) \geq \delta\} \subset E$ has positive measure. Then, using the representation

$$F = \bigcup_{j \in \mathbb{Z}} \left\{ x \in F: \frac{j\alpha}{2} \leq f(x) < \frac{(j+1)\alpha}{2} \right\},$$

we find a set

$$G = \left\{ x \in F: \frac{j_0\alpha}{2} \leq f(x) < \frac{(j_0+1)\alpha}{2} \right\} \subset F \quad (42)$$

having positive measure. Combining (40), (41) and (42), we will have

$$\delta_{R_\text{dyadic}}(x,f) > \alpha, \quad x \in G, \quad (43)$$

$$\left| \frac{1}{|R|} \int_R f - f(x) \right| < \frac{\alpha}{2}, \quad \text{if } x \in R \cap G, \ R \in \mathcal{R}_\Delta^{\text{dyadic}}, \ \text{len}(R) < \delta, \quad (44)$$

$$\sup_{x,y \in G} |f(x) - f(y)| \leq \frac{\alpha}{2}. \quad (45)$$

Since almost all points of $G$ are density points, we may fix $x_0 \in G$ with

$$\lim_{\text{len}(R) \to 0, x_0 \in R \in \mathcal{R}_\Delta^{\text{dyadic}}} \frac{|R \cap G|}{|R|} = 1. \quad (46)$$

Using this relation and (43), we find a rectangle

$$R' = \left[ \frac{p-1}{2^n}, \frac{p}{2^n} \right] \times \left[ \frac{q-1}{2^m}, \frac{q}{2^m} \right],$$

such that

$$x_0 \in R' \in \mathcal{R}_\Delta^{\text{dyadic}}, \ \text{len}(R') < \delta, \quad (46)$$

$$\left| \frac{1}{|R'|} \int_{R'} f - f(x_0) \right| > \alpha, \quad (47)$$

$$|R' \cap G| > (1 - 4^{-\gamma})|R'|, \quad (48)$$

where $\gamma$ is the number (38). Besides, we may suppose

$$\nu_{k_t-1} < n \leq \nu_{k_t}, \quad \nu_{k_s-1} < m \leq \nu_{k_s}, \quad (49)$$

for some integers $t, s$. This and (38) imply that $R'$ is a union of rectangles of the form

$$\left[ \frac{i-1}{2^{\nu_{k_t}}}, \frac{i}{2^{\nu_{k_t}}} \right] \times \left[ \frac{j-1}{2^{\nu_{k_s}}}, \frac{j}{2^{\nu_{k_s}}} \right] \in \mathcal{R}_\Delta^{\text{dyadic}},$$
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and from (47) it follows that at least for one of these rectangles, say \( R'' \), we have

\[
(50) \quad \left| \frac{1}{|R''|} \int_{R''} f - f(x_0) \right| > \alpha.
\]

From (38) and (49) we get

\[
|R''| = \frac{1}{2^{\alpha_1 + \nu_s}} \geq \frac{1}{2^{\alpha_1 - \nu_{s-1} - \nu_s}} \cdot \frac{1}{2^{n+m}} \geq |R'| \cdot 4^{-\gamma}.
\]

From this and (48) we obtain \( R'' \cap G \neq \emptyset \). Take a point \( x_1 \in R'' \cap G \). From (45) and (50) we get

\[
(51) \quad \left| \frac{1}{|R''|} \int_{R''} f - f(x_1) \right| > \left| \frac{1}{|R''|} \int_{R''} f - f(x_0) \right| - |f(x_1) - f(x_0)| > \frac{\alpha}{2}.
\]

On the other hand we have \( x_1 \in R'' \cap G, R'' \in \mathcal{R}_\Delta^\text{dyadic}, \text{len}(R'') \leq \text{len}(R') < \delta_0 \), and therefore by (44) we obtain

\[
\left| \frac{1}{|R''|} \int_{R''} f - f(x_1) \right| < \alpha/2.
\]

The last relation together with (51) gives a contradiction, which completes the proof of the first part of our theorem.

