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# The Martingale Method in the Theory of Random Fields

*Linda Khachatryan, Boris S. Nahapetian*

Institute of Mathematics, National Academy of Science of Armenia

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## Martingales

A stochastic sequence of random variables $S_1, S_2, ..., S_k, ...$ is called *martingale* if for any $k \geq 1$

$$E\,|S_k| < \infty \qquad \text{and} \qquad E\left(S_{k+1}/\sigma\,(S_i, 1 \leq i \leq k)\right) = S_k \text{ (a.s.)}$$

Basic example 1. Let $\eta$ be a random variable, $E\left|\eta\right| < \infty$, and let $\{\Im_k, k \geq 1\}$ be a family of $\sigma$-algebras, $\Im_k \subset \Im_{k+1}$. Then the sequence of random variables

$$S_k = E\left(\eta/\Im_k\right), \qquad k = 1, 2, 3, ...$$

forms a martingale.

Basic example 2. Let $\eta_1, \eta_2, ..., \eta_k, ...$ be a random variables. Then the sequence of random variables

$$S_k = \sum_{j=1}^{k-1} \Big(\eta_j - E\Big(\eta_j/\sigma\left(\eta_i, 1 \leq i \leq j-1\right)\Big)\Big), \qquad k = 1, 2, 3, ...$$

forms a martingale.

## Convergence theorems

**Theorem.** *Let* $S_1, S_2, ..., S_n, ...$ *be a martingale such that*

$$\sup_n E\,|S_n| < \infty.$$

*Then with probability 1 there exists limit*

$$\lim_{n\to\infty} S_n = S_\infty,$$

*and* $E\,|S_\infty| < \infty$*.*

# Limit theorems

The sequence $\xi_1, \xi_2, ..., \xi_k, ...$ of random variables is called a *martingale–difference* if for any $k \geq 1$

$$E\,|\xi_k| < \infty \qquad \text{and} \qquad E\left(\xi_{k+1}/\sigma\,(\xi_i, 1 \leq i \leq k)\right) = 0 \text{ (a.s.)}$$

**Theorem** (Billingsli, Ibragimov)**.** *Let* $\xi_1, \xi_2, ..., \xi_k, ...$ *be a stationary ergodic process which forms a martingale-difference, and* $0 < E\xi_0^2 < \infty$*. Then*

$$\lim_{n\to\infty} P\left(\frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{n}\xi_k < x\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-u^2/2}du, \qquad x \in \mathbb{R}.$$

# Multidimensional martingales

*Krikeberg K. (1956)* ''Convergence of martingales with directed index set'', Trans. Amer. Math. Soc. 83

*Helms L.L. (1958)* ''Mean convergence of martingales'', Trans. Amer. Math. Soc. 87

*Chow Y. S. (1960)* ''Martingales in a $\sigma$-finite measure space indexed by directed sets'', Trans. Amer. Math. Soc. 97

*Cairoly R., Walsh J. (1975)* ''Stochastic integrals in the plane'', Acta Math. 134

$\mathbb{Z}^d$ — $d$-dimensional integer lattice

Let $\left\{S_V, V \subset \mathbb{Z}^d, |V| < \infty\right\}$ be a family of random variables on $(\Omega, \Im, \mathsf{P})$ parameterized by finite subsets of $\mathbb{Z}^d$.

Let $\left\{\Im_V, V \subset \mathbb{Z}^d\right\}$ be a partially ordered by inclusion set of $\sigma$-subalgebras of $\Im$, i.e.

$$\Im_V \subset \Im, \qquad \Im_I \subset \Im_V \text{ as } I \subset V, \qquad \Im_\varnothing = \{\varnothing, \Omega\}$$

The family of random variables $S = (S_V, \Im_V)$ is called martingale, if for any finite $V \subset \mathbb{Z}^d$ $S_V$ is $\Im_V$-measurable and for any $I \subset V$

$$E\left(S_V / \Im_I\right) = S_I \text{ (a.s.)}$$

## Martingale–difference random fields


*Nahapetyan B.S., Petrosyan A.N. (1992)* ''Martingale-difference Gibbs random fields and central limit theorem'', Ann. Acad. Sci. Fennicae, Ser. A. I. Math. 17

*Nahapetyan B.S., Petrosyan A.N. (1995)* ''Martingale-difference random fields. Limit theorems and some applications'', Vienna, Preprint ESI 283

*Nahapetian B.S. (1995)* ''Billingsley-Ibragimov Theorem for martingale-difference random fields and its applications to some models of classical statistical physics'', C. R. Acad. Sci. Paris 320

*Nahapetian B.S. (1997)* ''Models with even potential and the behaviour of total spin at the critical point'', Commun. Math. Phys. 189

# Subsequent studies of the limit theorems for multidimensional martingales

*Pogosyan S., Roelly S. (1998)* ''Invariance principle for martingale-difference random fields'', Statist. Probab. Lett. 38

*Dedecer J. (1998)* ''A central limit theorem for stationary random fields'', Probab. Theory Relat. Fields 110

*Comets F., Janžura M. (1998)* ''A central limit theorem for conditionally centred random fields with an application to Markov fields'', J. Appl. Prob. 35

*El Machkouri M., Volný D. (2004)* ''On the central and local limit theorem for martingale difference sequences'', Stochastics and Dynamics 4

*Banys P. (2011)* ''CLT for linear random fields with stationary martingale-difference innovations'', Lith. Math. J.

