ON RIEMANN SUMS AND MAXIMAL FUNCTIONS IN $\mathbb{R}^n$

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Abstract. In this paper we investigate problems on almost everywhere convergence of subsequences of Riemann sums

$$R_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right), \quad x \in \mathbb{T}.$$

We establish a relevant connection between Riemann and ordinary maximal functions, which allows to use techniques and results of the theory of differentiations of integrals in $\mathbb{R}^n$ in mentioned problems. In particular, we prove that for a definite sequence of infinite dimension n_k Riemann sums $R_{n_k} f(x)$ converge almost everywhere for any $f \in L^p$ with $p > 1$.

1. Introduction

We consider the Riemann sums operators

$$R_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right), \quad x \in \mathbb{T},$$

for the functions defined on the torus $\mathbb{T} = [0,1] = \mathbb{R}/\mathbb{Z}$. It is not hard to observe that if f is continuous then these sums converge to the integral of f uniformly and they converge in $L^1(\mathbb{T})$ while f is Lebesgue integrable. In this paper we investigate certain problems concerning the almost everywhere convergence of subsequences of Riemann operators. B. Jessen's classical theorem in [1] is the first result in this concern.

Theorem A (Jessen)**.** *Let $\{n_k\}$ be an increasing sequence of positive integers such that n_k divides n_{k+1}. Then*

$$\lim_{k\to\infty} R_{n_k} f(x) = \int_0^1 f(t)dt, \ \text{a.e.} \tag{1.1}$$

for any function $f \in L^1(\mathbb{T})$. Moreover

$$|\{x \in \mathbb{T} : \sup_k R_{n_k}|f(x)| > \lambda\}| \le \frac{1}{\lambda}\|f\|_{L^1}, \quad \lambda > 0. \tag{1.2}$$

The next fundamental result in this direction due W. Rudin [2]. He has constructed an example of a bounded function with divergent Riemann sums. Moreover it was proved

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Theorem B (W. Rudin)**.** *Let D be a sequence of positive integers which contains the sets D_n $(n = 1, 2, \ldots)$, each consisting of n terms, such that no member of D_n divides the least common multiple of the other members of D_n. Then for every $\varepsilon > 0$ there exists a bounded measurable function f, such that $0 \le f \le 1$, and such that*

$$\limsup_{n\to\infty,\, n\in D} R_n f(x) \ge \frac{1}{2}$$

for all x, although $\int f < \varepsilon$.

For example, D could be any sequence of primes. Using the Dirichlet's theorem on primes in arithmetic progressions W. Rudin in [2] has constructed a sequence $\{n_k\}$ which satisfies the hypothesis of Jessen's theorem such that $\{1 + n_k\}$ is a sequence of primes. Thus $R_{n_k} f(x)$ converges a.e., although $R_{1+n_k} f(x)$ need not do so. This observation shows that in a.e. convergence of operators $R_{n_k} f(x)$ arithmetic properties of $\{n_k\}$ are crucial.

Following L. Dubins and J. Pitman [3], we define a chain to be an increasing sequence of natural numbers $\{n_k\}$ for which n_k divides n_{k+1}. For families of natural numbers $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_d$ we denote by $[\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_d]$ the set of all naturals which are least common multiple of some numbers $n_1 \in \mathcal{S}_1, n_2 \in \mathcal{S}_2, \ldots, n_d \in \mathcal{S}_d$. We will say a set $\mathcal{S}$ has dimension d, if d is the least possible integer such that $\mathcal{S}$ is the subset of $[\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_d]$ for some chains $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_d$. An example of a set of dimension d is the set of integers having the factorization

$$p_1^{k_1} p_2^{k_2} \ldots p_d^{k_d}, \quad k_1, k_2, \ldots, k_d \in \mathbb{N}, \tag{1.3}$$

for fixed different primes $p_1, p_2, \ldots, p_d$. L. Dubins and J. Pitman in [3] extended the Jessen's theorem proving

Theorem C. *If the set of positive integers has dimension d and $f \in L \log^{d-1} L(\mathbb{T})$ then*

$$\lim_{n\to\infty, n\in\mathcal{S}} R_n(f)(x) = \int_0^1 f(x)dx \ \ a.e.$$

and moreover

$$m\{x \in \mathbb{T} : \sup_{n\in\mathcal{S}} |R_n(f)(x)| > \lambda\} \le \frac{C_d}{\lambda} \int_0^1 |f| \log^{d-1}(1 + |f|). \tag{1.4}$$

In the original proof of this theorem the martingale theory was used. There is a rather elementary and short proof of (1.4) given by an unknown referee of the article Y. Bugeaud and M. Weber [4]. More precisely, the maximal operator in (1.4) is estimated by d iterations of the operator in (1.2). Then the inequality (1.4) is derived by using an interpolation theorem ([5], chap. 12, theorem 4.34). Another elementary proof of this theorem has also suggested by R. Nair in [6]. Y. Bugeaud and M. Weber in [4] proved that Theorem C is nearly sharp.

Theorem D. *For any integer $d \ge 2$ and for any real number $\varepsilon > 0$ with $0 < \varepsilon < 1$, there exist a sequence n_k of dimension d and a function $f \in L \log^{d-1-\varepsilon} L(\mathbb{T})$ such that $R_{n_k} f(x)$ is almost everywhere divergent .*

The proof of this theorem is based on the method of R. C. Baker [7], where author has proved a weaker version of this theorem. As it is mentioned in [4] Theorem D does not answer precisely whether the class $L \log^{d-1} L(\mathbb{T})$ in the theorem is optimal or not. In Theorem 1 we prove that this class in fact is exact and divergence can be everywhere.

In present paper we establish a direct connection between Riemann maximal functions and ordinary maximal functions in Euclidian spaces $\mathbb{R}^d$. Moreover it turns out, that Riemann maximal function corresponding to a given finite set of indexes D is equivalent to a maximal function in Euclidian spaces $\mathbb{R}^d$ with respect to certain d-dimensional rectangles which is the content of Theorem 4 in Section 3. Theorem 4 makes possible to use many results and methods of maximal functions in this theory. Many constructions used for Riemann sums get rather simple geometric interpretation in $\mathbb{R}^d$. As applications of Theorem 4 we obtain below solutions of some problems on Riemann sums. To figure out the key point of our observation in Section 3 we display an alternative proof to Jessen's theorem using a covering property of some sets associated with Riemann sums. We will see a resemblance between this proof and the proof of Hardy-Littlewood maximal inequality where a covering lemma for intervals is used. In the last section we deduce Rudin's theorem from Theorem 4 using a simple geometry of multidimensional rectangles. In the same section we prove that for a general class of operator sequences the strong sweeping out and δ-sweeping out properties are equivalent.

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Denote by $L^\Phi(\mathbb{T})$ the class of functions f on $\mathbb{T}$ with $\Phi(|f|) \in L^1(\mathbb{T})$. If Φ satisfies Δ_2-condition $\Phi(2x) \le k\Phi(x)$ then L^Φ is Banach space with the norm $\|f\|_{L^\Phi} = \|f\|_\Phi$ to be the least $c > 0$ for which the inequality

$$\int_{\mathbb{T}} \Phi\left(\frac{|f|}{c}\right) \le 1$$

holds. The following theorem makes correction in the last theorem and shows that the class $L \log^{d-1} L$ in Theorem C is exact.

Theorem 1. *Let n_k be the increasing sequence formed the numbers (1.3) with fixed different primes $p_1, p_2, \ldots, p_d$. If an increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition*

$$\lim_{x\to\infty} \frac{\phi(x)}{x \ln^{d-1} x} = 0,$$

then there exists a function $f(x) \in L^\phi$ such that the sequence $R_{n_k} f(x)$ is everywhere divergent.

According to the Theorem C, Riemann sums corresponding to a set of finite dimension converge a.e. in L^p classes with $p > 1$. As for the sets of infinite dimension it was a problem wether there exists a sequence of infinite dimension $\{n_k\}$ such that $R_{n_k} f(x)$ converges for any function $f \in L^p(\mathbb{T})$ with $p > 1$. In [4] Y. Bugeaud and M. Weber discussed a particular sequence of infinite dimension E consist of all integers defined

$$E = \{p_1 \ldots p_{j-1} \check{p}_j p_{j+1} \ldots p_k : \quad k = 2, 3, \ldots, 1 \le j \le k\} \tag{1.5}$$

where $p_1 < p_2 < \ldots$ is the sequence of primes and the symbol $\check{}$ means p_j must be excluded in the product. As it is proved in [3] E has infinite dimension. In [4] (see also [8]) it is proved the almost everywhere convergence of Riemann sums $R_{n_k} f(x)$ where $\{n_k\} = E$ for the functions $f \in L^2(\mathbb{T})$ with Fourier coefficients satisfying

$$\sum_{n>3} a_n^2 \left(\frac{\ln n}{\ln \ln n}\right) < \infty.$$

It is proved also

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}R_{n_k}f(x)=\int_0^1 f(x)dx \text{ a.e.}$$

for any $f\in L^2(\mathbb{T})$. We proved that Riemann sums associated to the set E converge a.e. in any L^p, $p>1$. Moreover, a.e. convergence holds in the Orlicz class L^Φ corresponding to the function

$$\Phi(x)=\frac{x\ln(1+x)}{\ln\ln(3+x)},\quad x\geq 0, \tag{1.6}$$

and this class is the optimal one for the set E. So we prove the following theorems.

Theorem 2. *Let E be the set defined in (1.5) and $\Phi(x)$ is the function (1.6). Then for any $f\in L^\Phi$ we have*

$$\lim_{n\to\infty,\, n\in E}R_nf(x)=\int_0^1 f \text{ a.e. } .$$

Moreover

$$|\{x\in\mathbb{T}:\sup_{n\in E}|R_nf(x)|>\lambda\}|\leq\int_0^1\Phi\left(\frac{c|f|}{\lambda}\right),\quad \lambda>0, \tag{1.7}$$

where $c>0$ is an absolute constant.

This theorem immediately implies

Corollary. *There exists an infinite set $E\subset\mathbb{N}$ such that for any $f\in L^p(\mathbb{T})$ with $p>1$ Riemann sums $R_nf(x)$, $n\in E$ converge a.e.*

Theorem 3. *If the sequence $n_1<n_2<\ldots$ consists of all the integers of the set E and the increasing function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the condition*

$$\lim_{x\to\infty}\frac{\phi(x)\ln\ln x}{x\ln x}=0, \tag{1.8}$$

then there exists a function $f(x)\in L^\phi$ such that the sequence $R_{n_k}f(x)$ is everywhere divergent.

2. Notations

We recall some definitions in measure theory (see [9]). Let X to be an arbitrary set. A family Ω of subsets of X is called algebra if it is closed with respect to the operations of union, intersection and difference and $X\in\Omega$. If the algebra is closed also with respect to countable union it is called σ-algebra. The set A is called atom for the algebra Ω if there is no nonempty $B\in\Omega$ so that $B\subset A$. We note that if the algebra Ω is finite then any set from Ω is a union of some atoms of Ω. If there is also a measure μ on Ω we denote this measure space by (X,Ω,μ). It is said the measure spaces (X,Ω,μ) and (Y,Δ,ν) are isomorph if there exists a one to one mapping $\gamma:\Omega\to\Delta$ called isomorphism such that

$$\gamma(A-B)=\gamma(A)-\gamma(B),\quad \gamma\left(\bigcup_{k=1}^\infty A_k\right)=\bigcup_{k=1}^\infty\gamma(A_k),$$

and

$$\nu(\gamma(A))=\mu(A),$$

for any sets A, B and A_k, $k = 1, 2, \ldots$, from Ω. If Ω is not σ-algebra we suppose in addition $\cup_{k=1}^{\infty} A_k \in \Omega$. We will say $f : X \to Y$ is an isomorphism function if the set function $\gamma(A) = \{y \in Y : y = f(x), x \in A\}$ determines one to one mapping between Ω and Δ which is an isomorphism. Suppose the algebras Ω and Δ are finite and have atoms $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ respectively. It is clear if $\nu(A_i) = \mu(B_i)$, $i = 1, 2, \ldots, n$, then the measure spaces (Ω, μ) and (Δ, ν) are isomorph.

