Local limit theorem for conditionally independent random fields

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Abstract

In this paper we introduce the conditions under which the local limit theorem for random fields with weakly dependent components follows from the central one. Obtained result can be applied to Gibbs and martingale–difference random fields.

Key words: local limit theorem, conditional independence, Gibbs random field, martingale–difference random field

Introduction

The validity of the central limit theorem (CLT) for random fields with weakly dependent components was considered in many works (see, for example, [6, 10], where were discussed different methods of proving the CLT, and references therein). There are a lot of results on this subject under different types of dependency conditions for random variables - so called mixing conditions. However the question of validity of the local limit theorem (LLT) for weakly dependent random fields practically was not under consideration. The possible explanation is that classical mixing conditions are probably not enough to obtain the LLT.

In the same time this problem is very important from the point of view of statistical physics, particularly concerning the problem of equivalence of ensembles. The importance of the LLT first was discussed in [8], where the LLT was proved for number of particles in the case of the ideal gas. The notion of Gibbs random field predetermined further development of the theory of limits theorems for random fields. The LLT for Gibbs random fields was a subject of consideration in many works (see [1–5, 9]).

For us the work [5] of Dobrushin and Tirozzi is of special interest. In this work it was shown that the LLT for Gibbs random fields with finite-range potentials follows from the CLT. Dobrushin and Tirozzi's approach to prove LLT essentially uses the finite-range condition of interaction potential. In our paper we introduce the notion of conditionally independent random field (not necessary Gibbsian). For such random fields we present general conditions under which the LLT follows from the CLT. The result can be applied to Gibbs and martingale-difference random fields.

Let us also note the work [7] where the similar result was obtained in one-dimensional case for random processes with finite-range dependence.

1 Preliminaries

In this paper we consider random fields on d-dimensional integer lattice $\mathbb{Z}^d$, $d \geq 1$ with finite phase space $X \subset \mathbb{Z}$, $1 \leq |X| < \infty$,[1] i.e. collections of random variables $(\xi_s) = (\xi_s, s \in \mathbb{Z}^d)$, each of which takes value in X.

For any $S \subset \mathbb{Z}^d$, $x_s \in X$ we denote by $X^S = \{(x_s, s \in S)\}$ the space of all configurations on S. If $S = \emptyset$, we assume that the space $X^{\emptyset} = \{\emptyset\}$. For any $S, T \subset \mathbb{Z}^d$ such that $S \cap T = \emptyset$ and any configurations $x \in X^S$ and $y \in X^T$, we denote by xy the concatenation of x and y, that is, the configuration on $T \cup S$ equal to x on S and to y on T. For any $S \subset T$, $x \in X^T$, we denote by x_S the restriction of x on S. Also we denote by $W = \left\{V \subset \mathbb{Z}^d, |V| < \infty\right\}$ the set of all finite subsets of $\mathbb{Z}^d$.

Let $\Im^{\mathbb{Z}^d}$ be the σ-algebra, generated by cylinder subsets of the set $X^{\mathbb{Z}^d}$. The distribution of a random field (ξ_s) is the probability measure P on $(X^{\mathbb{Z}^d}, \Im^{\mathbb{Z}^d})$, such, that

$$\Pr\{(\xi_s, s \in \mathbb{Z}^d) \in B\} = P(B), \qquad B \in \Im^{\mathbb{Z}^d}.$$

For the random filed (ξ_s) and any $V \in W$ denote by $\Im_V$ a σ-algebra, generated by ξ_s, $s \in V$.