# Martingale-difference random fields

A collection of random variables $(\xi_t) = \left(\xi_t, t \in \mathbb{Z}^d\right)$, each of which takes value in $X$ will call a random field defined on $\mathbb{Z}^d$ with phase space $X$, $X \subset \mathbb{R}$

A random field $(\xi_t)$ is called a *martingale–difference random* field if for any $t \in \mathbb{Z}^d$

$$E\left|\xi_t\right| < \infty \qquad \text{and} \qquad E\left(\xi_t / \sigma\left(\xi_s, s \in \mathbb{Z}^d \setminus \{t\}\right)\right) = 0 \text{ a.s.}$$

Let $(\xi_t)$ be a martingale difference random field. Then the family of random variables $S_V = \sum\limits_{t \in V} \xi_t$ forms a martingale with respect to any increasing sequence $\{V_n\}$ of finite subsets of $\mathbb{Z}^d$ such that $V_n \subset V_{n+1}$ and $\bigcup\limits_{n=1}^{\infty} V_n = \mathbb{Z}^d$.

## The Central Limit Theorem for martingale–difference random fields

**Theorem** (Nahapetian (1995)). *Let* $(\xi_t)$ *be a homogeneous ergodic martingale-difference random field such that* $0 < E\eta_0^2 < \infty$. *Then*

$$\lim_{n\to\infty} P\left(\frac{S_{V_n}}{\sqrt{DS_{V_n}}} < x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du, \qquad x \in \mathbb{R},$$

*where* $S_V = \sum\limits_{t\in V} \xi_t$, *and* $V_n$ *is a* $d$*-dimensional cube with side length* $n$, $n = 1, 2, \ldots$.

Homogenous random field $(\xi_t)$ is called ergodic if for any finite $I, \Lambda \subset \mathbb{Z}^d$

$$\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{a\in V_n} P\left(\{\xi_t = x_t, t \in I\} \cap \left\{\xi_{s+a} = \bar{x}_s, s \in \Lambda\right\}\right) =$$

$$= P\left(\xi_t = x_t, t \in I\right) P\left(\xi_s = \bar{x}_s, s \in \Lambda\right),$$

where $x \in X^I$, $\bar{x} \in X^\Lambda$.

Homogenous random field $(\xi_t)$ satisfies the strong mixing condition with coefficient $\varphi_I$ if for any fixed finite $I \subset \mathbb{Z}^d$

$$\sup\left\{\left|P\left(A/B\right) - P\left(A\right)\right|, A \in \Im_I, B \in \Im_\Lambda, P\left(B\right) > 0\right\} \le \varphi_I\left(\rho\left(I, \Lambda\right)\right),$$

where function $\varphi_I(\rho)$, $\rho \in \mathbb{R}$ is such that $\varphi_I(\rho) \to 0$ as $\rho \to \infty$ and the finite set $I \subset \mathbb{Z}^d$ is fixed.

$$\Im_S = \sigma\left(\xi_s, s \in S\right)$$

# Exact asymptotic for the moments of sums of components

**Theorem** (Khachatryan, Nahapetian (2013)). *Let* $(\xi_t)$ *be a homogenous martingale–difference random field with phase space* $X$, $1 < |X| < \infty$. *Then for any* $k = 1, 2, ...$

$$E\left(S_{V_n}\right)^{2k-1} = C_{2k-1} \cdot |V_n|^{k-1},$$

*where constant* $C_{2k-1}$ *does not depend on* $n$.

*If, moreover, the random field* $(\xi_t)$ *satisfies the strong mixing condition with coefficient* $\boldsymbol{\varphi}_I$ *such that*

$$\boldsymbol{\varphi}_I(j) \le |I| \cdot \boldsymbol{\varphi}(j) \qquad \textit{and} \qquad \sum_{j=1}^{\infty} j^{d-1} \cdot \boldsymbol{\varphi}(j) < \infty,$$

*then for any* $k = 1, 2, ...$

$$E\left(S_{V_n}\right)^{2k} = (2k-1)!!\left(E\xi_0^2\right)^k |V_n|^k + C_{2k}|V_n|^{k-1},$$

*where constant* $C_{2k}$ *does not depend on* $n$.