We consider the probability space $(\mathbb{T}^\infty, \lambda) = \prod_{i=1}^{\infty} (\mathbb{T}_i, \lambda_i)$ where each $(\mathbb{T}_i, \lambda_i)$ is the Lebesgue probability space on $\mathbb{T}$. Remind that measurable sets in $\mathbb{T}^\infty$ is generated from all the products $A = \prod_{i=1}^{\infty} A_i$ where each A_i is Lebesgue measurable set in $\mathbb{T}$ and only finite number of them differ from $\mathbb{T}$ (see definition in [10], chap. III.3). The measure in $\mathbb{T}^\infty$ is the extension of the measure $\lambda(A) = \prod_{i=1}^{\infty} |A_i|$. We will use $|A|$ to indicate measure of $A \subset \mathbb{T}^\infty$.

Let $l \in \mathbb{N}$ and $D \subset \mathbb{N}$ is finite. We will write $D|l$ if any member of D divides l. We denote

$$l/D = \{n \in \mathbb{N} : \frac{l}{n} \in D\}. \tag{2.1}$$

An important subject in this paper is the relationship between three type of sets. Namely we will consider Riemann sets, integer arithmetic progressions and special rectangles in $\mathbb{T}^\infty$ having the following descriptions.

Riemann sets: We denote by $\mathcal{I}_l$ the algebra in $\mathbb{T}$ generated by intervals $[\frac{j}{l}, \frac{j+1}{l})$, $j = 0, 1, \ldots l-1$. Define Riemann sets

$$I_l(n,t) = \bigcup_{i=0}^{n-1} \left[\frac{t}{l} + \frac{i}{n}, \frac{t+1}{l} + \frac{i}{n}\right), \quad t = 0, 1, \ldots, \frac{l}{n}. \tag{2.2}$$

where n divides l. Certainly we have

$$I_l(n,t) \in \mathcal{I}_l, \quad \lambda(I_l(n,t)) = \frac{n}{l}.$$

For fixed l and n dividing l the collection (2.2) is a pairwise disjoint partition of $[0,1]$. It is easy to verify if $x \in [k/l, (k+1)/l)$ then

$$l \int_{k/l}^{(k+1)/l} R_n f(t) dt = \frac{1}{|I_l(n,t)|} \int_{I_l(n,t)} f(t) dt, \quad x \in I_l(n,t).$$

Thus, using Lebesgue's theorem on $\mathbb{R}$, we get

$$\lim_{l\to\infty,\, x \in I_l(n,t)} \frac{1}{|I_l(n,t)|} \int_{I_l(n,t)} f(t) dt = R_n f(x) \text{ a.e. }, \quad n = 1, 2, \cdots. \tag{2.3}$$

For any subset $D \subset \mathbb{N}$ we define

$$\mathcal{R}_D^l f(x) = \sup_{n \in D:\, n|l,\, x \in I_l(n,t)} \frac{1}{|I_l(n,t)|} \int_{I_l(n,t)} |f(t)| dt. \tag{2.4}$$

If D_n are finite subsets of D with $\cup_n D_n = D$ then from (2.3) it follows that

$$\mathcal{R}_D f(x) = \sup_{n \in D} R_n f(x) = \lim_{n\to\infty} \lim_{l\to\infty:\, D_n|l} \mathcal{R}_D^l f(x) \text{ a.e. }. \tag{2.5}$$

Observe that if $f(x)$ is $\mathcal{I}_l$-measurable then

$$\mathcal{R}_D f(x) = \mathcal{R}_D^l f(x). \tag{2.6}$$

Indeed, since from $l|l'$ follows $\mathcal{A}_l\subset\mathcal{A}_{l'}$ we derive

$$\mathcal{R}_D^{l'}f(x)=\mathcal{R}_D^{l}f(x) \text{ if } l|l',$$

and therefore by (2.5) we have

$$\mathcal{R}_Df(x)=\lim_{l'\to\infty:\, l|l'}\mathcal{R}_D^{l'}f(x)=\mathcal{R}_D^{l}f(x).$$

Arithmetic progressions: We shall say a set of integers A is l-periodic if $A=l+A$. We denote by $\mathcal{A}_l$ the family of all l-periodic sets of integers and $\mathcal{A}=\cup_{l\in\mathbb{N}}\mathcal{A}_l$. It is clear if $l|l'$ then any l-periodic set is l'-periodic, i.e. $\mathcal{A}_l\subset\mathcal{A}_{l'}$. Observe that $\mathcal{A}$ and each $\mathcal{A}_l$ are algebras. We define the measure of a set $A\in\mathcal{A}$ by

$$\delta(A)=\lim_{l\to\infty}\frac{\#(A\cap[0,l))}{l},$$

where $\#B$ denotes the cardinality of the finite set B. It is clear that the limit exists and if $A\in\mathcal{A}_l$ then

$$\delta(A)=\frac{\#(A\cap[0,l))}{l}.$$

Observe that δ is an additive measure on $\mathcal{A}$. Now consider the arithmetic progressions

$$A_l(t)=\{lj+t,\, j\in\mathbb{Z}\},\quad 0\le t<l. \tag{2.7}$$

It is clear

$$A_l(t)\in\mathcal{A}_l,\quad \delta\big(A_l(t)\big)=\frac{1}{l},$$

and any set from $\mathcal{A}_l$ can be written as a finite union of these arithmetic progressions. It means the sets in (2.7) are the atoms of the algebra $\mathcal{A}_l$.

Rectangles in $\mathbb{T}^\infty$: We denote $p_1<p_2\ldots<p_d<\ldots$ the sequence of all primes. Consider an integer l with factorization

$$l=p_1^{l_1}p_2^{l_2}\cdots p_d^{l_d}. \tag{2.8}$$

We do not exclude that some of the numbers l_k are zero. Define rectangles in $\mathbb{T}^\infty$ by

$$B_l(j_1,\cdots,j_d)=\left\{x\in\mathbb{T}^\infty:\ \frac{j_k}{p_k^{l_k}}\le x_k<\frac{j_k+1}{p_k^{l_k}},\ k=1,2,\ldots,d\right\}, \tag{2.9}$$

where

$$0\le j_k<p_k^{l_k},\quad k=1,2,\cdots,d,\quad x=(x_1,x_2,\ldots,x_k,\ldots).$$

We denote by $\mathcal{B}_l$ the algebra generated of all the finite unions of the rectangles (2.9). So the family

$$\mathcal{B}=\bigcup_{l\in\mathbb{N}}\mathcal{B}_l.$$

is an algebra in $\mathbb{T}^\infty$. We note that $\mathcal{B}_l\subset\mathcal{B}_{l'}$ while $l|l'$. We shall consider the measure space $(\mathcal{B},\lambda)$, where λ is the Lebesgue's measure on $\mathbb{T}^\infty$. It is clear

$$\lambda(B_l(j_1,\cdots,j_d))=\frac{1}{l}.$$

It is clear that $(\mathcal{I}_l,\lambda)$, $(\mathcal{A}_l,\delta)$ and $(\mathcal{B}_l,\lambda)$ are isomorph, because all have l atoms with with equal measures. In Section 4 we are going to construct a special isomorphism between $\mathcal{A}$ and $\mathcal{B}$ assigning the arithmetic progressions (2.7) to the rectangles (2.9).

3. An alternative proof of Jessen's theorem

Operators (2.5) play a significant role in the study of a.e. convergence of Riemann sums. To prove Jessen's theorem it is enough to prove the inequality (1.2), because (1.1) follows from (1.2) by using Banach principle. So we suppose $D = \{m_1, m_2, \cdots, m_d, \ldots\}$ where m_k divides m_{k+1}. We fix a finite subset $U = \{m_1, m_2, \cdots, m_d\} \subset D$ and an integer l divided by m_d and so all m_k, $1 \le k \le d$. It is clear

$$\{\mathcal{R}_U^l f(x) > \lambda\} = \bigcup_j I_j,$$

where I_j are Riemann sets form $\mathcal{I}_l$ with

$$\frac{1}{|I_j|}\int_{I_j} |f(t)|dt > \lambda.$$

We will prove that it may be chosen a subfamily of mutually disjoint sets $\{\tilde{I}_j\}$ such that

$$\bigcup_j I_j = \bigcup_j \tilde{I}_j. \tag{3.1}$$

We define priority(I)=n if I has the form (2.2). It is easy to observe that if priority(I) divides priority(J) and $I \cap J \neq \varnothing$ then we have $I \subseteq J$. We take $\tilde{I}_1$ to be some of I_j with highest priority. Suppose we have chosen $\tilde{I}_1, \tilde{I}_2, \cdots, \tilde{I}_m$. We consider all I_j's with $I_j \not\subseteq \bigcup_{j=1}^m \tilde{I}_j$ and so $I_j \cap \left(\cup_{j=1}^m \tilde{I}_j\right) = \varnothing$. We take $\tilde{I}_{m+1}$ among these sets having an highest priority. Certainly this process generates a subcollection $\{\tilde{I}_j\}$ of mutually disjoint sets with (3.1). Thus we obtain

$$|\{\mathcal{R}_B^l f(x) > \lambda\}| = \Big|\bigcup_j I_j\Big| = \sum_j |\tilde{I}_j| < \frac{1}{\lambda}\sum_j \int_{\tilde{I}_j} |f(t)|dt \le \frac{\|f\|_{L^1}}{\lambda}.$$

Since the inequality is true for any finite $U \subset D$, applying (2.5) we get (1.2).

4. An isomorphism between arithmetic progressions and rectangles

Let l be an integer with factorization (2.8) and

$$m = p_1^{m_1} p_2^{m_2} \cdots p_d^{m_d}, \quad 0 \le m_k \le l_k, \quad k = 1, 2, \cdots, d. \tag{4.1}$$

From the definition (2.9) it follows that

$$B_m(t_1, \ldots, t_d) = \bigcup_{p_k^{l_k - m_k} t_k \le s_k < p_k^{l_k - m_k}\left(t_k + 1\right)} B_l(s_1, \cdots, s_d). \tag{4.2}$$

For a fixed integer t we consider the set of integer vectors

$$S_t = \{(s_1, s_2, \cdots, s_d) : 0 \le s_k < p_k^{l_k}, \quad s_k = t \bmod p_k^{m_k}, \quad k = 1, 2, \cdots, d, \} \tag{4.3}$$

In fact S_t depends also on l and m.

