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Local limit theorem for conditionally independent random fields

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Central limit theorem

- random fields with independent components;
- random fields which satisfy mixing conditions with appropriate rate of decay of corresponding mixing coefficient
  - Gibbs random fields under conditions of smallness of the norm of a potential and certain conditions on the rate of its decay;
- ergodic martingale difference random fields.

Local limit theorem

LLT holds for random fields with independent components. Validity of the LLT for weakly dependent random fields practically was not under consideration.

LLT is very important from the point of view of statistical physics, particularly concerning the problem of equivalence of ensembles.
Khinchin A.Ya., Mathematical foundations of statistical mechanics. New York, Dover Publications, 1949 — LLT for number of particles in the case of the ideal gas.

The LLT for Gibbs random fields was a subject of consideration in many works including

Minlos R.A., Khalfina A.M., Two-dimensional limit theorem for the particle number and energy in the grand canonical ensemble. Izv. Akad. Nauk SSSR 34, 1970;

Del Grosso G., On the local central limit theorem for Gibbs processes. Commun. Math. Phys. 37, 1974;

Dobrushin R.L., Tirozzi B., The Central Limit Theorem and the Problem of Equivalence of Ensembles. Commun. math. Phys. 54, 1977;

Campanino M., Capocaccia D., Tirozzi B., The local central limit theorem for a Gibbs random field. Commun. Math. Phys. 70, 1979;

Campanino M., Del Grosso G., Tirozzi B.,, Local limit theorem for a Gibbs random field of particles and unbounded spins. J. Math. Phys. 20, 1979;

Arzumanyan V.A., Nakhapetyan B.S., Pogosyan S.K., Local limit theorem for the particle number in spin lattice systems. Theoretical and Mathematical Physics 89, 1991.

Kazanchyan T.P., The local limit theorem for the sequences of dependent random variables. Izvestia NAN Armenii 39, 2004

In our talk

- we introduce the notion of conditionally independent random field;

- for such random fields we present general conditions under which the LLT follows from the CLT;

- we discuss possible applications of the result to Gibbs and martingale-difference random fields.

Random field

Random field on d-dimensional integer lattice $\mathbb{Z}^d$, $d \geq 1$ with finite phase space $X \subset \mathbb{Z}$, $1 \leq |X| < \infty$ is a collection of random variables $(\xi_s) = (\xi_s, s \in \mathbb{Z}^d)$, each of which takes value in X.

The distribution of a random field (ξ_s) is the probability measure P on $(X^{\mathbb{Z}^d}, \Im^{\mathbb{Z}^d})$, such, that

$$P(B) = \mathsf{Pr}\{(\xi_s, s \in \mathbb{Z}^d) \in B\}, \qquad B \in \Im^{\mathbb{Z}^d},$$

where $\Im^{\mathbb{Z}^d}$ is the σ-algebra, generated by cylinder subsets of the set $X^{\mathbb{Z}^d}$.

Random field (ξ_s) is a homogenous random filed if for any finite $V \subset \mathbb{Z}^d$ and any $a \in \mathbb{Z}^d$

$$P(\xi_s = x_s, s \in V) = P(\xi_{s+a} = x_s, s \in V), \qquad x_s \in X, s \in V.$$

Classical limits theorems

Let $V \subset \mathbb{Z}^d$ be a finite space and put $S_V = \sum\limits_{t \in V} \xi_t$.

We say that for the random field (ξ_t) the CLT is valid if

$$\lim_{n\to\infty} P\left(\frac{S_{V_n} - ES_{V_n}}{\sqrt{DS_{V_n}}} < x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du, \qquad x \in \mathbb{R},$$

and the LLT is valid if

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}} \left| \sqrt{DS_{V_n}} P(S_{V_n} = j) - \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(S_{V_n} - ES_{V_n})^2}{2DS_{V_n}} \right\} \right| = 0,$$

where $V_n = [-n, n]^d$, $n = 1, 2, \ldots$.

Conditions of weak dependence

A homogeneous random field (ξ_s) satisfies

- *the ergodicity condition* if for any finite $I, V \subset \mathbb{Z}^d$ and $x \in X^I$, $y \in X^V$ the following relation holds

$$\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{a\in V_n} P(\{\xi_s = x_s, s \in I\} \cap \{\xi_{r+a} = y_r, r \in V\}) =$$
$$= P(\xi_s = x_s, s \in I) P(\xi_r = y_r, r \in V)$$

- *the uniform strong mixing condition with coefficient* φ_I if for any finite $I \subset \mathbb{Z}^d$

$$\sup_{A\in\Im_I, B\in\Im_V, P(B)>0} \{|P(A/B) - P(A)|\} \le \varphi_I(\rho(I,V)),$$

  where $\Im_V = \sigma(\xi_s, s \in V)$, $\rho(I,V)$ is the distance between I and V, and the function $\varphi_I(\rho)$, $\rho \in \mathbb{R}$, is such that $\varphi_I(\rho) \to 0$ as $\rho \to \infty$ and the set I is fixed

Gibbs random field

A set of functions $\Phi = \left\{\Phi_V\left(x\right), x \in X^V, V \in W\right\}$ satisfying the condition

$$\sum_{J \in W : J \neq \varnothing} \sup_{x \in X^J} \left|\Phi_J\left(x\right)\right| < \infty,$$

is called interaction potential.