**Theorem.** *Let* $(\xi_t)$ *be a homogenous martingale–difference random field with phase space* $X$*, satisfying the strong mixing condition with coefficient* $\varphi_I$ *such that*

$$\varphi_I(j) \leq |I| \cdot \varphi(j) \qquad \text{and} \qquad \sum_{j=1}^{\infty} j^{d-1} \cdot \varphi(j) < \infty,$$

*and let* $E\xi_0^2 > 0$*. Then for random field* $(\xi_t)$ *the CLT is valid and for any* $k = 1, 2, \ldots$

$$E\left(\frac{S_{V_n}}{\sqrt{DS_{V_n}}}\right)^k \to E\zeta^k \qquad \text{при } n \to \infty$$

*where* $\zeta$ *is a random variable with standard normal distribution.*

**Theorem** (Khachatryan, Nahapetian (2013)). *Let $(\xi_t)$ be a homogenous martingale–difference random field with phase space $X$, satisfying the strong mixing condition with coefficient $\varphi_I$ such that*

$$\varphi_I(j) \le |I| \cdot \varphi(j) \qquad \text{and} \qquad \sum_{j=1}^{\infty} j^{d-1} \cdot \varphi(j) < \infty,$$

*and let $E\xi_0^2 > 0$. Then*

$$\sup_{x\in\mathbb{R}} \left| P\left( \frac{S_{V_n}}{\sqrt{DS_{V_n}}} < x \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \right| \le C \cdot n^{-d/8},$$

*where positive constant $C$ does not depend on $n$.*

**Theorem.** *Under conditions of Theorem*

$$P\left( \limsup_{n\to\infty} \frac{S_{V_n}}{\sqrt{2DS_{V_n} \ln\ln |V_n|}} = 1 \right) = 1.$$

## Basic examples of martingale–difference random fields

Example 1.

Let $(\xi_t)$ be a positive random field with symmetric (with respect to zero) phase space $X$ and even finite dimensional probability distributions, i.e. for any finite $V \subset \mathbb{Z}^d$

$$P_V > 0$$

and

$$P_V(\theta_t x_t, t \in V) = P_V(x_t, t \in V),$$

for any $\theta_t \in \{1, -1\}$. Then the random field $(\xi_t)$ is a martingale–difference.

# Example 2.

Let $(\xi_t)$ be a random field with phase space $X$ for which there exists a partition $\Pi = \{X_1, X_2, ..., X_n\}$

$$X = \bigcup_{k=1}^{n} X_k, \qquad X_i \cap X_j = \varnothing, \quad i \neq j,$$

such that one-dimensional conditional probability distributions

$$q_t^{\bar{x}}(x) = P\left(\xi_t = x \big/ \xi_s = \bar{x}_s, s \in \mathbb{Z}^d \setminus \{t\}\right)$$

of random field $(\xi_t)$ take constant values on elements of the partition $\Pi$, i.e. for any $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ and $t \in \mathbb{Z}^d$

$$q_t^{\bar{x}}(x) = q_{t,k}^{\bar{x}}, \qquad x \in X_k, \quad k = \overline{1,n}.$$

If the partition $\Pi$ such that

$$\sum_{x \in X_k} x = 0 \qquad \text{for any } k = \overline{1,n},$$

then the random field $(\xi_t)$ is a martingale–difference.

## Example 3. Martingale–difference Markov random fields

Let $(\xi_t)$ be a Markov random field with phase space $X$ for which there exists a partition $\Pi = \{X_1, X_2, ..., X_n\}$ such that for $x, x' \in X_k$

$$P\left(\xi_t = x\right) = P\left(\xi_t = x'\right)$$

and

$$P\left(\xi_t = x / \xi_s = \bar{x}_s, s \in \partial t\right) = P\left(\xi_t = x' \middle/ \xi_s = \bar{x}_s, s \in \partial t\right)$$

for any $k = \overline{1, n}$ и $\bar{x} \in X^{\partial t}$, where $\partial t$ is neighborhood of a point $t$, $t \in \mathbb{Z}^d$. If the partition $\Pi$ such that

$$\sum_{x \in X_k} x = 0 \qquad \text{for any } k = \overline{1, n},$$

then the Markov random field $(\xi_t)$ is a martingale–difference.

# Conditional distribution of a random field

*Dobrushin R.L. (1968)* ''The description of random field by means of conditional probabilities and conditions of its regularity'', Theory Probab. Appl. 13

For a given random field $(\xi_t)$ with distribution $P$ the conditional probability $q_V^{\bar{x}}(x)$, $x \in X^V$ in finite volume $V \subset \mathbb{Z}^d$ with boundary conditions $\bar{x} \in X^{\mathbb{Z}^d \setminus V}$ is the limit

$$q_V^{\bar{x}}(x) = \lim_{\tilde{V} \uparrow \mathbb{Z}^d \setminus V} \frac{P_{V \cup \tilde{V}}\left(x_V \bar{x}_{\tilde{V}}\right)}{P_{\tilde{V}}\left(\bar{x}_{\tilde{V}}\right)},$$

which exists almost everywhere.