Lemma 1. *There exists a one to one correspondence from*

$$U = \{0, 1, \cdots, \frac{l}{m} - 1\} \tag{4.4}$$

to the set of vectors (4.3), such that the vector $(s_1, s_2, \cdots, s_d) \in S_t$ *assigned to* $u \in U$ *satisfies*

$$s_k = (mu + t) \bmod p_k^{l_k}, \quad k = 1, 2, \cdots, d. \tag{4.5}$$

Proof. We note that there are $p_k^{l_k-m_k}$ number of s_k's satisfying

$$0 \le s_k < p_k^{l_k}, \quad s_k = t \bmod p_k^{m_k}.$$

So we have

$$\# S_t = \prod_{k=1}^{d} p_k^{l_k-m_k} = \frac{l}{m} = \# U.$$

Thus, it is enough to prove that for any $u \in U$ there exists a vector $(s_1, \cdots, s_d) \in S$ with (4.5), and the images of different u's are different. To determine the vector $(s_1, \ldots, s_d)$ corresponding to u we define s_k to be the remainder when $mu + t$ is divided by $p_k^{l_k}$. Certainly $(s_1, \ldots, s_d)$ satisfies (4.5) and $0 \le s_k < p_k^{l_k}$. Since $p_k^{m_k} | m$ we get $s_k = t \bmod p_k^{m_k}$. So $(s_1, \ldots, s_d) \in S_t$. Now we suppose $(s_1, s_2, \cdots, s_d) \in S_t$ is assigned to different $u_1, u_2 \in U$. The numbers u_1 and u_2 satisfy the relation (4.5). Hence, we get

$$m(u_1 - u_2) = 0 \bmod p_k^{l_k}, \quad k = 1, 2, \cdots, d.$$

Since $0 \le u_1 - u_2 < \frac{l}{m}$, using (2.8) and (4.1), we conclude $u_1 - u_2 = 0$. □

Let $p \ge 2$ be an integer. Any nonnegative integer a has p-adic decomposition

$$a = a_0 p^k + a_1 p^{k-1} + \cdots + a_k, \quad 0 \le a_i < p.$$

We denote

$$(a)_p = a_k p^k + a_{k-1} p^{k-1} + \cdots + a_0,$$

the integer with revers arrangement of p-digits of a. We shall say that $(a)_p$ is the p-reverse of a. We note this action defines a one to one mapping of the set of integers $\{0, 1, \cdots, p^k - 1\}$ into itself. Notice if

$$s = p^i v + t, \quad 0 \le v < p^{j-i}, \quad 0 \le t < p^i, \, i \le j,$$

then

$$\bar{s} = p^{j-i} \bar{t} + \bar{v}. \tag{4.6}$$

It is easy to observe that for a fixed t the correspondence $s \to \bar{s}$ is a one to one mapping between the sets

$$\{s : \, s = p^i v + t, \, 0 \le v < p^{j-i}\} \text{ and } \{s : \, s = p^{j-i} \bar{t} + v, \, 0 \le v < p^{j-i}\}.$$

Lemma 2. *There exists an isomorphism α from the measure space $(\mathcal{A}, \delta)$ to $(\mathcal{B}, \lambda)$ assigning any progression (2.7) to a rectangle (2.9).*

Proof. At first we define α on the progressions (2.7). We take an arbitrary $A_l(t)$. Suppose

$$t_k = t \bmod p_k^{l_k}, \quad 0 \le t_k < p_k^{l_k}, \quad k = 1, 2, \ldots, d,$$

and denote by $\bar{t}_k$ the p_k-reverse of the integer t_k. We have $0 \le \bar{t}_k < p_k^{l_k}$. We define

$$\alpha\big(A_l(t)\big) = B_l(\bar{t}_1, \cdots, \bar{t}_d). \tag{4.7}$$

According the definition (2.7) for a given arithmetic progression $A_m(t)$ we have

$$A_m(t) = \bigcup_{u=0}^{l/m-1} A_l(mu + t). \tag{4.8}$$

We shall prove that

$$\alpha\big(A_m(t)\big) = \bigcup_{u=0}^{l/m-1} \alpha\big(A_l(mu+t)\big). \tag{4.9}$$

According to Lemma 2 there exists a one to one mapping between the sets U and S_t defined in (4.3) and (4.4). In addition, if $(s_1, s_2, \cdots, s_d) \in S_t$ is assigned to a given $u \in U$ then it satisfies the condition (4.5) and therefore by (4.7) we have

$$\alpha(A_l(mu+t)) = B_l(\bar{s}_1, \ldots, \bar{s}_d). \tag{4.10}$$

Now let t_k be the remainder when t is divided by $p_k^{m_k}$, i.e.

$$t_k = t \bmod p_k^{m_k}, \quad 0 \le t_k < p_k^{m_k}. \tag{4.11}$$

From (4.7) we get

$$\alpha\big(A_m(t)\big) = B_m(\bar{t}_1, \cdots, \bar{t}_d).$$

From (4.3) it follows that p^{m_k} divides $s_k - t$ and therefore by (4.11) it divides also $s_k - t_k$. So we have

$$s_k = p_k^{m_k} v_k + t_k, \quad 0 \le v_k < p_k^{l_k - m_k}, \quad 0 \le t_k < p_k^{m_k}.$$

Thus, according to (4.6), for the p_k-revers $\bar{s}_k$ of the integer s_k we have

$$\bar{s}_k = p_k^{l_k - m_k} \bar{t}_k + \bar{v}_k, \quad 0 \le \bar{v}_k < p_k^{l_k - m_k}, \quad 0 \le \bar{t}_k < p_k^{m_k},$$

where $\bar{v}_k$ and $\bar{t}_k$ are the p_k-reverses of v_k and t_k respectively. Hence for any $u \in U$ may be determined $(\bar{s}_1, \cdots, \bar{s}_d)$ with

$$p_k^{l_k - m_k} \bar{t}_k \le \bar{s}_k < p_k^{l_k - m_k}\big(\bar{t}_k + 1\big). \tag{4.12}$$

In addition, it is easy to check this correspondence is a one to one mapping from U to the set of vectors $(\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_d)$ with (4.12). Therefore, according to (4.8), (4.10) and (4.2), we get

$$\begin{aligned}\alpha\big(A_m(t)\big) &= B_m(\bar{t}_1, \cdots, \bar{t}_d)\\ &= \bigcup_{p_k^{l_k - m_k} \bar{t}_k \le \bar{s}_k < p_k^{l_k - m_k}\left(\bar{t}_k + 1\right)} B_l(\bar{s}_1, \cdots, \bar{s}_d) = \bigcup_{u=0}^{l/m-1} \alpha\big(A_l(mu+t)\big).\end{aligned}$$

So (4.9) is true. Now take an arbitrary set $A \in \mathcal{A}$. We have $A \in \mathcal{A}_l$ for some $l \in \mathbb{N}$. Since (2.7) are the atoms of $\mathcal{A}_l$, the set A is a union of some mutually disjoint atoms, i.e.

$$A = \bigcup_{i \in I} A_l(i). \tag{4.13}$$

We define

$$\alpha(A) = \bigcup_{i \in I} \alpha(A_l(i)). \tag{4.14}$$

Since A belongs to different algebras $\mathcal{A}_l$, there are different representations (4.13) corresponding to different l's. However, using (4.9), it is easy to verify that the right side of (4.14) does not depend on the representation (4.13). On the other hand α is measure preserving, because $\delta(A_l(i)) = \lambda(\alpha(A_l(i))) = 1/l$ by (4.7). So we conclude that α is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. In addition, according to (4.7) it assigns any progression (2.7) to a rectangle (2.9). The proof of Lemma 2 is complete. $\square$

For any l-periodic set of integers $A \in \mathcal{A}_l$ we define

$$\beta_l(A) = \bigcup_{k\in A}\left[\frac{k}{l}, \frac{k+1}{l}\right).$$

It is easy to check that β_l determines an isomorphism from the probability space $(\mathcal{A}_l, \lambda)$ to $(\mathcal{I}_l, \delta)$. Moreover

$$\beta((A_{l/n}(t)) = I_l(n,t).$$

Thus, the composition of $\alpha \circ \beta_l^{-1}$ where α is from Lemma 2 is an isomorphism from $(\mathcal{I}_l, \lambda)$ to $(\mathcal{B}_l, \lambda)$. Moreover the following lemma is true.

Lemma 3. *For any $l \in \mathbb{N}$ there exists a one to one mapping $\tau_l : \mathbb{T} \to \mathbb{T}^\infty$ such that*

(1) *τ_l is measure preserving, i.e. $|\tau(A)| = |A|$ for any Lebesgue measurable $A \subset \mathbb{T}$,*
(2) *τ_l is an isomorphism function between $(\mathbb{T}, \mathcal{I}_l, \lambda)$ and $(\mathbb{T}^\infty, \mathcal{B}_l, \lambda)$*
(3) *for any $I_l(n,t)$ from (2.2) the set $\gamma_l(I_l(n,t))$ is a rectangle of the form $B_m(i_1, \ldots, j_d)$ with $m = \frac{l}{n}$.*

Remark. *The existence of a mapping with the conditions (1) and (2) is trivial. The important part of the lemma is the fact that $\gamma_l(I_l(n,t))$ is a certain rectangle in $\mathbb{T}^\infty$.*

For any set of integers $D \subset \mathbb{N}$ we define the maximal function

$$\mathcal{M}_D g(x) = \sup_{m\in D:\, x\in B_m(j_1,\ldots,j_d)} \frac{1}{|B_m(j_1,\ldots,j_d)|}\int_{B_m(j_1,\ldots,j_d)} |g(t)|dt \tag{4.15}$$

where $g \in L^1(\mathbb{T}^\infty)$. We note that if l is a multiple for the numbers from D then the rectangles in (4.15) are in $\mathcal{B}_l$. This implies that for the conditional expectation $E^{\mathcal{B}_l}g(x)$ of $g(x)$ with respect to the algebra $\mathcal{B}_l$ we have

$$\mathcal{M}_D g(x) = \mathcal{M}_D E^{\mathcal{B}_l} g(x). \tag{4.16}$$

The following theorem clearly follows from Lemma 3. It creates an equivalency between Riemann maximal function $\mathcal{R}_D^l f(x)$ defined in (2.4) and $\mathcal{M}_{l/D}g(x)$, where l/D is defined in (2.1).

Theorem 4. *For any $l \in \mathbb{N}$ there exists a measure preserving mapping $\tau_l : \mathbb{T} \to \mathbb{T}^\infty$ such that if $f(x) \in L^1(\mathbb{T})$ and $g(x) = f(\tau_l^{-1}(x))$ then*

$$|\{x \in \mathbb{T} : \mathcal{R}_D^l f(x) > \lambda\}| = |\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/D} g(x) > \lambda\}|, \quad \lambda > 0.$$

Corollary. *Let D be a set of indexes and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Then*

$$\sup_{\|f\|_\Phi \le 1}\left|\left\{x \in \mathbb{T} : \mathcal{R}_D f(x) > \lambda\right\}\right| = \sup_{B\subset D,\, l\in\mathbb{N},\, \|g\|_\Phi\le 1}\left|\left\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g(x) > \lambda\right\}\right|, \tag{4.17}$$

for any $\lambda > 0$, where in sup *finite sets B are considered.*

Proof. Take $f \in L^\Phi(\mathbb{T})$. If τ_l is the mapping satisfying the conditions of Theorem 4 then the functions $g_l(x) = f(\tau_l^{-1}(x))$, $l = 1, 2, \ldots$, satisfy

$$|\{x \in \mathbb{T} : \mathcal{R}_B^l f(x) > \lambda\}| = \{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda\},$$

and $\|f\|_\Phi = \|g_l\|_\Phi$ since τ_l is measure preserving. Taking into account (2.5) we obtain

$$\left|\left\{x \in \mathbb{T} : \mathcal{R}_D f(x) > \lambda\right\}\right|$$

$$\leq \sup_{B\subset D,\, l\in\mathbb{N}} \left|\left\{x \in \mathbb{T} : \mathcal{R}^l_B f(x) > \lambda\right\}\right| = \sup_{B\subset D,\, l\in\mathbb{N}} \left|\left\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda\right\}\right|.$$

Since $f \in L^\Phi$ is arbitrary and $\|f\|_\Phi = \|g_l\|_\Phi$ we get

$$\sup_{\|f\|_\Phi\leq 1} \left|\left\{x \in \mathbb{T} : \mathcal{R}_D f(x) > \lambda\right\}\right| \leq \sup_{B\subset D,\, l\in\mathbb{N},\, \|g\|_\Phi\leq 1} \left|\left\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g(x) > \lambda\right\}\right|.$$