**Sufficiency:** Now we suppose (2) doesn’t hold, that means there exists a sequence of integers \( p_k \nearrow \infty \) such that

\[
(52) \quad \lim_{k \to \infty} (\nu_{p_k+1} - \nu_{p_k}) = \infty.
\]

Using this relation, we may find sequences of integers \( L_k, l_k, k = 1, 2, \ldots \), such that

\[
(53) \quad l_{k+1} > l_k + \alpha(L_k), \quad k = 1, 2, \ldots,
\]

\[
(54) \quad \nu_{p_k} < l_k < l_k + \alpha(L_k) < \nu_{p_{k+1}}, \quad k = 1, 2, \ldots,
\]

\[
(55) \quad L_{k+1} > 2^k \cdot (\beta(L_k) + k) \quad k = 1, 2, \ldots,
\]

where \( \alpha(L) \) and \( \beta(L) \) are the constants taken from Lemma 3. Applying Lemma 3 for the numbers \( L = L_k, l = l_k \) and for the square

\[
Q = Q_{ij}^k = \left[ \frac{i-1}{2^{l_k}}, \frac{i}{2^{l_k}} \right) \times \left[ \frac{j-1}{2^{l_k}}, \frac{j}{2^{l_k}} \right), \quad 1 \leq i, j \leq 2^{l_k},
\]

we get functions \( f_{ij}^k(x) \in L^\infty(\mathbb{R}^2) \) satisfying the conditions

\[
(56) \quad \text{supp } f_{ij}^k \subset Q_{ij}^k,
\]

\[
(57) \quad \|f_{ij}^k\|_\infty \leq \beta(L_k),
\]

\[
(58) \quad |\text{supp } f_{ij}^k| \leq \frac{2 |Q_{ij}^k|}{\beta(L_k)},
\]

\[
(59) \quad \text{wd}(\text{supp } f_{ij}^k) \geq 2^{-l_k-\alpha(L_k)}
\]