## Gibbs random fields

*Dachian S., Nahapetian B.S. (2009)* ''On Gibbsiannes of Random Fields'', Markov Processes and Related Fields 15

A random field $(\xi_t)$ with distribution $P$ is called *Gibbs random field*, if

1. $P_V(x) > 0$ for any finite $V \subset \mathbb{Z}^d$ and $x \in X^V$;

2. the limits

$$q_t^{\bar{x}}(x) = \lim_{V \uparrow \mathbb{Z}^d \setminus \{t\}} \frac{P_{\{t\} \cup V}(x\bar{x})}{P_V(\bar{x})}, \qquad x \in X, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}, t \in \mathbb{Z}^d$$

   exist, are positive, and the convergence is uniform with respect to $\bar{x}$.

*Dobrushin R.L. (1968)* ''Gibbs random fields for lattice systems with pair–wise interaction'', Funct. Anal. Appl. 2

*Gibbs random field* corresponding to the potential $\Phi$ is a random field which has a version of conditional distribution almost everywhere coinciding with Gibbs specification

$$\mathcal{Q} = \left\{ q_V^{\bar{x}}, \bar{x} \in X^{\mathbb{Z}^d \setminus V}, V \subset \mathbb{Z}^d \right\},$$

where

$$q_V^{\bar{x}}(x) = \frac{\exp\left\{-U_V^{\bar{x}}(x)\right\}}{\sum\limits_{z \in X^V} \exp\left\{-U_V^{\bar{x}}(z)\right\}}, \qquad x \in X^V,$$

$$U_V^{\bar{x}}(x) = \sum_{J \subset V : J \neq \varnothing} \sum_{\tilde{J} \subset \mathbb{Z}^d \setminus V} \Phi_{J \cup \tilde{J}}\left(x_J \bar{x}_{\tilde{J}}\right)$$

$\Phi = \left\{ \Phi_V(x), x \in X^V, V \subset \mathbb{Z}^d, |V| < \infty \right\}$ is an interaction potential

$$\|\Phi\| = \sum_{0 \in V} \sup_{x \in X^V} |\Phi_V(x)| < \infty$$

## Example 4. Martingale–difference Gibbs random fields

**Theorem** (Nahapetian, Petrosian (1992)). *Let potential* $\Phi$ *with spin space* $X$ *be even, i.e. for any finite* $V \subset \mathbb{Z}^d$

$$\Phi_V(\theta_t x_t, t \in V) = \Phi_V(x_t, t \in V).$$

*If* $X$ *is symmetric with respect to zero, then Gibbs random field corresponding to the potential* $\Phi$ *is a martingale–difference.*

## Example 4'. Martingale–difference Gibbs random fields

**Theorem** (Khachatryan, Nahapetian (2013)). *Let potential* $\Phi$ *takes constant values on elements of partition* $\Pi = \{X_1, X_2, ..., X_n\}$ *of spin space* $X$*, i.e. for any finite* $V \subset \mathbb{Z}^d$ *and* $t \in \mathbb{Z}^d \backslash V$

$$\Phi_{\{t\}\cup V}(x\bar{x}) = \Phi_{\{t\}\cup V}\left(x'\bar{x}\right) \qquad \text{for any } x, x' \in X_k,$$

*where* $\bar{x} \in X^V$*,* $k = \overline{1,n}$*. If the partition* $\Pi$ *is such that*

$$\sum_{x \in X_k} x = 0, \qquad k = \overline{1,n},$$

*then Gibbs random field corresponding to the potential* $\Phi$ *is a martingale–difference.*

# Application of martingale method in statistical physics

## Limit Theorems for Gibbs random fields

Conditions on the potential:
a smallness of the norm of the potential;
conditions on the rate of convergence of the potential.