Now suppose $g \in L^\Phi(\mathbb{T}^\infty)$, $B \subset D$ is finite and $l \geq 2$ is arbitrary integer. According to (4.16) there exists $\mathcal{A}_l$-measurable function g_l such that

$$\mathcal{M}_{l/B} g(x) = \mathcal{M}_{l/B} g_l(x). \tag{4.18}$$

According to Theorem 4 for $f_l(x) = g_l(\tau_l(x))$ we have

$$|\{x \in \mathbb{T} : \mathcal{R}^l_B f_l(x) > \lambda\}| = \{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g_l(x) > \lambda\}|. \tag{4.19}$$

From (2.6) we have

$$\mathcal{R}_B f_l(x) = \mathcal{R}^l_B f_l(x).$$

So, using also (4.18), (4.19) and relation $B \subset D$, we get

$$\begin{aligned}\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/B} g(x) > \lambda\}| &= |\{x \in \mathbb{T} : \mathcal{R}^l_B f_l(x) > \lambda\}| \\ &= |\{x \in \mathbb{T} : \mathcal{R}_B f_l(x) > \lambda\}| \leq |\{x \in \mathbb{T} : \mathcal{R}_D f_l(x) > \lambda\}|\end{aligned}$$

and therefore

$$\sup_{B\subset D,\, l\in\mathbb{N},\, \|g\|_\Phi\leq 1} \left|\left\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/D} g(x) > \lambda\right\}\right| \leq \sup_{\|f\|_\Phi\leq 1} \left|\left\{x \in \mathbb{T} : \mathcal{R}_D f(x) > \lambda\right\}\right|.$$

□

5. A covering lemma

The covering lemma we establish in this section is needed to prove Theorem 2. We consider the function

$$\alpha(x) = \begin{cases} x^{x-1}, & \text{if} \quad x > 1, \\ x, & \text{if} \quad 0 \leq x \leq 1. \end{cases} \tag{5.1}$$

This is an increasing continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$. It is easy to observe its inverse satisfies the condition

$$\lim_{x\to\infty} \frac{\alpha^{-1}(x)\ln x}{\ln\ln x} = 1. \tag{5.2}$$

Define the functions

$$\Psi(x) = \int_0^{|x|} \alpha(t)dt, \quad \Phi(x) = \int_0^{|x|} \alpha^{-1}(t)dt, \quad x \in \mathbb{R}. \tag{5.3}$$

These are complementary N-functions (see definition in [11], chap. 1, par. 2). Performing simple estimations we get

$$\frac{x\ln(x/2)}{2\ln\ln(x/2)} < \Phi(x) < \frac{x\ln x}{\ln\ln x}, \quad x > \gamma, \tag{5.4}$$

where γ is an absolute constant. According to the Young's inequality ([11], (2.6)) we have

$$uv \le \Phi(u) + \Psi(v), \quad u > 0, v > 0. \tag{5.5}$$

Everywhere below we will use notation $a \lesssim b$ for the inequality $a \le c \cdot b$ with an absolute constant $c > 0$. The following lemma is a variant of the lemma 3 from [12].

Lemma 4. *If $A_1, A_2, \dots A_n$ and A are independent sets in some probability space and $\sum_{k=1}^n |A_k| \le 1/2$ then*

$$\int_E \Psi\left(\frac{1}{3}\cdot\left(1+\sum_{k=1}^n \mathbb{I}_{A_k}(x)\right)\right) \lesssim |E|, \tag{5.6}$$

where

$$E = A \bigcap \left(\bigcup_{k=1}^n A_k\right). \tag{5.7}$$

Proof. To prove (5.6) it is enough to get

$$m(\lambda) = \left|\left\{x \in A :\, 1 + \sum_{k=1}^n \mathbb{I}_{A_k}(x) > \lambda\right\}\right| \lesssim |E|\cdot\left(\frac{2}{\lambda-1}\right)^{\frac{\lambda-1}{2}}, \quad \lambda > 3.$$

Indeed, using the relation $\Psi'(x) = \alpha(x)$, $x > 0$, as a consequence of (5.3), combined with (5.1), we obtain

$$\int_E \Psi\left(\frac{1}{3}\left(1+\sum_{k=1}^n \mathbb{I}_{A_k}(x)\right)\right) dx = \frac{1}{3}\int_0^\infty \Psi'\left(\frac{\lambda}{3}\right) m(\lambda) d\lambda$$
$$= \frac{1}{3}\int_0^\infty \alpha\left(\frac{\lambda}{3}\right) m(\lambda) d\lambda \lesssim |A| \int_0^\infty \alpha\left(\frac{\lambda}{3}\right)\left(\frac{2}{\lambda}\right)^{\frac{\lambda}{2}} \lesssim |A|.$$

Putting $\delta_k = |A_k|$, we have $\sum_{k=1}^n \delta_k < 1/2$. Then using the independence, we get

$$|E| = |A \cap A_1| + |A \cap (A_2 \setminus A_1)| + \dots + |A \cap (A_n \setminus \cup_{k=1}^{n-1} A_k)|$$
$$= \delta_1|A| + \delta_2|A|(1-|A_1|) + \dots + \delta_n|A|(1 - |\cup_{k=1}^{n-1} A_k|)$$
$$\ge \delta_1|A| + \frac{\delta_2}{2}|A| + \dots + \frac{\delta_n}{2}|A| \ge \frac{1}{2}|A|(\delta_1+\delta_2+\dots+\delta_n).$$

We assume $\lambda > 3$. Hence

$$\begin{aligned} m(\lambda) &= \\ &= \sum_{k=[\lambda]}^{n} \sum_{i_1<\dots<i_k} \left| A \cap A_{i_1} \cap \ldots \cap A_{i_k} \cap \left(\bigcap_{j \notin \{i_1,\ldots,i_k\}} (A_j)^c \right) \right| \\ &= \sum_{k=[\lambda]}^{n} \sum_{i_1<\dots<i_k} |A| \cdot |A_{i_1}| \cdot \ldots \cdot |A_{i_k}| \prod_{j \notin \{i_1,\ldots,i_k\}} (1 - |A_j|) \\ &= |A| \sum_{k=[\lambda]}^{n} \sum_{i_1<\dots<i_k} \delta_{i_1} \cdots \delta_{i_k} \prod_{j \notin \{i_1,\ldots,i_k\}} (1 - \delta_j) \\ &\le |A| \sum_{k=[\lambda]}^{n} \sum_{i_1<\dots<i_k} \delta_{i_1} \cdots \delta_{i_k} \le |A| \sum_{k=[\lambda]}^{\infty} \frac{(\delta_1 + \cdots + \delta_n)^k}{k!} \\ &< |A|(\delta_1 + \cdots + \delta_n) \sum_{k=[\lambda]}^{\infty} \frac{1}{k!} \le 2|E| \sum_{k=[\lambda]}^{\infty} \frac{1}{\left[\frac{k}{2}\right]!\left(\left[\frac{k}{2}\right]+1\right)\cdots k} \\ &\le 2|E| \left(\frac{2}{\lambda - 1} \right)^{\frac{\lambda-1}{2}} \sum_{k=[\lambda]}^{\infty} \frac{1}{\left[\frac{k}{2}\right]!} \lesssim |E| \left(\frac{2}{\lambda - 1} \right)^{\frac{\lambda-1}{2}}. \end{aligned}$$

The proof is complete. □

For a set of indexes $S \subset \mathbb{N}$ we denote by $\mathcal{R}(S)$ the algebra generated by the rectangles (2.9) with $l_i = 0$, $i \notin S$. For any set $R \subset \mathcal{R}$ we define its spectrum $\operatorname{sp}(R)$ to be the smallest set of indexes S for which $R \subset \mathcal{R}(S)$. That is

$$\operatorname{sp}(R) = \bigcap_{S:\, R \in \mathcal{R}(S)} S. \tag{5.8}$$

It is easy to observe

(5.9)
if $\operatorname{sp}(B_1), \ldots, \operatorname{sp}(B_k)$ are mutually disjoint, then $B_1, \ldots, B_k$ are independent,

(5.10)
if $\operatorname{sp}(R) \subseteq \operatorname{sp}(Q)$ and $Q \not\subseteq R$ then $R \cap Q = \varnothing$.

We denote

$$l = l_d = p_1 p_2 \ldots p_d, \tag{5.11}$$

$$E_d = \{m \in \mathbb{N} : \, m = p_\nu p_\mu p_{\mu+1} \ldots p_d, \, 1 \le \nu < \mu\}. \tag{5.12}$$

Let $\mathcal{F}_d$ be the family of all rectangles from $\mathcal{B}_l$ defined

$$\mathcal{F}_d = \{B_m(j_1, \ldots, j_d) : \, m \in E_d\} \tag{5.13}$$

According to (5.13) any $B \in \mathcal{F}_d$ has the form

$$B = \left\{ x \in \mathbb{T}^\infty : \, \frac{j_k}{p_k} \le x_k < \frac{j_k + 1}{p_k}, \, k \in \{\nu\} \cup \{\mu, \mu+1, \ldots, d\} \right\}, \tag{5.14}$$

where $0 \le j_k < p_k$, $k = 1, 2, \ldots$. In the case $\mu = d + 1$ we understand $\{\mu, \mu + 1, \ldots, d\} = \varnothing$. As μ and ν in (5.14) are uniquely determined for a given $B \in \mathcal{F}_d$, sometimes we will use $\mu(B)$, $\nu(B)$ for them. We define the base $\mathrm{bs}\,(B)$ and the tail $\mathrm{tl}\,(B)$ of B by

$$\mathrm{bs}\,(B)=\left\{x\in\mathbb{T}^\infty:\ \frac{j_k}{p_k}\le x_k<\frac{j_k+1}{p_k},\ k\in\{\mu,\mu+1,\ldots,d\}\right\},\tag{5.15}$$

and

$$\mathrm{tl}\,(B)=\left\{x\in\mathbb{T}^\infty:\ \frac{j_\nu}{p_\nu}\le x_\nu<\frac{j_\nu+1}{p_\nu}\right\}.\tag{5.16}$$

Obviously for any $B\in\mathcal{F}_d$ we have

$$B=\mathrm{bs}\,(B)\cap\mathrm{tl}\,(B).\tag{5.17}$$

Observe that if $A,B\in\mathcal{F}_d$ then

$$\mathrm{bs}\,(A)\cap\mathrm{bs}\,(B)\neq\varnothing \quad\Rightarrow\quad \mathrm{bs}\,(A)\subseteq\mathrm{bs}\,(B)\text{ or }\mathrm{bs}\,(B)\subseteq\mathrm{bs}\,(A),\tag{5.18}$$

$$\mathrm{bs}\,(A)\subset\mathrm{bs}\,(B) \quad\Rightarrow\quad \mu(A)<\mu(B),\tag{5.19}$$

$$\mathrm{bs}\,(A)\subset\mathrm{bs}\,(B),\quad A\not\subset B \quad\Rightarrow\quad \mathrm{tl}\,(A)\neq\mathrm{tl}\,(B).\tag{5.20}$$

Lemma 5. *Any collection of rectangles $\Theta=\{A_\alpha\}\subset\mathcal{F}_d$ contains a finite subcollection $\tilde\Theta=\{\tilde A_1,\ldots,\tilde A_m\}$ with*

$$\left|\bigcup_{j=1}^m\tilde A_j\right|\ge\frac{1}{5}\left|\bigcup_\alpha A_\alpha\right|,\tag{5.21}$$

$$\int_{\mathbb{T}^\infty}\Psi\left(\frac{1}{3}\sum_{j=1}^m\mathbb{I}_{\tilde A_j}(x)\right)dx\lesssim 1.\tag{5.22}$$