Gibbs random field with potential Φ is a random field which has a version of conditional distribution a.e. coinciding with Gibbs specification $Q = \left\{q_V^{\bar{x}}, V \subset \mathbb{Z}^d, |V| < \infty, \bar{x} \in X^{\mathbb{Z}^d \setminus V}\right\}$, where

$$q_V^{\bar{x}}(x) = \frac{\exp\{H_V^{\bar{x}}(x)\}}{\sum\limits_{z \in X^V} \exp\{H_V^{\bar{x}}(z)\}}, \qquad x \in X^V,$$

and

$$H_V^{\bar{x}}(x) = \sum_{J \subset V : J \neq \varnothing} \sum_{\tilde{J} \in \mathbb{Z}^d \setminus V} \Phi(x_J \bar{x}_{\tilde{J}}).$$

LLT for Gibbs random fields with finite-range potentials

The potential Φ is called finite-range if there exists $R > 0$ such that for any $x \in X^V$ and finite $V \subset \mathbb{Z}^d$

$$\Phi_V(x) = 0 \qquad \text{as soon as } \sup_{t,s \in V} |t - s| > R.$$

Theorem 1 (Dobrushin, Tirozzi (1977)) *If Φ is a finite-range potential then for corresponding Gibbs random field the LLT follows from the CLT.*

Conditionally independent random fields

We say that a homogenous random field (ξ_s) is *conditionally independent* with coefficient β_I if for any $I, V, \Lambda \in W$ such that $I \cap V = \emptyset$ and $I, V \subset \Lambda$, and any random variables η_1, η_2 which are $\Im_I$- and $\Im_V$-measurable correspondingly, $|\eta_1|, |\eta_2| < C$, $0 < C < \infty$, the following relation holds

$$|E(\eta_1 \cdot \eta_2 / \Im_{\Lambda \setminus \{I \cup V\}}) - E(\eta_1 / \Im_{\Lambda \setminus \{I \cup V\}}) \cdot E(\eta_2 / \Im_{\Lambda \setminus \{I \cup V\}})| \leq \\ \leq C\, \beta_I(\rho(I, V)),$$

where $\rho(I, V)$ is the distance between I and V, and $\beta_I(\rho) \to 0$ as $\rho \to \infty$ (and, hence, $\Lambda \uparrow \mathbb{Z}^d$) and I is fixed.

LLT for conditionally independent random fields

Theorem 2 *Let (ξ_s) be a homogenous random field with phase space $X \subset \mathbb{Z}$. If*

1. $DS_V = \sigma^2 |V|(1 + o(1)),$ *as $V \uparrow \mathbb{Z}^d$, $\sigma > 0$;*

2. *(ξ_s) is conditionally independent with coefficient β_I such that*

$$\beta_I(\rho) \leq |I|\beta(\rho) \qquad \text{and} \qquad \beta(\rho) = \frac{\mu(\rho)}{\rho^{3d/2}},$$

   where $\mu(\rho) \to 0$ arbitrarily slow as $\rho \to \infty$;

3. *there exists $\gamma > 0$ such that for any finite $I \subset V \subset \mathbb{Z}^d$*

$$P(S_I = y / \Im_{V \setminus I}) \geq \gamma \qquad \text{for any possible value } y \text{ of } S_I;$$

then for the random field (ξ_s) the LLT follows from the CLT.

Applications to Gibbs random fields

Gibbs random fields with finite-range R potentials are conditionally independent, since for such random fields $\beta_I(\rho) = 0$ as soon as $\rho > R$. Hence the Theorem 2 generalizes Dobrushin and Tirozzi's result.

Applications to martingale-difference random filed

A random field (ξ_s) is called a martingale–difference if for any $s \in \mathbb{Z}^d$

$$E\,|\xi_s| < \infty \qquad \text{and} \qquad E\left(\xi_s/\Im_{\mathbb{Z}^d\setminus\{s\}}\right) = 0 \text{ (a.s.)}.$$

Theorem 3 (Nahapetian (1995)) *Let* (ξ_s) *be a homogenous ergodic martingale–difference random field such that* $0 < M\xi_s^2 < \infty$*. Then for* (ξ_s) *the CLT is valid.*

Theorem 4 *Let* (ξ_s) *be a homogenous martingale–difference random field with phase space* $X \subset \mathbb{Z}$*, and let there exists* $\gamma > 0$ *such that for any finite* $I \subset V \subset \mathbb{Z}^d$

$$P(S_I = y/\Im_{V \setminus I}) \geq \gamma \qquad \text{for any possible value } y \text{ of } S_I.$$

If, in addition, (ξ_s) *is a conditionally independent with coefficient* β_I *such that*

$$\beta_I(\rho) \leq |I|\beta(\rho) \qquad \text{and} \qquad \beta(\rho) = \frac{\mu(\rho)}{\rho^{3d/2}},$$

where $\mu(\rho) \to 0$ *arbitrarily slow as* $\rho \to \infty$*, then for the martingale–difference random field* (ξ_s) *the LLT is valid.*

Thank you

for your attention