**Theorem** (Dobrushin, Tirozzi (1977)). *If the potential $\Phi$ is bounded then the local limit theorem for the corresponding Gibbs random field follows from the central limit theorem.*

# Limit Theorems for martingale–difference Gibbs random fields

**Theorem.** *Let potential* $\Phi$ *with spin space* $X$ *be translation–invariant, ergodic, and let potential* $\Phi$ *takes constant values on elements of a partition* $\Pi = \{X_1, X_2, ..., X_n\}$ *of set* $X$*, i.e. for any finite* $V \subset \mathbb{Z}^d$ *and* $t \in \mathbb{Z}^d \backslash V$

$$\Phi_{\{t\}\cup V}(x\bar{x}) = \Phi_{\{t\}\cup V}\left(x'\bar{x}\right) \qquad \text{for any } x, x' \in X_k,$$

*where* $\bar{x} \in X^V$*,* $k = \overline{1,n}$*. If the partition* $\Pi$ *is such that*

$$\sum_{x \in X_k} x = 0, \qquad k = \overline{1,n},$$

*then for Gibbs random field corresponding to the potential* $\Phi$ *CLT and LLT are valid.*

**Theorem.** *Let potential* $\Phi$ *with spin space* $X$ *be translation–invariant, ergodic, and let potential* $\Phi$ *takes constant values on elements of a partition* $\Pi = \{X_1, X_2, ..., X_n\}$ *of set* $X$*, and*

$$\sum_{x \in X_k} x = 0, \qquad k = \overline{1, n}.$$

*Let further the potential* $\Phi$ *satisfies the following conditions*

$$\frac{1}{2} e^{4\|\Phi\|} \left( e^{4\|\Phi\|} - 1 \right) < 1,$$

$$\sum_{0 \in V \subset \mathbb{Z}^d} |V| \left( diamV \right)^{\gamma} \sup_{x \in X^V} |\Phi_V(x)| < \infty, \qquad \gamma > d - 1,$$

*where*

$$\|\Phi\| = \sum_{0 \in V \subset \mathbb{Z}^d} |V| \sup_{x \in X^V} |\Phi_V(x)|, \qquad diamV = \sup_{t,s \in V} |t - s| .$$

*Then for Gibbs random field corresponding to the potential* $\Phi$

1.
$$M\left(\frac{S_{V_n}}{\sqrt{DS_{V_n}}}\right)^k \to M\zeta^k \qquad \text{as } n \to \infty,$$
*for any* $k = 1, 2, \ldots,$ *where* $\zeta$ *is a random variable with standard normal distribution;*

2.
$$\sup_{x\in\mathbb{R}}\left|P\left(\frac{S_{V_n}}{\sqrt{DS_{V_n}}} < x\right) - \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-u^2/2}du\right| \leq C \cdot n^{-d/8},$$
*where the positive constant* $C$ *does not depend on* $n$*;*

3. *the law of iterated logarithm is valid.*

# The Ising ferromagnetic model

$$\Phi_V(x) = \begin{cases} -\beta h \cdot x_t, & V = \{t\} \\ -\beta \cdot x_t x_s, & V = \{t, s\} \text{ и } \|t - s\| = 1 \\ 0, & \text{in other cases} \end{cases}$$

where

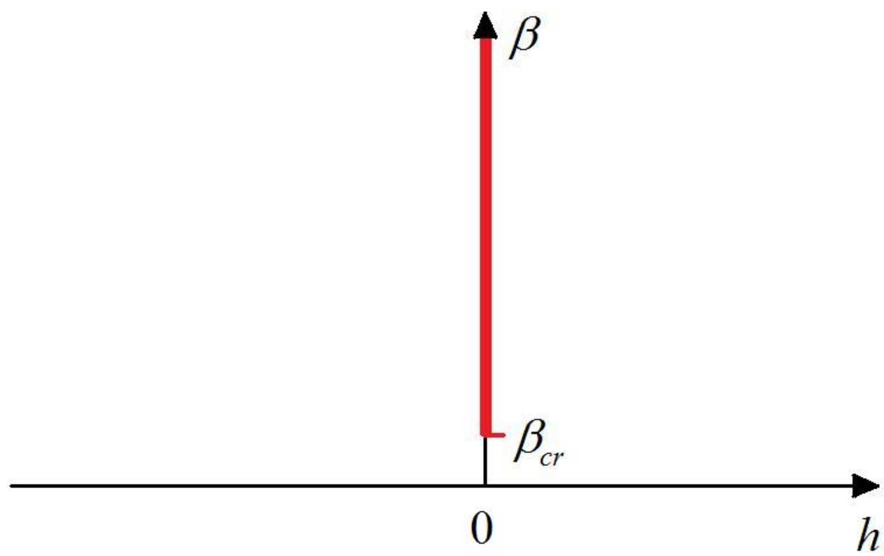
$$x_t, x_s \in X = \{-1, 1\},$$

$$\|t - s\| = \sum_{i=1}^{d} \left| t^{(i)} - s^{(i)} \right|, \qquad t, s \in \mathbb{Z}^d,$$

$h \in \mathbb{R}$ — the external field,

$\beta > 0$ — the inverse temperature.