Proof. Since $\mathcal{F}_d$ is finite and $\tilde\Theta\subset\mathcal{F}_d$ we can assume $\Theta=\{A_1,A_2,\ldots,A_n\}$ and $\mu(A_i)\ge\mu(A_{i+1})$ for any i. The subcollection $\tilde\Theta$ will be chosen from $\{A_1,A_2,\ldots,A_n\}$ as follows. We choose $\tilde A_1=A_1$. If the sets $\tilde A_1=A_{l_1},\ldots,\tilde A_k=A_{l_{k-1}}$ with $l_1<\ldots<l_{k-1}$ have been chosen then we select $\tilde A_k$ to be the first set among $A_{l_{k-1}+1},\ldots,A_n$ satisfying the conditions

$$\tilde A_k\not\subset\tilde A_1\cup\ldots\cup\tilde A_{k-1},\tag{5.23}$$

$$\left|\bigcup_{j\le k,\mathrm{tl}\,(\tilde A_j)\cap\mathrm{bs}\,(\tilde A_k)\neq\varnothing,\ \mathrm{bs}\,(\tilde A_j)\supseteq\mathrm{bs}\,(\tilde A_k)}\mathrm{tl}\,(\tilde A_j)\right|<\frac{3}{4}.\tag{5.24}$$

This process generates a sequence $\tilde A_1,\tilde A_2,\ldots,\tilde A_m$. According to (5.24), for any fixed k we have

$$\left|\bigcup_{1\le j\le m,\ \mathrm{tl}\,(\tilde A_j)\cap\mathrm{bs}\,(\tilde A_k)\neq\varnothing,\ \mathrm{bs}\,(\tilde A_j)\supseteq\mathrm{bs}\,(\tilde A_k)}\mathrm{tl}\,(\tilde A_j)\right|<\frac{3}{4}.\tag{5.25}$$

We consider a base $U=\mathrm{bs}\,(\tilde A_k)$ satisfying the inequality

$$\left|\bigcup_{\mathrm{tl}\,(\tilde A_j)\cap U\neq\varnothing,\ \mathrm{bs}\,(\tilde A_j)\supseteq U}\mathrm{tl}\,(\tilde A_j)\right|\ge\frac{1}{4}.\tag{5.26}$$

It is easy to observe that from

$$\operatorname{tl}(\tilde{A}_j)\cap U\neq\varnothing,\quad \operatorname{bs}(\tilde{A}_j)\supseteq U,$$

it follows that $\nu(\tilde{A}_j)<\mu(\tilde{A}_k)$. Therefore the sets

$$U,\qquad \bigcup_{\operatorname{bs}(\tilde{A}_j)\supseteq U}\operatorname{tl}(\tilde{A}_j)$$

have disjoint spectrums and so they are independent according to (5.9). Thus, using (5.26) we conclude

$$\begin{aligned}(5.27)\qquad \left|U\bigcap\left(\bigcup_{j=1}^m\tilde{A}_j\right)\right| &\geq \left|U\bigcap\left(\bigcup_{\operatorname{bs}(\tilde{A}_j)\supseteq U}\tilde{A}_j\right)\right|\\ &=\left|U\bigcap\left(\bigcup_{\operatorname{bs}(\tilde{A}_j)\supseteq U}\operatorname{tl}(\tilde{A}_j)\right)\right|=|U|\cdot\left|\bigcup_{\operatorname{bs}(\tilde{A}_j)\supseteq U}\operatorname{tl}(\tilde{A}_j)\right|\geq\frac{1}{4}|U|.\end{aligned}$$

We denote by $U_1,U_2,\ldots,U_\gamma$ the family of all maximal bases $U=\operatorname{bs}(\tilde{A}_k)$ satisfying (5.26). It is clear they are mutually disjoint their union contains all U satisfying (5.26). Thus, using (5.27) we get

$$(5.28)\qquad \left|\bigcup_{i=1}^{\gamma}U_i\right|\leq 4\left|\bigcup_{j=1}^m\tilde{A}_j\right|.$$

Now suppose A_t is an arbitrary set which is not in the subcollection $\{\tilde{A}_k\}$. We have $l_{k-1}<t<l_k$ for some k. According to the process of the selection we have either

$$(5.29)\qquad A_t\subset\bigcup_{i=1}^{k-1}\tilde{A}_i$$

or

$$\left|\operatorname{tl}(\tilde{A}_t)\bigcup\left(\bigcup_{j<k,\operatorname{tl}(\tilde{A}_j)\cap\operatorname{bs}(\tilde{A}_t)\neq\varnothing,\,\operatorname{bs}(\tilde{A}_j)\supseteq\operatorname{bs}(\tilde{A}_t)}\operatorname{tl}(\tilde{A}_j)\right)\right|\geq\frac{3}{4}.$$

Since $\operatorname{tl}(\tilde{A}_t)\leq\frac{1}{2}$ we obtain

$$\left|\bigcup_{j<k,\operatorname{tl}(\tilde{A}_j)\cap\operatorname{bs}(\tilde{A}_t)\neq\varnothing,\,\operatorname{bs}(\tilde{A}_j)\supseteq\operatorname{bs}(\tilde{A}_t)}\operatorname{tl}(\tilde{A}_j)\right|\geq\frac{1}{4},$$

which means $\operatorname{bs}(A_t)\subseteq U=\operatorname{bs}(\tilde{A}_{k-1})$ where U satisfies (5.26). Hence we have either (5.29) or

$$A_t\subset\cup_{i=1}^{\gamma}U_i,$$

and therefore, applying (5.28), we get

$$\left|\bigcup_t A_t\right|\leq\left|\bigcup_{j=1}^m\tilde{A}_j\right|+\left|\bigcup_{i=1}^{\gamma}U_i\right|\leq 5\left|\bigcup_{j=1}^m\tilde{A}_j\right|.$$

which gives (5.21). To prove (5.22) denote

$$B_k = \mathrm{bs}\,(\tilde{A}_k) \setminus \left(\bigcup_{\mathrm{bs}\,(\tilde{A}_i) \subset \mathrm{bs}\,(\tilde{A}_k)} \mathrm{bs}\,(\tilde{A}_i) \right), \quad k = 1, 2, \ldots, m. \tag{5.30}$$

It is clear $B_1, B_2, \cdots, B_m$ are pairwise disjoint. We note some of this sets can be empty. Using (5.17) we have

$$\bigcup_{k=1}^{m} B_k = \bigcup_{k=1}^{m} \mathrm{bs}\,(\tilde{A}_k) \supset \bigcup_{k=1}^{m} \tilde{A}_k.$$

Thus, to obtain (5.22), it is enough to prove

$$I_k = \int_{B_k} \Psi \left(\frac{1}{3} \sum_{j=1}^{m} \mathbb{I}_{\tilde{A}_j}(x) \right) dx \lesssim |B_k|. \tag{5.31}$$

Observe that

$$I_k = \int_{B_k} \Psi \left(\frac{1}{3} \sum_{j:\, \mathrm{bs}\,(\tilde{A}_j) \supseteq \mathrm{bs}\,(\tilde{A}_k)} \mathbb{I}_{\tilde{A}_j}(x) \right) dx. \tag{5.32}$$

Indeed, according to (5.18), any $\tilde{A}_j$ satisfies one of the relations

$$\mathrm{bs}\,(\tilde{A}_j) \cap \mathrm{bs}\,(\tilde{A}_k) = \varnothing, \tag{5.33}$$

$$\mathrm{bs}\,(\tilde{A}_j) \subset \mathrm{bs}\,(\tilde{A}_k), \tag{5.34}$$

$$\mathrm{bs}\,(\tilde{A}_j) \supseteq \mathrm{bs}\,(\tilde{A}_k). \tag{5.35}$$

In the case (5.33) or (5.34), using (5.30), we have $\tilde{A}_j \cap B_k = \varnothing$. So the integral (5.31) depends only on the sets $\tilde{A}_j$ with (5.35), which implies (5.32). If $\mathrm{bs}\,(\tilde{A}_j) \supseteq \mathrm{bs}\,(\tilde{A}_k)$ then by (5.30) $\mathrm{bs}\,(\tilde{A}_j) \supseteq B_k$. Thus, such that $\tilde{A}_j = \mathrm{bs}\,(\tilde{A}_j) \cap \mathrm{tl}\,(\tilde{A}_j)$ (see (5.17)) from (5.32) we get

$$I_k = \int_{B_k} \Psi \left(\frac{1}{3} \sum_{\mathrm{bs}\,(\tilde{A}_j) \supseteq \mathrm{bs}\,(\tilde{A}_k)} \mathbb{I}_{\mathrm{tl}\,(\tilde{A}_j)}(x) \right) dx.$$

Now denote

$$C_\nu = \bigcup_{j:\, \nu(\tilde{A}_j) = \nu,\, \mathrm{tl}\,(\tilde{A}_j) \cap \mathrm{bs}\,(\tilde{A}_k) \neq \varnothing,\, \mathrm{bs}\,(\tilde{A}_j) \supseteq \mathrm{bs}\,(\tilde{A}_k)} \mathrm{tl}\,(\tilde{A}_j) \tag{5.36}$$

and consider all nonempty sets $C_{\nu_1}, C_{\nu_2}, \ldots, C_{\nu_p}$, with decreasing numbering $\nu_1 > \nu_2 > \ldots > \nu_p$. From (5.25) it follows that

$$\left| \bigcup_{i=1}^{p} C_{\nu_i} \right| < \frac{3}{4}. \tag{5.37}$$

Observe that if the sets $\tilde{A}_j$ and $\tilde{A}_i$ satisfy the relations

$$\mathrm{bs}\,(\tilde{A}_j) \supseteq \mathrm{bs}\,(\tilde{A}_i) \text{ and } \nu(\tilde{A}_j) \geq \mu(\tilde{A}_i) \tag{5.38}$$

then

$$\mathrm{tl}\,(\tilde{A}_j) \cap \mathrm{bs}\,(\tilde{A}_i) = \varnothing. \tag{5.39}$$

Indeed, from (5.38) and the definition of the set $\mathcal{F}_d$ in (5.13) it follows that

$$\mathrm{sp}\,(\tilde{A}_j) \subseteq \{\nu(\tilde{A}_j), \nu(\tilde{A}_j)+1, \dots d\} \subseteq \{\mu(\tilde{A}_i), \mu(\tilde{A}_i)+1, \dots d\} = \mathrm{sp}\,(\mathrm{bs}\,(\tilde{A}_i)).$$

Thus, using (5.10) we will have either $\tilde{A}_j \supseteq \mathrm{bs}\,(\tilde{A}_i) \supset \tilde{A}_i$ or $\tilde{A}_j \cap \mathrm{bs}\,(\tilde{A}_i) = \varnothing$. The first inclusion is not possible because of (5.23). So we have $\tilde{A}_j \cap \mathrm{bs}\,(\tilde{A}_i) = \varnothing$. Therefore, since $\tilde{A}_j = \mathrm{bs}\,(\tilde{A}_j) \cap \mathrm{tl}\,(\tilde{A}_j)$ and $\mathrm{bs}\,(\tilde{A}_j) \supseteq \mathrm{bs}\,(\tilde{A}_i)$ (see (5.38)) we get (5.39). Combining (5.39) with (5.36) we get

$$C_{\nu_j} \cap \mathrm{bs}\,(\tilde{A}_i) = \varnothing,$$

provided

$$\mathrm{bs}\,(\tilde{A}_k) \supseteq \mathrm{bs}\,(\tilde{A}_i), \quad \mu(\tilde{A}_i) \le \nu_j.$$

Therefore by (5.30)

$$\begin{aligned} &B_k \cap (C_{\nu_j} \setminus \cup_{s=1}^{j-1} C_{\nu_s}) \\ &\qquad = \left(\mathrm{bs}\,(\tilde{A}_k) \setminus \bigcup_{\mathrm{bs}\,(\tilde{A}_i) \subset \mathrm{bs}\,(\tilde{A}_k),\, \mu(\tilde{A}_i) > \nu_j} \mathrm{bs}\,(\tilde{A}_i) \right) \cap (C_{\nu_j} \setminus \cup_{s=1}^{j-1} C_{\nu_s}). \end{aligned}$$