\[
(60) \quad \int_R f_{ij}^k(x) dx = 0, \quad R \in \mathcal{R}_\Delta^\text{dyadic}, \quad R \notin Q_{ij}^k,
\]
and for any point \( x \in Q_{ij}^k \) there exists a dyadic rectangle \( R_k(x) \subset Q_{ij}^k \) with
\[
wd(R_k(x)) \geq 2^{-l_k - \alpha(L_k)},
\]
\[
\frac{1}{|R_k(x)|} \left| \int_{R_k(x)} f_{ij}^k(t) dt \right| \geq L_k.
\]
Define the function
\[
F_k(x) = \sum_{i,j=1}^{2^k} f_{ij}^k(x).
\]
From the relations (56) - (62) we conclude
\[
|\text{supp} F_k| \leq \frac{2}{\beta(L_k)},
\]
\[
wd(\text{supp} F_k) \geq 2^{-l_k - \alpha(L_k)},
\]
\[
\|F_k\|_{\infty} \leq \beta(L_k),
\]
\[
\int_R F_k(x) dx = 0, \quad R \in \mathcal{R}^{\text{dyadic}}, \quad \text{len}(R) \geq 2^{-l_k},
\]
and for any point \( x \in [0, 1)^2 \) there exists a dyadic rectangle \( R_k(x) \subset [0, 1)^2 \) such that
\[
2^{-l_k} > \text{len}(R_k(x)) \geq wd(R_k(x)) \geq 2^{-l_k - \alpha(L_k)},
\]
\[
\frac{1}{|R_k(x)|} \left| \int_{R_k(x)} F_k(t) dt \right| \geq L_k.
\]
Denote
\[
F(x) = \sum_{k=1}^{\infty} \frac{F_k(x)}{2^k}.
\]
From (63) and (54) it follows that \( \|F_k\|_1 \leq 2 \) and so \( \|F\|_1 \leq 2 \). Let \( x \in [0, 1)^2 \) be an arbitrary point. From the relations (53) and (67) we get \( \text{len}(R_k(x)) \geq 2^{-l_{k+1}} \geq 2^{-l_j} \) if \( j > k \). Thus, using (66), we obtain
\[
\int_{R_k(x)} F_j(t) dt = 0, \quad j > k.
\]
On the other hand the relations (65) and (55) imply
\[
\left| \int_{R_k(x)} \frac{1}{R_k(x)} \sum_{j=1}^{k-1} \frac{F_j(t)}{2^j} dt \right| \leq \beta(L_{k-1}) < \frac{L_k}{2}, \quad k \geq 2.
\]
From (68), (70) and (71) we get the inequality
\[
\left| \frac{1}{|R_k(x)|} \int_{R_k(x)} F(t) dt \right| \geq \frac{1}{|R_k(x)|} \left| \int_{R_k(x)} F_k(t) dt \right| - \frac{L_k}{2} > \frac{L_k}{2},
\]
which yields
\[
\limsup_{\text{len}(R) \rightarrow 0, \ x \in R \in \mathcal{R}^{\text{dyadic}}} \left| \frac{1}{|R|} \int_R F(t) dt \right| = \infty, \quad x \in [0, 1)^2.
\]
Now take an arbitrary rectangle \( R \in \mathcal{R}^{\text{dyadic}}_\Delta \). We have
\[
\text{len}(R) = 2^{-\nu_k} \geq \text{wd}(R) = 2^{-\nu_k}.
\]
From (66) we get
\[
\int_R F_j(t)dt = 0 \quad \text{if } l_j \geq \nu_k.
\]
On the other hand if \( l_j < \nu_k \), then from (54) it follows that
\[
l_j + \alpha(L_j) < \nu_k
\]
and therefore by (64) we get
\[
\text{wd}(\text{supp}(F_j)) \geq 2^{-l_j-\alpha(L_j)} \geq 2^{-\nu_k}.
\]
Thus, using simple properties of dyadic rectangles, we conclude that
\[
l_j < \nu_k, \quad R \not\subset \text{supp}(F_j) \Rightarrow R \cap \text{supp}(F_j) = \emptyset.
\]
Consider the sets
\[
G_1 = \{x \in [0,1]^2 : \delta_R(x,F_k) = 0, \ k = 1,2,\ldots\},
\]
\[
G_2 = \bigcup_{k=1}^{\infty} \bigcap_{j: l_j \geq \nu_k} (0,1)^2 \setminus \text{supp}(F_j),
\]
\[
G = G_1 \cap G_2.
\]
Since \( F_k(x) \) is bounded, the equality \( \delta_R(x,F_k) = 0 \) holds almost everywhere and so \( |G_1| = 1 \). From (63) it follows that \( |G_2| = 1 \) and therefore we get \( |G| = 1 \). Take an arbitrary point \( x \in G \). We have
\[
x \not\in \text{supp}(F_j), \quad j > k_0,
\]
for some \( k_0 \). Consider the rectangle \( R \in \mathcal{R}^{\text{dyadic}}_\Delta \) such that \( x \in R \). Suppose we have (73) and \( k > k_0 \). Then form (76) and (77) we get
\[
R \cap \text{supp}(F_j) = \emptyset, \quad \text{if } j > k_0 \text{ and } l_j < \nu_k.
\]
From (74) and (78) we conclude
\[
\frac{1}{|R|} \int_R F(t)dt = \sum_{j=1}^{k_0} \frac{1}{2^j} \cdot \frac{1}{|R|} \int_R F_j(t)dt.
\]
Thus we obtain
\[
\lim_{\text{len}(R) \to 0, x \in R \in \mathcal{R}^{\text{dyadic}}_\Delta} \frac{1}{|R|} \int_R F(t)dt = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}.
\]
On the other hand (77) implies
\[
F(x) = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}.
\]
From (72), (79) and (80) we conclude the relation \( F \in \mathcal{F} \left( \mathcal{R}^{\text{dyadic}}_\Delta \right) \setminus \mathcal{F}(\mathcal{R}^{\text{dyadic}}_\Delta) \), which completes the proof of the theorem. \( \square \)
References


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