# Phase diagram for the Ising model




$(0, \beta_{cr})$ — the critical point for the Ising model

Nahapetian, 1997

## Martingale model

$$\tilde{\Phi}_V(y) = \begin{cases} -\beta h \cdot |y_t|, & V = \{t\} \\ -\beta |y_t| \cdot |y_s|, & V = \{t, s\} \text{ и } \|t - s\| = 1 \\ 0, & \text{in other cases} \end{cases}$$

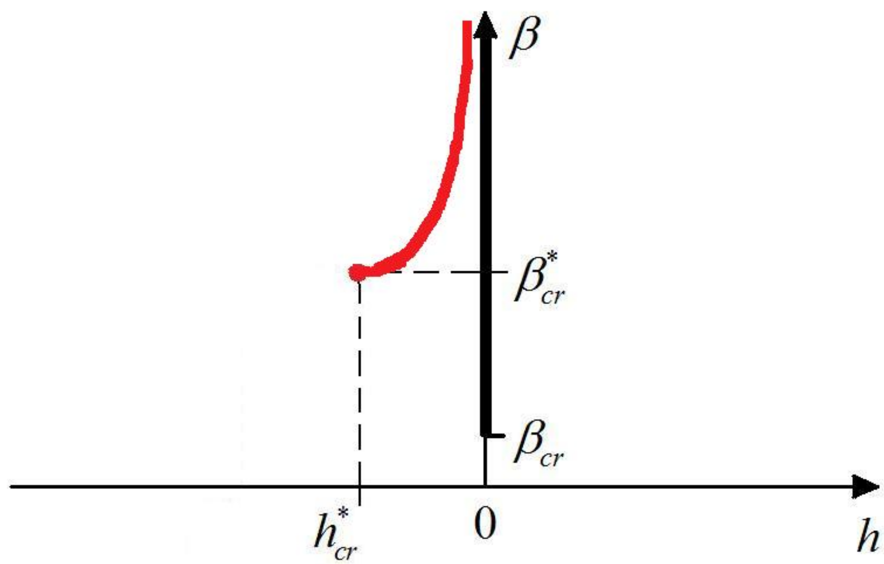
where

$$y_t, y_s \in Y = \{-1, 0, 1\}$$

$$\varphi : Y \to X$$

$$x = \varphi(y) = 2|y| - 1$$

# Phase diagram for the martingale model




Coordinates of the critical point for the martingale model

$$\beta^*_{cr} = 4\beta_{cr}, \qquad h^*_{cr} = -d - \frac{\ln 2}{\beta^*_{cr}},$$

where $\beta_{cr}$ is the critical temperature for the Ising model

## The connection formulas of total spins probability distribution

Denote for any finite $V \subset \mathbb{Z}^d$

$S_V^{Is}$ — total spin of the Ising model

$S_V^{ev}$ — total spin of the model with even potential

$$P\left(S_V^{ev} = k\right) = \sum_{j=0}^{\frac{|V|-k}{2}} 2^{-(k+2j)} \begin{pmatrix} k+2j \\ 2j \end{pmatrix} P\left(S_V^{Is} = 2k + 4j - n\right),$$

$$P\left(S_V^{ev} = k\right) = P\left(S_V^{ev} = -k\right)$$

$k = 0, 1, ..., |V|$

$$P\left(S_V^{Is}=k\right)=2^{-k}\Big[P\left(S_V^{ev}=k\right)-$$

$$-\sum_{j=1}^{(|V|-k)/2}\left(\begin{array}{c}k+2j\\2j\end{array}\right)P\left(S_V^{ev}=k+2j\right)+$$

$$+\sum_{j=1}^{(|V|-k)/2}\left(\begin{array}{c}k+2j\\2j\end{array}\right)2^{2j}\times$$

$$\times\sum_{s=1}^{(|V|-k)/2-j}\left(\begin{array}{c}k+2\left(j+s\right)\\2s\end{array}\right)2^{2s}P\left(S_V^{ev}=k+2\left(j+s\right)\right)-$$

$$-\sum_{j=1}^{(|V|-k)/2}\left(\begin{array}{c}k+2j\\2j\end{array}\right)2^{2j}\sum_{s=1}^{(|V|-k)/2-j}\left(\begin{array}{c}k+2\left(j+s\right)\\2s\end{array}\right)2^{2s}\times$$

$$\times\sum_{l=1}^{(|V|-k)/2-j-s}\left(\begin{array}{c}k+2\left(j+s+l\right)\\2l\end{array}\right)2^{2l}P\left(S_V^{ev}=k+2\left(j+s+l\right)\right)+\ldots\Big]$$

# Associated martingale-difference random fields

**Theorem.** *Let there be given a random field* $(\xi_t)$ *with phase space* $X$, *a set* $Y$, *a surjective map* $\varphi : Y \to X$ *and a set of randomizations* $R = \left\{R_t, t \in \mathbb{Z}^d\right\}$. *Then there exists an* associated *random field* $(\eta_t)$ *with phase space* $Y$ *such that for any* $t \in \mathbb{Z}^d$

$$\xi_t = \varphi(\eta_t).$$

*Finite dimensional probability distributions of random field* $(\eta_t)$ *have a form*

$$P(\eta_t = y_t, t \in V) = \prod_{t \in V} R_t^{x_t}(y_t) \cdot P(\xi_t = x_t, t \in V),$$

*where* $y_t \in \varphi^{-1}(x_t)$, $x_t \in X$, $t \in V$, *and* $V$ *is a finite subset of* $\mathbb{Z}^d$.