Since $\mathrm{sp}\,(C_{\nu_s}) = \nu_s$, $\nu_p < \nu_{p-1} < \dots < \nu_1$ and $\mathrm{sp}\,(\mathrm{bs}\,(\tilde{A}_i)) = \{\mu(\tilde{A}_i), \mu(\tilde{A}_i) + 1, \dots, d\}$ (see (5.15)), each set on the right has spectrum in $\{\nu_j, \nu_j+1, \dots, d\}$. So we have

$$\mathrm{sp}\,\big(B_k \cap (C_{\nu_j} \setminus \cup_{s=1}^{j-1} C_{\nu_s})\big) \subset \{\nu_j, \nu_j+1, \dots, d\}.$$

Hence the sets

$$B_k \cap (C_{\nu_i} \setminus \cup_{s=1}^{i-1} C_{\nu_s}), C_{\nu_{i+1}}, \dots, C_{\nu_p}$$

have mutually disjoint spectrums, so they are independent by (5.9). According to (5.37) these sets satisfy the hypothesis of Lemma 4. Hence, applying (5.6), we get

$$\int_{B_k \cap (C_{\nu_i} \setminus \cup_{s=1}^{i-1} C_{\nu_s})} \Psi\left(\frac{1}{3} \left(1 + \sum_{t=i+1}^{p} \mathbb{I}_{C_{\nu_t}}(x) \right) \right) dx \lesssim |B_k \cap (C_{\nu_i} \setminus \cup_{s=1}^{i-1} C_{\nu_s})|$$

and therefore

$$\begin{aligned} I_k &= \int_{B_k} \Psi\left(\frac{1}{3} \sum_{i=1}^{p} \mathbb{I}_{C_{\nu_i}}(x) \right) dx \\ &= \sum_{i=1}^{p} \int_{B_k \cap (C_{\nu_i} \setminus \cup_{s=1}^{i-1} C_{\nu_s})} \Psi\left(\frac{1}{3} \left(1 + \sum_{t=i+1}^{p} \mathbb{I}_{C_{\nu_t}}(x) \right) \right) dx \\ &\lesssim \sum_{i=1}^{p} |B_k \cap (C_{\nu_i} \setminus \cup_{s=1}^{i-1} C_{\nu_s})| \le |B_k|, \end{aligned}$$

where $C_{\nu_0} = \varnothing$. Hence the inequality (5.31) and so the lemma is proved. □

In the following lemma $E \subset Z$ is the set defined in (1.5) and $\mathcal{M}_{l/E} f(x)$ is the maximal function from (4.15).

Lemma 6. *If $\Phi(t)$ is the function from (5.3) then*

$$|\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/E} f(x) > \lambda\}| \lesssim \frac{1}{\lambda} \left(1 + \int_{\mathbb{T}^\infty} \Phi(f(t)) dt \right), \quad \lambda > 0, \tag{5.40}$$

for any $f \in L^\Phi(\mathbb{T}^\infty)$ and $l \in \mathbb{N}$.

Proof. We suppose l has the factorization (5.11). From (5.12) and (1.5) we get $l/E = E_d$. So taking into account (5.13) we have

$$\mathcal{M}_{l/E}f(x) = \sup_{F\in\mathcal{F}_d\colon F\ni x} \frac{1}{|F|}\int_F |f(t)|dt, \quad x\in\mathbb{T}^\infty, \quad f\in L^1\big(\mathbb{T}^\infty\big).$$

Hence, for any $\lambda > 0$ there exists a collection $F = \{F_k\}$ from $\mathcal{F}_d$ such that

$$\{x\in\mathbb{T}^\infty : \mathcal{M}_{l/E}f(x) > \lambda\} = \bigcup_k F_k,$$
$$\frac{1}{|F_k|}\int_{F_k} f(t)dt > \lambda.$$

According to Lemma 5 we can choose a subfamily $\{\tilde{F}_k\}$ such that

$$\left|\bigcup_k \tilde{F}_k\right| \ge \frac{1}{5}\left|\bigcup_k F_k\right|, \tag{5.41}$$

$$\int_{\mathbb{T}^\infty} \Psi\left(\sum_k \mathbb{I}_{\tilde{F}_k}(x)\right)dx \lesssim 1. \tag{5.42}$$

Thus, applying (5.41),(5.42) and (5.5) we obtain

$$\begin{aligned}
&|\{x\in\mathbb{T}^\infty : \mathcal{M}_{l/E}f(x) > \lambda\}|\\
&\quad = \left|\bigcup_k F_k\right| \le 5\sum_k |\tilde{F}_k| \le \sum_k \frac{5}{\lambda}\int_{\tilde{F}_k} f(t)dt = \frac{5}{\lambda}\int_{\mathbb{T}^\infty} f(t)\sum_k \mathbb{I}_{\tilde{F}_k}(t)dt\\
&\le \frac{5}{\lambda}\left(\int_{\mathbb{T}^\infty}\Phi(f(x))dx + \int_{\mathbb{T}^\infty}\Psi\left(\sum_k \mathbb{I}_{\tilde{F}_k}(x)\right)dx\right) \lesssim \frac{5}{\lambda}\left(1+\int_{\mathbb{T}^\infty}\Phi(f(t))dt\right).
\end{aligned}$$

□

6. Proofs of Theorems

To avoid of the repetition of the same standard argument in the proofs of the theorems we will use E. M. Stein's well-known weak type maximal functions principle (see. [13] or [14] chap. X, par. 3.6). Consider a sequence of convolution operators

$$T_j = f * \mu_j : L^1(\mathbb{T}) \to \{\text{measurable functions on } \mathbb{R}\}$$

where μ_j are positive finite measures on $\mathbb{T}$.

Lemma 7 (E. M. Stein). *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ to be an increasing convex function such that $\Phi(\sqrt{x})$ is concave. Then if for every $f \in \Phi(L)$*

$$Mf(x) = \sup_j |T_j f(x)| < \infty$$

on a set of positive measure then

$$|\{x\in\mathbb{R} : Mf(x) > \lambda\}| \le \int_{\mathbb{R}} \Phi\left(\frac{c|f|}{\lambda}\right), \quad \lambda > 0,$$

where $c > 0$ is a constant.

Proof of Theorem 2. We suppose $B \subset E$ is an arbitrary finite set. If l is a multiple for the members of B then $l/B \subset l/E$, and so by (4.15) we obtain

$$\mathcal{M}_{l/B} f(x) \le \mathcal{M}_{l/E} f(x).$$

Hence, according to (5.40) we have

$$|\{x \in \mathbb{T} : \mathcal{M}_{l/B} f(x) > \lambda\}| < \frac{c}{\lambda}\left(1 + \int_{\mathbb{T}} \Phi(f(t))dt\right),$$

for any finite $B \subset E$ and $f \in L^\Phi$. Combining this with the corollary after Theorem 4 we obtain

$$|\{x \in \mathbb{T} : \mathcal{R}_E f(x) > \lambda\}| \le \frac{c}{\lambda}\left(1 + \int_{\mathbb{T}^\infty} \Phi(f(t))dt\right), \tag{6.1}$$

where $c > 0$ is an absolute constant. We have each $B_m f(x)$ is a convolution operator with the kernel

$$\mu_m = \frac{1}{m}\sum_{i=1}^{m} \delta_{i/m}$$

where δ_a is the unit measure (Dirac function) concentrated at a. It is easy to check as well Φ satisfies the hypothesis of Stein's lemma. Therefore applying the Stein's principle from (6.1) we get (1.7). The proof is thus complete. $\square$

Suppose $f(x) \in L^1(\mathbb{T})$, D is a finite set of naturals and l is a common multiple for the members of D. Consider the conditional expectation $E^{\mathcal{I}_l} f(x)$ of the function $f(x)$ with respect the algebra $\mathcal{I}_l$ defined. For any convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ we have

$$\|E^{\mathcal{I}_l} f(x)\|_\phi \le \|f\|_\phi. \tag{6.2}$$

To deduce *everywhere* divergence in Theorem 1 and Theorem 3 we use the following general lemma.

Lemma 8. *Let D be a set of indexes and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex increasing function. If there exists a function $f \in L^\phi$ such that $\mathcal{R}_D f(x) = \infty$ on a set of positive measure, then it can be found a function $\tilde{f} \in L^\phi$ with $\mathcal{R}_D \tilde{f}(x) = \infty$ everywhere.*

Proof. Suppose for some $f \ge 0$ we have

$$\mathcal{R}_D f(x) = \infty, \quad x \in E,$$

and $|E| > 0$. According to Borel-Cantelli lemma (see. [14], p. 442 or [5], section XIII, 1.24) there exists a sequence $x_k \in \mathbb{T}$ such that $\sum_k \mathbb{I}_{E+x_k}(x) = \infty$ a.e.. Denoting $\tilde{f}(x) = \sum_k 2^{-k} f(x + x_k)$, we get $\tilde{f} \in L^\phi$ and

$$\mathcal{R}_D \tilde{f}(x) = \infty \text{ a.e. .}$$

Hence by (2.5) there exist a sequence of finite sets $D_1 \subset D_2 \subset \dots$ with $\cup_n D_n = D$ and a integers l_n divided by the members of D_n such that

$$|\{x \in \mathbb{T} : \mathcal{R}^{l_n}_{D_n} \tilde{f}(x) > n^3\}| > 1 - \frac{1}{\phi(n^3)}.$$

Since $\mathcal{R}^{l_n}_{D_n} \tilde{f}(x)$ is $\mathcal{I}_{l_n}$-measurable, so the set

$$A_n = \{x \in \mathbb{T} : \mathcal{R}^{l_n}_{D_n} \tilde{f}(x) > n^3\}$$

is. Hence we get

$$|A_n^c| \le 1/\phi(n^3),$$
$$\mathcal{R}_{D_n}^{l_n}\mathbb{I}_{A_n^c}(x) = 1, \quad x \in A_n^c.$$

Thus, denoting

$$f_n(x) = \tilde{f}(x) + n^3 \cdot \mathbb{I}_{A_n^c}(x),$$

we have

$$\|f_n\|_\phi \le \|f_n\|_\phi + \|n^3 \cdot \mathbb{I}_{A_n^c}\|_\phi = \|f_n\|_\phi + 1 = \|\tilde{f}\|_\phi + 1,$$
$$\mathcal{R}_{D_n}^{l_n} f_n(x) > n^3, \text{ for all } x \in \mathbb{T}^\infty.$$

Now denote

$$g(x) = \sum_{n=1}^\infty \frac{1}{n^2} \cdot E^{\mathcal{I}_{l_n}} f_n(x).$$

According to (6.2) we have

$$\|g\|_\phi \le \sum_{n=1}^\infty \frac{1}{n^2} \cdot \|E^{\mathcal{I}_{l_n}} f_n\|_\phi \le \sum_{n=1}^\infty \frac{1}{n^2} \cdot \|f_n\|_\phi < \infty,$$

and using (2.6) we get

$$\mathcal{R}_D g(x) \ge \mathcal{R}_{D_n} g(x) \ge \frac{1}{n^2}\mathcal{R}_{D_n} E^{\mathcal{I}_{l_n}} f_n(x) = \frac{1}{n^2}\mathcal{R}_{D_n}^{l_n} f_n(x) > n, \quad x \in \mathbb{T}^\infty,$$

for any $n \in \mathbb{N}$, i.e. $\mathcal{R}_D g(x) = \infty$ everywhere on $\mathbb{T}$. The proof is complete. □

Proof of Theorem 3. We consider the rectangles

$$B_i^k = \left\{ x \in \mathbb{T}^\infty : \frac{i}{p_k} \le x_k < \frac{i+1}{p_k} \right\}, \quad i = 0, 1, \ldots, p_k - 1.$$