A random field (ξ_s) is called a homogeneous random field if for any $V \in W$ and $a \in \mathbb{Z}^d$

$$P(\xi_s = x_s, s \in V) = P(\xi_{s+a} = x_s, s \in V), \qquad x_s \in X, s \in V,$$

and called ergodic if for any $I, V \in W$ and $x \in X^I$, $y \in X^V$ the following relation holds

$$\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{a \in V_n} P(\{\xi_s = x_s, s \in I\} \cap \{\xi_{r+a} = y_r, r \in V\}) = P(\xi_s = x_s, s \in I) P(\xi_r = y_r, r \in V),$$

where $V_n = [-n, n]^d$, $n = 1, 2, ...$

For a given random field (ξ_s) denote $S_V = \sum_{s \in V} \xi_s$, $V \in W$. We say that for the random field (ξ_s) the CLT is valid if

$$\lim_{n\to\infty} P\left(\frac{S_{V_n} - ES_{V_n}}{\sqrt{DS_{V_n}}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \qquad x \in \mathbb{R},$$

and the LLT is valid if

$$\lim_{n\to\infty} \sup_{j \in \mathbb{Z}} \left| \sqrt{DS_{V_n}} P(S_{V_n} = j) - \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(S_{V_n} - ES_{V_n})^2}{2DS_{V_n}} \right\} \right| = 0.$$

2 Main result and some of its applications

Let us introduce the following condition of weak dependence of components of a random field.

We say that a homogenous random field (ξ_s) is *conditionally independent* with coefficient β_I if for any $I, V, \Lambda \in W$ such that $I \cap V = \emptyset$ and $I, V \subset \Lambda$, and any random variables η_1, η_2 which are $\Im_I$- and $\Im_V$-measurable correspondingly the following relation holds

$$|E(\eta_1 \cdot \eta_2 / \Im_{\Lambda \setminus \{I \cup V\}}) - E(\eta_1 / \Im_{\Lambda \setminus \{I \cup V\}}) \cdot E(\eta_2 / \Im_{\Lambda \setminus \{I \cup V\}})| \leq \beta_I(\rho(I, V)),$$

where $\rho(I, V)$ is the distance between I and V, and $\beta_I(\rho) \to 0$ as $\rho \to \infty$ (and, hence, $\Lambda \uparrow \mathbb{Z}^d$) and I is fixed.

[1] Here and below the symbol $|X|$ is used to denote the power of the finite set X.

Introduced condition of weak dependence of components of random fields seems stronger than classical mixing conditions. Also it seems more suitable to use conditions of this type instead of classical ones for random fields which are described by means of their conditional distributions such as Markov, Gibbs and martingale–difference random fields.

In the next theorem conditions under which for conditionally independent random field the LLT follows from the CLT are presented.

Theorem 1. *Let (ξ_s) be a homogenous random field with phase space $X \subset \mathbb{Z}$. If*

1. $DS_V = \sigma^2 |V|(1 + o(1)), \qquad$ *as* $V \uparrow \mathbb{Z}^d$, $\sigma > 0$;
2. (ξ_s) *is conditionally independent with coefficient* β_I *such that*

$$\beta_I(\rho) \leq |I| \beta(\rho) \qquad \text{and} \qquad \beta(\rho) = \frac{\mu(\rho)}{\rho^{3d/2}},$$

where $\mu(\rho) \to 0$ *arbitrarily slow as* $\rho \to \infty$;

3. *there exists* $\gamma > 0$ *such that for any* $I \subset V \in W$

$$P(S_I = y / \Im_{V \setminus I}) \geq \gamma \qquad \text{for any possible value } y \text{ of } S_I;$$

then for the random field (ξ_s) *the LLT follows from the CLT.*

Let us make some remarks on possible applications of this theorem. First, let us note, that Gibbs random fields with finite-range R potentials are conditionally independent, since for such random fields $\beta_I(\rho) = 0$ as soon as $\rho > R$. Hence the Theorem 1 generalizes Dobrushin and Tirozzi's result.