# Properties of associated random fields

Associated random fields inherit properties of a given random field such as

1. homogeneity;
2. ergodicity;
3. weak dependence;
4. a random field associated with Gibbs random field is also a Gibbsian.

## Associated martingale–difference random fields

**Theorem.** *Let* $(\xi_t)$ *be a random field with phase space* $X$*, and let* $(\eta_t)$ *be a random field with phase space* $Y$ *associated with* $(\xi_t)$ *by means of map* $\varphi$ *and set of randomizations* $R$*. If for any* $x \in X$ *and* $t \in \mathbb{Z}^d$

$$\sum_{y \in \varphi^{-1}(x)} y \cdot R_t^x(y) = 0,$$

*then the random field* $(\eta_t)$ *is a martingale–difference.*

## Limit theorems for martingale-difference (Gibbs) random fields

**Theorem.** *Let* $(\xi_t)$ *be a homogenous ergodic (Gibbs) random field. Then there exists a martingale-difference (Gibbs) random field* $(\eta_t)$*, associated with random field* $(\xi_t)$*, for which the CLT is valid.*

# Application of the martingale method to the Ising model

Let $(\xi_t)$ be the homogenous Gibbs random field with phase space $X = \{0, 1\}$, and let $(\eta_t)$ be an associated random field with phase space $Y = \{-1, 0, 1\}$ such that

$$\xi_t = \varphi(\eta_t) = \eta_t^2, \qquad t \in \mathbb{Z}^d$$

and for any $t \in \mathbb{Z}^d$

$$P(\eta_t = 1) = P(\eta_t = -1) = \frac{1}{2} P(\xi_t = 1), \qquad P(\eta_t = 0) = P(\xi_t = 0).$$

The random field $(\eta_t)$ is a martingale-difference; and for any finite $V \subset \mathbb{Z}^d$

$$P(\eta_t = y_t, t \in V) = 2^{-\sum\limits_{t \in V} x_t} P(\xi_t = x_t, t \in V),$$

$y_t \in \varphi^{-1}(x)$, $x_t \in X$, $t \in V$.

# The connection formulas of total spins probability distributions

For any finite $V \subset \mathbb{Z}^d$

$$P\left(S_V^\eta = k\right) = \sum_{j=0}^{\frac{|V|-k}{2}} 2^{-(k+2j)} \begin{pmatrix} k+2j \\ 2j \end{pmatrix} P\left(S_V^\xi = k+2j\right),$$

$$P\left(S_V^\eta = -k\right) = P\left(S_V^\eta = k\right),$$

$k = 1, 2, ..., |V|$.

For any finite $V \subset \mathbb{Z}^d$

$$P\left(S_V^{\xi} = k\right) = 2^k \sum_{j=0}^{\frac{|V|-j}{2}} (-1)^j \frac{k+2j}{k+j} \left( \begin{array}{c} k+j \\ j \end{array} \right) P\left(S_V^{\eta} = k+2j\right),$$

$k = 0, 1, ..., |V|$

## Characteristic function of total spin of given r.f. by means of total spin distributions of associated r.f.

For any finite $V \subset \mathbb{Z}^d$

$$f_{S_V^\xi}(t) = \sum_{j=-|V|}^{|V|} \cos\left(j \cdot \operatorname{arccos} e^{it}\right) P\left(S_V^\eta = j\right)$$

# The connection formula of moments of total spins

For any $k = 1, 2, ...$

$$E\left(S_V^{\xi}\right)^k = \sum \left( \frac{k!}{m_1! m_2! \cdot ... \cdot m_k! \, (1!)^{m_1} (2!)^{m_2} \cdot ... \cdot (k!)^{m_k}} \cdot \right.$$

$$\left. \cdot \frac{1}{(2m-1)!!} \sum_{i=1}^{m} a_{m,i} E\left(S_V^{\eta}\right)^{2i} \right)$$

where sum is taken by all integers $m_1, m_2, ..., m_k \geq 0$ such that

$$1 \cdot m_1 + 2 \cdot m_2 + ... + k \cdot m_k = k,$$

$m = \sum\limits_{i=1}^{k} m_i$, and coefficients $a_{m,i}$ are defined by the following relation

$$\prod_{s=0}^{m-1} \left(x^2 - s^2\right) = \sum_{i=1}^{m} a_{m,i} x^{2i}$$

## Investigation of the Ising model total spin at the critical point

$(0, \beta_{cr})$ — critical point for the Ising model

$(h^*_{cr}, \beta^*_{cr})$ — critical point for the martingale model, where

$$\beta^*_{cr} = 4\beta_{cr}, \qquad h^*_{cr} = -d - \frac{\ln 2}{\beta^*_{cr}},$$

For the martingale model the CLT and the LLT hold at the critical point $(h^*_{cr}, \beta^*_{cr})$.