Since $\operatorname{sp}(B_i^k) = \{k\}$ we have $B_i^k \in \mathcal{F}_{2d}$ if $1 \le k \le 2d$. Denote

$$G_k = \bigcup_{0 \le i < \left[\frac{p_k}{d}\right]} B_i^k, \quad k = 1, 2, \ldots, 2d, \tag{6.3}$$

$$G = \bigcup_{k=d+1}^{2d} G_k, \quad C = \bigcap_{k=d+1}^{2d} G_k. \tag{6.4}$$

(6.5)

It is clear $p_{d+1} > 2d$. Since the number of B_i^k in the union (6.3) is $\left[\frac{p_k}{d}\right]$ and $|B_i^k| = 1/p_k$ we conclude

$$\frac{1}{d} \ge |G_k| = \left[\frac{p_k}{d}\right]\frac{1}{p_k} > \frac{1}{d}\left(1 - \frac{d}{p_k}\right) > \frac{1}{2d}, \text{ if } k > d. \tag{6.6}$$

Because of independence of the sets G_k we get

(6.7)

$$|G| = |\bigcap_{k=d+1}^{2d} G_k| = 1 - \prod_{k=d+1}^{2d}(1-|G_k|) > 1-(1-(2d)^{-1})^d > 1 - \frac{1}{\sqrt{e}} > \frac{1}{3},$$

(6.8)

$$|C| = \prod_{k=d+1}^{2d} |G_k| \le d^{-d}.$$

Choose an arbitrary $x \in G$. We have $x \in G_k$ for some k and therefore $x \in B_i^k$ for some $0 \le i < \left[\frac{p_k}{d}\right]$ and $d < k \le 2d$. On the other hand, using (6.6) and the independence of the sets G_j, $d < j \le 2d$, $j \ne k$, with B_i^k, we obtain

$$|C \bigcap B_i^k| = \left|\left(\bigcap_{d<j\le 2d,\, j\ne k} G_k\right)\bigcap B_i^k\right| = |B_i^k| \prod_{d<j\le 2d,\, j\ne k} |G_k| > \frac{|B_i^k|}{(2d)^{d-1}}.$$

From this we get

$$\frac{1}{|B_i^k|}\int_{B_i^k} \mathbb{I}_C(x)dx > (2d)^{1-d}.$$

So we conclude

$$\mathcal{M}_{l_{2d}/E}\mathbb{I}_C(x) > (2d)^{1-d}, \quad x \in G, \tag{6.9}$$

where l_{2d} is defined in (5.11). Taking into account (1.8) and (6.8), we have

$$\int_{\mathbb{T}^\infty} \phi((2d)^{d-1}\mathbb{I}_C(x))dx = \phi((2d)^{d-1})|C| < d^{-d}\phi((2d)^{d-1}) \overset{d\to\infty}{\to} 0,$$

Thus, we may find a sequence $c_d \to \infty$ such that the function

$$g_d(x) = c_d(2d)^{d-1}\mathbb{I}_C(x). \tag{6.10}$$

satisfies

$$\int_{\mathbb{T}^\infty} \phi(g_d(x))dx \le 1.$$

From (6.10) we get

$$\mathcal{M}_{l_{2d}/E}g_d(x) = c_d(2d)^{d-1}\mathcal{M}_{l_{2d}/E}\mathbb{I}_C(x)$$

and so, using (6.7) and (6.9), we obtain

$$|\{x \in \mathbb{T}^\infty : \mathcal{M}_{l_{2d}/E}g_d(x) > c_d\}| = |\{x \in \mathbb{T}^\infty : \mathcal{M}_{l_{2d}/E}\mathbb{I}_C(x) > (2d)^{1-d}\}| \ge |G| > \frac{1}{3}.$$

Applying (4.17) we may find sequence of functions f_d on $\mathbb{T}$ with

$$\|f_d\|_\Phi = \|g_d\|_\Phi \le \int_{\mathbb{T}^\infty} \phi(g_d(x))dx \le 1$$

such that

$$|\{x \in \mathbb{T} : \mathcal{R}_E f_d(x) > c_d\}| > \frac{1}{3}.$$

Hence, according to Stein's principle there exists a function $f \in L^\Phi(\mathbb{T})$ such $\mathcal{R}_E f(x) = \infty$ a.e.. To get everywhere divergence it remains to use Lemma 8. Theorem 3 is proved. □

The proof of Theorem 1 is based on some results in the Theory of Differentiation of Integrals in $\mathbb{R}^n$. According to well known Jessen-Marcinkiewicz-Zygmund theorem (see [15] or [16] chapter 2)

$$\lim_{\operatorname{diam} R\to 0,\, x\in R}\frac{1}{|R|}\int_R f(t)dt = f(x), \text{ a.e} \tag{6.11}$$

for any $f\in L\log^{n-1}L(\mathbb{R}^n)$, where R are rectangles with sides parallel to the axis. On the other hand S. Saks in [17] has proved that in this theorem the Orlicz class $L\log^{d-1}L$ is the optimal. Certainly the relation (6.11) is true also if we consider the rectangles (2.9) with fixed d instead of all rectangles in $\mathbb{R}^n$. As for the divergence theorem the proof is not immediate. However there is a generalization of Saks theorem due A. Stokolos [18](see also [19]). According to this theorem if ϕ satisfies (1.8) then there exists a function $f\in L^\phi(\mathbb{R}^n)$ such that

$$\lim_{\operatorname{diam} R\to 0,\, x\in R}\frac{1}{|R|}\int_R f(t)dt = \infty, \tag{6.12}$$

for any $x\in\mathbb{R}^n$, where R are the rectangles of the form (2.9) with fixed d. Moreover, it can be taken any integers greater than or equal 2 instead of primes $p_1,p_2,\ldots,p_d$. We note that all this theorems can be stated also on $\mathbb{T}^\infty$.

Proof of Theorem 1. Suppose D is the set of all integers of the form

$$p_1^{m_1}p_2^{m_2}\ldots p_d^{m_d},\quad m_k\in\mathbb{Z}^+,\, k=1,2,\ldots d.$$

Consider a sequence of subsets $D_n\subset D$ defined

$$D_n=\{m=p_1^{m_1}p_2^{m_2}\ldots p_d^{m_d}:\, 0\le m_k\le n,\, k=1,2,\ldots d\},$$

and denote

$$l_n=p_1^np_2^n\ldots p_d^n.$$

We have $\cup_n D_n = D$ and $l_n/D_n = D_n$. Therefore if the function $f\in L^\phi(\mathbb{T}^\infty)$ satisfies the condition (6.12) then

$$\lim_{k\to\infty}\mathcal{M}_{l_n/D_n}f(x)=\infty, \text{ a.e on } \mathbb{T}^\infty.$$

Applying (4.17), we get $\mathcal{R}_Dg_n(x)\to\infty$ a.e. for a sequence of functions g_n with $\|g_n\|_\Phi\le 1$. Using Stein's principle, we will get a function g with $\mathcal{R}_Dg(x)=\infty$ a.e., and the existence of a function with *everywhere* divergence Riemann sums follows from Lemma 8. □

7. On Rudin's theorem and sweeping out properties

In this section we establish equivalency between strong sweeping out and δ-sweeping out properties of operator sequences, which seems to be interesting in view of the papers [20],[21]. Then we will deduce Rudin's theorem in general settings from Theorem 4.

Let (X,m) be a probability space. We consider linear operators

$$T: L^1(X,m)\to\{\text{measurable functions on X}\}. \tag{7.1}$$

Definition 1. *A sequence of linear operators T_n is said to be strong sweeping out if given $\varepsilon>0$ there is a set E with $mE<\varepsilon$ such that* $\limsup_{n\to\infty}T_n\mathbb{I}_E(x)=1$ *a.e. and* $\liminf_{n\to\infty}T_n\mathbb{I}_E(x)=0$ *a.e..*

Definition 2. *Let $0 < \delta \le 1$. A sequence of linear operators T_n is said to be δ-sweeping out if given $\varepsilon > 0$ there is a set $E \subset X$ with $mE < \varepsilon$ such that $\limsup_{n\to\infty} T_n \mathbb{I}_E(x) \ge \delta$ a.e..*

Definition 3. *Let $0 < \delta \le 1$. A sequence of linear operators T_n is said to be weak δ-sweeping out if given $r > 0$ there is a set E such that*

$$m\{x \in X : \sup_{n\in\mathbb{N}} T_n \mathbb{I}_E(x) \ge \delta\} > r \cdot mE.$$

It turns out that these definitions are equivalent for the sequences of linear operators having the following settings

(1) if $f \ge 0$ then $Tf \ge 0$,
(2) $T(\mathbb{I}_X) = 1$,
(3) for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subset X$ and $m(E) < \delta$ then

$$m\{x \in X; T\mathbb{I}_E(x) > \varepsilon\} < \varepsilon.$$

Theorem 5. *If the sequence of linear operators T_n satisfying (1)-(3) is δ-sweeping out for any $0 < \delta < 1$ then it is strong sweeping out.*

Proof. Assume $\{T_n\}$ satisfies the hypothesis of the theorem. Using a standard argument, one can easily choose a sequence of integers $1 = n_0 < n_1 < n_2 < \ldots$ and measurable sets $E_k \subset X$ such that

$$mE_k < \varepsilon 2^{-k}, \tag{7.2}$$

$$m\{x \in X; \sup_{n_{k-1}\le m\le n_k} T_m \mathbb{I}_{E_k}(x) > 1 - 2^{-k}\} > 1 - 2^{-k}, \tag{7.3}$$

$$m\{x \in X; \sup_{n_{k-1}\le m\le n_k} T_m \left(\sum_{j=k+1}^{\infty} \mathbb{I}_{E_j}(x)\right) > 2^{-k}\} < 2^{-k}. \tag{7.4}$$

The selection of n_k and E_k is realized in this order: $E_1, n_1, E_2, n_2, \ldots$. To avoid big expressions we use the notation $U_k = \sup_{n_{k-1}\le m\le n_k} T_m$. Denote

$$\begin{aligned}
&\tilde{E}_k = E_k \setminus \cup_{j=k+1}^{\infty} E_j, \quad E = \cup_{j=0}^{\infty} \tilde{E}_{2j+1},\\
&A_k = \left\{ x \in X : U_k \left(\sum_{j=k+1}^{\infty} \mathbb{I}_{E_j}(x)\right) \le 2^{-k} \right\},\\
&B_k = \left\{ x \in X : U_k \mathbb{I}_{E_k}(x) > 1 - 2^{-k} \right\},\\
&G = (\liminf_{k\to\infty} A_k) \cap (\liminf_{k\to\infty} B_k).
\end{aligned}$$

From (7.4) and (7.3) we get $mG = 1$. Given an arbitrary $x \in G$ we have

$$x \in A_k \cap B_k, \quad k > k_0,$$

and consequently

$$U_k \left(\sum_{j=k+1}^{\infty} \mathbb{I}_{E_j}(x)\right) \le 2^{-k}, \quad U_k \mathbb{I}_{E_k}(x) > 1 - 2^{-k}, \quad k > k_0. \tag{7.5}$$

Thus

$$U_{2k+1}\mathbb{I}_E(x) \ge U_{2k+1}\mathbb{I}_{\tilde{E}_{2k+1}}(x)$$
$$\ge U_{2k+1}\mathbb{I}_{E_{2k+1}}(x) - U_{2k+1}\left(\sum_{j=2k+2}^{\infty}\mathbb{I}_{E_j}(x)\right) > 1-2^{-(2k+1)}-2^{-(2k+1)} = 1-2^{-2k}.$$

This implies

$$\limsup_{m\to\infty} T_m\mathbb{I}_E(x) = 1, \quad x \in E.$$

It is easy to observe $E \cap E_{2k} = \varnothing$. So we have $E \subset \tilde{E}_{2k}^c$ and from (7.5) we derive

$$U_{2k}\mathbb{I}_E(x) \le U_{2k}\mathbb{I}_{\tilde{E}_{2k}^c}(x) = 1 - U_{2k}\mathbb{I}_{\tilde{E}_{2k}}(x)$$
$$\le 1 - U_{2k}\mathbb{I}_{E_{2k}}(x) + U_{2k}\left(\sum_{j=2k+1}^{\infty}\mathbb{I}_{E_j}(x)\right) < 1-(1-2^{-2k}) + 2^{-2k} = 2^{-2k+1}.$$

Hence

$$\liminf_{m\to\infty} T_m\mathbb{I}_E(x) = 0, \quad x \in E,$$

and the proof is complete. □

Now suppose (X, m) in (7.1) coincides with $(\mathbb{T}, \lambda)$. In the next theorem we consider translation invariant operators T_n defined

$$T_n f_x(t) = T_n f(x+t),$$

where $f_x(t) = f(x+t)$.