Now let us consider martingale–difference random fields. A random field (ξ_s) is called a martingale–difference (see [12]) if for any $s \in \mathbb{Z}^d$

$$E\,|\xi_s| < \infty \qquad \text{and} \qquad E(\xi_s / \xi_r, r \in \mathbb{Z}^d \backslash \{s\}) = 0 \text{ a.s..}$$

For homogenous ergodic martingale–difference random fields the CLT is valid (see [11]). It is easy to see that if a random field is conditionally independent than it is ergodic. Hence we can formulate the following result.

Theorem 2. *Let (ξ_s) be a homogenous martingale–difference random field with phase space $X \subset \mathbb{Z}$, and let there exists $\gamma > 0$ such that for any $I \subset V \in W$*

$$P(S_I = y / \Im_{V \setminus I}) \geq \gamma \qquad \text{for any possible value } y \text{ of } S_I.$$

If, in addition, (ξ_s) is a conditionally independent with coefficient β_I such that

$$\beta_I(\rho) \leq |I| \beta(\rho) \qquad \text{and} \qquad \beta(\rho) = \frac{\mu(\rho)}{\rho^{3d/2}},$$

where $\mu(\rho) \to 0$ arbitrarily slow as $\rho \to \infty$, then for the martingale–difference random field (ξ_s) the LLT is valid.

3 Proof of the main theorem

In this section we give the proof of the Theorem 1.

Proof of the Theorem 1. For any $n \in \mathbb{N}$ set $V_n = [-n, n]^d$, $S_n = \sum_{s \in V_n} \xi_s$, and let $f_n(t)$ be a characteristic function of S_n. Then

$$P(S_n = j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itj} f_n(t) dt.$$

Denote $z_{nj} = \dfrac{j - ES_n}{\sqrt{DS_n}}$. Then $j = z_{nj}\sqrt{DS_n} + ES_n$, and we can write

$$P(S_n = j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itz_{nj}\sqrt{DS_n} - itES_n} f_n(t) dt = \frac{1}{2\pi\sqrt{DS_n}} \int_{-\pi\sqrt{DS_n}}^{\pi\sqrt{DS_n}} e^{-itz_{nj} - itES_n/\sqrt{DS_n}} f_n\left(\frac{t}{\sqrt{DS_n}}\right) dt.$$

Also we have

$$\frac{1}{\sqrt{2\pi}} e^{-z_{nj}^2/2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-t^2/2} e^{-itz_{nj}} dt \qquad \text{for any } z_{nj} \in \mathbb{R}.$$

Hence we can write

$$2\pi \sup_{j \in \mathbb{Z}} \left| \sqrt{DS_n} P(S_n = j) - \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(j - ES_n)^2}{2DS_n} \right\} \right| =$$

$$= \sup_{j \in \mathbb{Z}} \left| \int_{-\pi\sqrt{DS_n}}^{\pi\sqrt{DS_n}} e^{-itz_{nj} - itES_n/\sqrt{DS_n}} f_n\left(\frac{t}{\sqrt{DS_n}}\right) dt - \int_{-\infty}^{+\infty} e^{-itz_{nj} - t^2/2} dt \right| \leq$$

$$\leq \int_{-T}^{T} \left|e^{-itz_{nj}}\right| \cdot \left| \left(e^{-itES_n/\sqrt{DS_n}} E \exp\left\{ \frac{itS_n}{\sqrt{DS_n}} \right\} - e^{-t^2/2} \right) \right| dt + \int_{|t| \geq T} \left|e^{-itz_{nj}}\right| \cdot \left|e^{-t^2/2}\right| dt +$$

$$+ \int_{T \leq |t| \leq \pi\sqrt{DS_n}} \left|e^{-itz_{nj}}\right| \cdot \left| e^{-itES_n/\sqrt{DS_n}} E \exp\left\{ \frac{itS_n}{\sqrt{DS_n}} \right\} \right| dt \leq$$

$$\leq \int_{-T}^{T} \left| E \exp\left\{ it \frac{S_n - ES_n}{\sqrt{DS_n}} \right\} - e^{-t^2/2} \right| dt + \int_{|t| \geq T} e^{-t^2/2} dt + \int_{T \leq |t| \leq \pi\sqrt{DS_n}} \left| E \exp\left\{ it \frac{S_n - ES_n}{\sqrt{DS_n}} \right\} \right| dt,$$

where T, $0 < T < \pi\sqrt{DS_n}$, can be chosen big enough for sufficiently large n.