Obtained formulas can be applied for discovering a limit distribution of the Ising model total spin at its critical point $(0, \beta_{cr})$.

**Thank you for your attention!**

## References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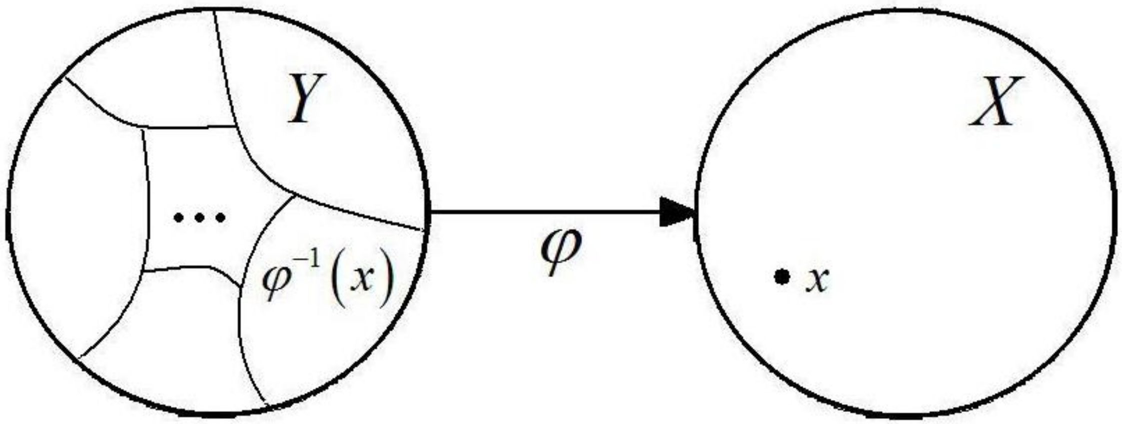
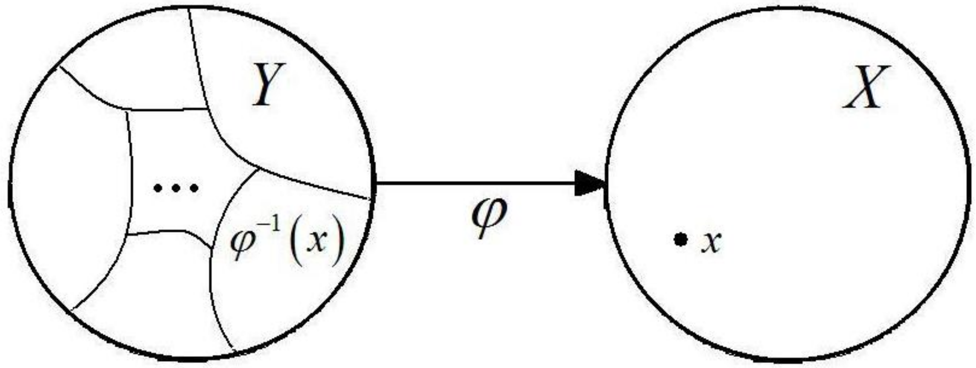
# Randomization

$X, Y \subset \mathbb{R}$ — finite sets,
$\varphi : Y \to X$ — a surjective map,
$\varphi^{-1}(x) = \{y \in Y : \varphi(y) = x\}$, $x \in X$




$$Y = \bigcup_{x \in X} \varphi^{-1}(x)$$

$P$ — probability distribution on $X$




Randomization $R^{\varphi} = \{R^{\varphi,x}, x \in X\}$ is the set of probability distributions on $Y$ such that for any $x \in X$

$$R^{\varphi,x}(y) > 0, \quad y \in \varphi^{-1}(x) \qquad \text{and} \qquad R^{\varphi,x}(y) = 0, \quad y \notin \varphi^{-1}(x).$$

Put

$$\widehat{P}(y) = R^{\varphi,x}(y)\, P(x) \qquad \text{as } y \in \varphi^{-1}(x)$$

Then $\widehat{P}$ is a probability distribution on $Y$ and

$$P(x) = \sum_{y \in \varphi^{-1}(x)} \widehat{P}(y), \quad x \in X$$