Theorem 6. *If the sequence of translation invariant operators $\{T_n\}$ with (1)-(3) is weak δ-sweeping out for any $0 < \delta < 1$ then it is strong sweeping out.*

Proof. According to the previous theorem it is enough to proof that $\{T_n\}$ is δ-sweeping out for any $0 < \delta < 1$. By weak δ-sweeping property we may choose measurable sets F_k such that

$$\frac{|\{x \in X : \sup_{n>k} T_n\mathbb{I}_{F_k}(x) \ge 1 - \frac{1}{k}\}|}{|F_k|} \to \infty.$$

Taking subsequences of F_k (with possible repetitions) allows us to find a sequence of sets E_k, a sequences $\delta_k \nearrow 1$, and $n_k \to \infty$ so that, taking

$$A_k = \{x \in X : \sup_{n>n_k} T_n\mathbb{I}_{E_k}(x) \ge \delta_k\},$$

we have

$$\sum_{k=1}^{\infty}|A_k| = \infty, \quad \sum_{k=1}^{\infty}|E_k| < \varepsilon.$$

Applying Borel-Cantelli lemma, we can choose a sequence x_k so that

$$|\limsup_{k\to\infty}(A_k + x_k)| = 1.$$

Since T_n are translation invariant operators, denoting

$$E = \bigcup_{k=1}^{\infty}(E_k + x_k)$$

we get

$$|\{x \in X : \limsup_{n\to\infty} T_n \mathbb{I}_E(x) = 1\}|$$
$$\geq |\limsup_{k\to\infty} \{x \in X : \sup_{n>n_k} T_n \mathbb{I}_{E_k+x_k}(x) \geq \delta_k\}| = |\limsup_{k\to\infty}(A_k + x_k)| = 1,$$

and

$$|E| \leq \sum_{j=1}^{\infty} |E_k + x_k| = \sum_{j=1}^{\infty} |E_k| < \varepsilon.$$

□

Clearly Riemann sums operators satisfy the conditions (1)-(3). (1) and (2) are clear. Let us verify (3). If for $E \subset \mathbb{T}$ we have $|E| < \delta = \varepsilon^2$ then

$$\int_{\mathbb{T}} R_n \mathbb{I}_E(x) dx = |E| < \varepsilon^2$$

and therefore, using Chebishev's inequality, we get

$$|\{x \in \mathbb{T} : R_n \mathbb{I}_E(x) > \varepsilon\}| < \varepsilon,$$

which proves (3). Analyzing Rudin's proof one can easily understand it allows to get δ-sweeping out property for any $0 < \delta < 1$. Thus applying Theorem 5 we conclude that if $\{n_k\}$ satisfies the hypothesis of Rudin's theorem then R_{n_k} is strong sweeping out. We note that this assertion for Riemann sums was proved by M. Akcoglu, A. Bellow, R. Jones, V. Losert, K. Reinhold-Larsson and M. Wierdl in [20] by using Rudin's ideas. Now consider the operators

$$\frac{1}{n}\sum_{j=1}^{n} f(jx). \tag{7.6}$$

J. M. Marstrand in [22], solving Kinchine's conjecture, has proved this sequence has 1-sweeping out property. Applying Theorem 5 we get the sequence (7.6) is strong sweeping out. We note that alternate proofs of Rudin's and Marstrand's theorems follows from Bourgain Entropy Theorem [**?**] a general tool for investigation of divergence of certain operator sequences.

Proof of Rudin's theorem based on Theorem 4. Fix a number $0 < \delta < 1$. According to the conditions of Rudin's theorem for any $k \in \mathbb{N}$ there exists a collection $D_k = \{n_1, n_2, \ldots, n_k\} \subset D$ such that no member of D_k divides the least common multiple of the others. It means we can choose primes $p_{\nu_1}, p_{\nu_2}, \ldots, p_{\nu_k}$ such that $p_{\nu_j} | n_{\nu_j}$ and $p_{\nu_j} \not| n_{\nu_i}$ if $i \neq j$. Let l be the least common multiple of the numbers $n_1, n_2, \ldots, n_k$. Denoting $q_j = l/n_j$ we have

$$l/D_k = \{q_1, q_2, \ldots, q_k\}.$$

In addition

$$q_j = p_{\nu_1}^{m_1(j)} \cdot p_{\nu_2}^{m_2(j)} \ldots p_{\nu_k}^{m_k(j)} \cdot \gamma_j$$

where

$$m_j(j) = 0, \quad m_i(j) > 0, \text{ if } i \neq j.$$

Denote by Q_j the collection of rectangles (2.9) corresponding to $l = q_j$ and suppose $Q = \cup_{j=1}^{k} Q_k$. According to (4.15) we have

$$\mathcal{M}_{l/D_k} f(x) = \sup_{B \in Q:\, x \in B} \frac{1}{|B|} \int_B |f(t)| dt.$$

On the other hand any rectangle of the form

$$\left\{x \in \mathbb{T}^\infty : \left[\frac{t_i}{p_{\nu_i}}, \frac{t_i+1}{p_{\nu_i}}\right), 1 \le j \le k, j \ne i\right\},$$
$$0 \le t_i < p_{\nu_i}, \quad 1 \le j \le k, j \ne i,$$

can be represented as a disjoint union of rectangles from Q_i. Thus the same assertion is true also for the set

$$C_i = \{x \in \mathbb{T}^\infty : 0 \le x_{\nu_j} < \frac{r_j}{p_{\nu(j)}}, 1 \le j \le k, j \ne i\},, \quad r_j = [\delta p_{\nu(j)}] + 1.$$

Denote

$$C = \bigcap_{j=1}^{k} C_j = \{x \in \mathbb{T}^\infty : 0 \le x_{\nu_j} < \frac{r_j}{p_{\nu(j)}}, j = 1, 2, \ldots, k\}.$$

It is easy to observe if $B \in Q_j$ and $B \subset C_j$ then

$$|B \cap C| = \frac{r_j}{p_{\nu(j)}}|B|.$$

Therefore, since $\frac{r_j}{p_{\nu(j)}} > \delta$ we obtain

$$\mathcal{M}_{l/D_k}\mathbb{I}_C(x) > \delta, \quad x \in \bigcup_{j=1}^{k} C_j$$

On the other hand we have

$$\left|\bigcup_{j=1}^{k} C_j\right| = |C|\left(1 + \sum_{j=1}^{k}\frac{p_{\nu(j)}}{r_j}\right) > (k+1)|C|.$$

Thus we get

$$|\{x \in \mathbb{T}^\infty : \mathcal{M}_{l/D_k}\mathbb{I}_C(x) > \delta\}| > (k+1)|C|$$

According Theorem 4 for some $G \subset \mathbb{T}$ we get

$$|\{x \in \mathbb{T} : \mathcal{R}^l_{D_k}\mathbb{I}_G(x) > \delta\}| > (k+1)|G|.$$

In addition, since C is $\mathcal{B}_l$-measurable we have G is $\mathcal{I}_l$-measurable. Thus from (2.6) we conclude

$$\begin{aligned}|\{x \in \mathbb{T} : \mathcal{R}_D\mathbb{I}_G(x) > \delta\}| &\ge |\{x \in \mathbb{T} : \mathcal{R}_{D_k}\mathbb{I}_G(x) > \delta\}| \\ &= |\{x \in \mathbb{T} : \mathcal{R}^l_{D_k}\mathbb{I}_G(x) > \delta\}| \ge (k+1)|G|, \quad k = 1, 2, \ldots.\end{aligned}$$

This implies the sequence $R_n f(x)$, $n \in D$, has weak δ sweeping out property for any $0 < \delta < 1$. Applying Theorem 6 we obtain it has strong sweeping out property. The proof is complete. □

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References

[1] B. Jessen, On the approximation of Lebesgue integrals by Riemann sums, Annals of Math., v. 35, 1934, p. 248–251
[2] W.Rudin,An arithmetic property of Riemann sums, Proc. Amer. Math. Soc., t. 15, 1964, p. 321–324.
[3] L. E. Dubins and J. Pitman. A pointwise ergodic theorem for the group of rational rotations, Trans. Amer. Math. Soc., v. 251, 1979, p. 299–308.
[4] Y. Bugeaud and M. Weber, Examples and counterexamples for Riemann sums, Indag. Math. (NS) 9 (1998), No. 1, 11–13.
[5] A. Zygmund, Trigonometric series,v. 2, 1959.
[6] R. Nair, On Riemann Sums and Lebesgue Integrals, Monatsh. Math., t. 120, 1995, p. 49–54.
[7] R. C. Baker, Riemann sums and Lebesgue integrals, Quart. J. Math. Oxford Ser., v. 27, 1976, p. 191-198.
[8] J. J. Ruch and M. Weber, On Riemann sums, Note di Matematica 26, No. 2, 2006, 150.
[9] P. R. Halmos, Mesure Theory, New York, 1950.
[10] J. Neveu, Bases Mathémathiques du calculus des probabilités, Paris, 1964. Russian translation: Matematicheskie osnovy teorii veroyatnostei, Moscow "Mir", 1969.
[11] Krasnosel'skii M.A., Rutickii Ya.B., Convex Functions and Orlicz Spaces. P.Noordhoff Ltd, Groningen, 1961.
[12] G. A. Karagulyan, On the order of growth $o(\log\log n)$ of the partial sums of Fourier-Stieltjes series of random measures , Math. Sbornik , 1994, V. 78, No 1, 11–33.
[13] E. M. Stein, On limits of sequences of operators, Ann. of Math.,1961, Vol. 74, 140–170
[14] E. M. Stein, Harmonic Analisys, Princeton Unevirsity Press, 1993.
[15] B. Jessen, J. Marcinkiewicz, and A. Zygmund, Note on the differentiability of multiple integrals, Fund. Math. 25 (1935), 217–234.
[16] M. de Guzman, Differentiation of Integrals in $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer, 1975.
[17] Saks S., Remark on the differentiability of the Lebesgue indefinite integral, Fund. Math., 1934, v. 22, 257–261.
[18] A. Stokolos, On weak type inequalities for rare maximal function in $\mathbb{R}^n$, Colloq. Math., 104 (2006), 311–315.
[19] K. Hare, A. Stokolos, On weak type inequalities for rare maximal functions, Colloq. Math. 83 (2000), No. 2, 173–182.
[20] M. Akcoglu, A. Bellow, R. L. Jones, V. Losert, K. Reinhold-Larsson, and M. Wierdl, The strong sweeping out property for lacunary sequences, Riemann sums, convolution powers, and related matters, Ergodic Theory and Dynamical Systems, 16(1996), 207-253.
[21] M. Akcoglu, M. D. Ha, and R. L. Jones, Sweeping out properties for operator sequences, Canadian J. Math., 49(1997), 3-23.
[22] J. M. Marstrand, On Kinchine's conjecture about strong uniform distribution, Proc. London Math. Soc., 21(1970), 540–556.
[23] J. Bourgain, Almost sure convergence and bounded entropy, Israel J. Math , 62(1988), 79-97.

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