Let $\varepsilon > 0$ be fixed. If the CLT for (ξ_s) is valid, then for sufficiently large T we have

$$\int_{-T}^{T} \left| \left(E \exp\left\{ it \frac{S_n - ES_n}{\sqrt{DS_n}} \right\} - e^{-t^2/2} \right) \right| dt \leq \varepsilon.$$

Choosing T big enough we obtain also

$$\int\limits_{|t|\geq T} e^{-t^2/2}dt \leq \varepsilon.$$

It remains to show that

$$\int\limits_{T\leq|t|\leq\pi\sqrt{DS_n}} \left|E\exp\left\{it\frac{S_n-ES_n}{\sqrt{DS_n}}\right\}\right| dt \leq \varepsilon. \tag{1}$$

Let us make the sectionalization of V_n. Set $p = p(n) = o(n)$, $q = q(n) = o(p) = n\cdot\lambda(n)$, where $\lambda(n) = \mu^{2/(3d)-\tau}(n)$, $\tau > 0$. Set also

$$I_n(j) = [-n + jp + jq; -n + (j+1)p + jq], \qquad j = 1, 2, ..., \left[\frac{2n+1}{p+q}\right]^2,$$

$$I_n = \bigcup_{j=0}^{\left[\frac{2n+1}{p+q}\right]} I_n(j),$$

and let I_n^d be a Cartesian product of d copies of I_n. Then I_n^d is a union of $k_n = \left[\frac{2n+1}{p+q}\right]^d$ d-dimensional cubes $\Delta_j^{(n)}$ with side-length p, which are numerated in some way: $I_n^d = \bigcup\limits_{j=1}^{k_n}\Delta_j^{(n)}$.

Denote by $\Im_n = \Im_{V_n\backslash I_n^d} = \sigma(\xi_s, s\in V_n\backslash I_n^d)$. We have

$$E\exp\left\{it\frac{S_n-ES_n}{\sqrt{DS_n}}\right\} = E\left(E\left(\exp\left\{it\frac{S_{I_n^d}-ES_{I_n^d}}{\sqrt{DS_n}}\right\}\cdot\exp\left\{it\frac{S_{V_n\backslash I_n^d}-ES_{V_n\backslash I_n^d}}{\sqrt{DS_n}}\right\}/\Im_n\right)\right) =$$

$$= E\left(\exp\left\{it\frac{S_{V_n\backslash I_n^d}-ES_{V_n\backslash I_n^d}}{\sqrt{DS_n}}\right\}E\left(\exp\left\{it\frac{S_{I_n^d}-ES_{I_n^d}}{\sqrt{DS_n}}\right\}/\Im_n\right)\right).$$

Hence

$$\left|E\exp\left\{it\frac{S_n-ES_n}{\sqrt{DS_n}}\right\}\right| \leq E\left|E\left(\exp\left\{it\frac{S_{I_n^d}-ES_{I_n^d}}{\sqrt{DS_n}}\right\}/\Im_n\right)\right| = E\left|E\left(\prod_{j=1}^{k_n}\exp\left\{it\frac{S_j-ES_j}{\sqrt{DS_n}}\right\}/\Im_n\right)\right|,$$

where $S_j = \sum\limits_{s\in\Delta_j^{(n)}}\xi_s$. Denote $\eta_j = \dfrac{S_j-ES_j}{\sqrt{DS_n}}$, and consider

$$E\left(\prod_{j=1}^{k_n}e^{it\eta_j}/\Im_n\right) = E\left(\prod_{j=1}^{k_n}e^{it\eta_j}/\Im_n\right) - \prod_{j=1}^{k_n}E\left(e^{it\eta_j}/\Im_n\right) + \prod_{j=1}^{k_n}E\left(e^{it\eta_j}/\Im_n\right).$$

Using the following relation (see, for example, Lemma 3.3.1 in [10])

$$E\left(\prod_{j=1}^{k_n}e^{it\eta_j}/\Im_n\right) - \prod_{j=1}^{k_n}E\left(e^{it\eta_j}/\Im_n\right) =$$

$$= \sum_{r=2}^{k_n}\left(\prod_{m=r+1}^{k_n}E\left(e^{it\eta_m}/\Im_n\right)\right)\cdot\left(E\left(\prod_{j=1}^{r}e^{it\eta_j}/\Im_n\right) - E\left(e^{it\eta_r}/\Im_n\right)\prod_{j=1}^{r-1}E\left(e^{it\eta_j}/\Im_n\right)\right),$$

[2] Here and below by $[z]$ we denote the integer part of z.

we obtain

$$\left|E\left(\prod_{j=1}^{k_n} e^{it\eta_j}/\Im_n\right)\right| \le \left|\prod_{j=1}^{k_n} E\left(e^{it\eta_j}/\Im_n\right)\right| +$$

$$+\sum_{r=2}^{k_n}\left(\prod_{m=r+1}^{k_n} E\left(\left|e^{it\eta_m}\right|/\Im_n\right)\right)\cdot\left|E\left(\prod_{j=1}^{r} e^{it\eta_j}/\Im_n\right) - E\left(e^{it\eta_r}/\Im_n\right)\prod_{j=1}^{r-1} E\left(e^{it\eta_j}/\Im_n\right)\right| \le$$

$$\le \prod_{j=1}^{k_n}\left|E\left(e^{it\eta_j}/\Im_n\right)\right| + \sum_{r=2}^{k_n}\beta_{\Delta_r^{(n)}}\left(\rho\left(\Delta_r^{(n)}, I_n^d\backslash\Delta_r^{(n)}\right)\right) \le \prod_{j=1}^{k_n}\left|E\left(e^{it\eta_j}/\Im_n\right)\right| + k_n p^d\beta(q).$$

For any $j=\overline{1,k_n}$ we can write

$$\left|E\left(e^{it\eta_j}/\Im_n\right)\right| = \left|E\left(\exp\left\{it\frac{S_j - E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}\cdot\exp\left\{it\frac{E(S_j/\Im_n) - ES_j}{\sqrt{DS_n}}\right\}/\Im_n\right)\right| \le$$

$$\le \left|\exp\left\{it\frac{E(S_j/\Im_n) - ES_j}{\sqrt{DS_n}}\right\}\right|\cdot\left|E\left(\exp\left\{it\frac{S_j - E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}/\Im_n\right)\right| \le$$

$$\le \left|E\left(\exp\left\{it\frac{S_j - E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}/\Im_n\right)\right|.$$

Further,

$$E\left(\exp\left\{it\frac{S_j - E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}/\Im_n\right) = 1 - \frac{t^2}{2DS_n}D(S_j/\Im_n) + \frac{O(t^2)}{DS_n}.$$

Let Y be the set of all possible values of S_j. Using the third condition of the theorem, we can write

$$D(S_j/\Im_n) = \sum_{y\in Y}(y - E(S_j/\Im_n))^2 P(S_j = y/\Im_n) \ge \gamma\sum_{y\in Y}(y - E(S_j/\Im_n))^2 \ge \gamma\alpha,$$

where $\alpha > 0$. Hence

$$E\left(\exp\left\{it\frac{S_j - E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}/\Im_n\right) \le 1 - \frac{\gamma\alpha}{2DS_n}t^2 + \frac{O(t^2)}{DS_n}.$$

On the other hand

$$e^{-\frac{\gamma\alpha t^2}{2DS_n}} = 1 - \frac{\gamma\alpha}{2DS_n}t^2 + \sum_{k=2}^{\infty}\frac{1}{k!}\left(\frac{-\gamma\alpha t^2}{2DS_n}\right)^k.$$

From here it follows that there exists $\delta > 0$ (which is independent of j) such that for any $|t| \le \delta\pi\sqrt{DS_n}$

$$E\left(\exp\left\{it\frac{S_j - E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}/\Im_n\right) \le \exp\left\{-\frac{\gamma\alpha t^2}{2DS_n}\right\}.$$

Then

$$\prod_{j=1}^{k_n}\left|E\left(e^{it\eta_j}/\Im_n\right)\right| \le \prod_{j=1}^{k_n}\exp\left\{-\frac{\gamma\alpha t^2}{2DS_n}\right\} = \exp\left\{-\frac{\gamma\alpha}{2}\cdot\frac{k_n}{DS_n}\cdot t^2\right\} \le e^{-c_n t^2},$$

where, due to the first condition of the theorem,

$$c_n = \frac{\gamma\alpha}{2}\cdot\left[\frac{2n+1}{p+q}\right]^d\cdot\frac{1}{\sigma^2(1+o(1))n^d} = \frac{C}{(p+q)^d},$$

and C is a positive constant.

Hence for the integral in left-hand side of (1) we can write

$$\int\limits_{T\le|t|\le\pi\sqrt{DS_n}} \left|E\exp\left\{it\frac{S_n-ES_n}{\sqrt{DS_n}}\right\}\right|dt \le \int\limits_{T\le|t|\le\delta\pi\sqrt{DS_n}} e^{-c_nt^2}dt + \int\limits_{T\le|t|\le\pi\sqrt{DS_n}} k_np^d\beta(q)dt +$$

$$+ \int\limits_{\delta\pi\sqrt{DS_n}\le|t|\le\pi\sqrt{DS_n}} \prod_{j=1}^{k_n}\left|E\left(\exp\left\{it\frac{S_j-E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}/\Im_n\right)\right|dt.$$

Let us show that for sufficiently big T and n and sufficiently small δ each summand on the right-hand part of the obtained inequality can be made smaller then $\varepsilon/3$. By doing this, we conclude the proof. It is obvious for the second summand. For the first one, taking into account the second condition of the theorem, we can write

$$\int\limits_{T\le|t|\le\pi\sqrt{DS_n}} k_np^d\beta(q)dt = 2k_np^d\beta(q)(\pi\sqrt{DS_n}-T) = \left[\tfrac{2n+1}{p+q}\right]^d\cdot p^d\cdot\tfrac{\mu(q)}{q^{3d/2}}\cdot C'n^{d/2} \le$$

$$\le C'\tfrac{\mu(n)\cdot n^{3d/2}}{\lambda^{3d/2}(n)\cdot n^{3d/2}} = C'\lambda^{3d\tau/2}(n),$$

where C' is a positive constant. Since $\lambda^{3d\tau/2}(n)\to 0$ as $n\to\infty$, for sufficiently large n

$$\int\limits_{T\le|t|\le\pi\sqrt{DS_n}} k_np^d\beta(q)dt \le \frac{\varepsilon}{3}.$$

Consider the third summand. We have

$$\int\limits_{\delta\pi\sqrt{DS_n}\le|t|\le\pi\sqrt{DS_n}} \prod_{j=1}^{k_n}\left|E\left(\exp\left\{it\frac{S_j-E(S_j/\Im_n)}{\sqrt{DS_n}}\right\}/\Im_n\right)\right|dt \le \sqrt{DS_n}\int\limits_{\delta\pi\le|t|\le\pi}\prod_{j=1}^{k_n}\left|E\left(e^{itS_j}/\Im_n\right)\right|dt.$$

Denote $a_j = \left|E\left(e^{itS_j}/\Im_n\right)\right|$. We can write

$$\prod_{j=1}^{k_n}a_j = \exp\left\{\ln\prod_{j=1}^{k_n}a_j\right\} = \exp\left\{\frac{1}{2}\sum_{j=1}^{k_n}\ln a_j^2\right\} \le \exp\left\{\frac{1}{2}\sum_{j=1}^{k_n}(a_j^2-1)\right\}.$$

Further for any j we have

$$a_j^2-1 = \left|E\left(e^{itS_j}/\Im_n\right)\right|^2 - 1 = \sum_{x,y\in Y}(\cos t(x-y)-1)P(S_j=x/\Im_n)P(S_j=y/\Im_n) =$$

$$= -2\sum_{x,y\in Y}\sin^2\frac{t(x-y)}{2}P(S_j=x/\Im_n)P(S_j=y/\Im_n) \le -2\gamma^2\sum_{x,y\in Y}\sin^2\frac{t(x-y)}{2},$$

where we used the third condition of the theorem. Hence

$$\prod_{j=1}^{k_n}\left|E\left(e^{itS_j}/\Im_n\right)\right| \le \exp\left\{-k_n\gamma^2\sum_{x,y\in X}\sin^2\frac{t(x-y)}{2}\right\} = \exp\left\{-k_n\cdot g(t)\right\},$$

where

$$g(t) = -\gamma^2 \sum_{x,y\in X} \sin^2 \frac{t(x-y)}{2}$$

is continuous positive function. Using the mean value theorem for integral, we obtain

$$\sqrt{DS_n} \int\limits_{\delta\pi\le|t|\le\pi} \prod_{j=1}^{k_n} \left|E\left(e^{itS_j}/\Im_n\right)\right| dt \le \sqrt{DS_n} \int\limits_{\delta\pi\le|t|\le\pi} e^{-k_n g(t)} dt \le \frac{(\pi-\delta\pi)\sqrt{DS_n}}{e^{k_n g(t_0)}} \le \frac{\varepsilon}{3}$$

for any fixed δ, some $t_0 \in [\delta\pi;\pi]$ and n big enough.

The theorem is proved.

$\square$

In conclusion let us note that for homogenous martingale–difference random fields the proof of the theorem can be simplified in some way. Indeed, for such fields one have

$$ES_n = 0 \qquad \text{and} \qquad DS_n = ES_n^2 = E\xi_0^2 \cdot |V_n| > 0.$$

To estimate the integral in (1) we can write

$$E\exp\left\{it\frac{S_n - ES_n}{\sqrt{DS_n}}\right\} = Ee^{itS_n/\sqrt{DS_n}} = 1 + \frac{it}{\sqrt{DS_n}}ES_n - \frac{t^2}{2DS_n}ES_n^2 + \frac{O(t^2)}{DS_n} = 1 - \frac{t^2}{2} + \frac{O(t^2)}{DS_n}.$$

Hence there exists $\delta > 0$ such that for any $|t| \le \delta\pi\sqrt{DS_n}$

$$|Ee^{itS_n/\sqrt{DS_n}}| \le e^{-t^2/2}.$$

Therefore

$$\int\limits_{T\le|t|\le\delta\pi\sqrt{DS_n}} \left|E\exp\left\{it\frac{S_n}{\sqrt{DS_n}}\right\}\right| dt \le \int\limits_T^\infty e^{-t^2/2} dt \le \frac{\varepsilon}{2},$$

for T and n sufficiently large. It remains to show that

$$\int\limits_{\delta\pi\sqrt{DS_n}\le|t|\le\pi\sqrt{DS_n}} \left|E\exp\left\{it\frac{S_n}{\sqrt{DS_n}}\right\}\right| dt \le \frac{\varepsilon}{2},$$

which can be done in the same way it was done above.

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