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Martingale method in the theory of random fields and its application in statistical physics

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Martingale method

We introduce a new approach to prove limit theorems for random fields, which essentially based on properties of martingale–difference random fields.

Using this method we extend the range of validity of various limit theorems (the central, the functional and the local limit theorems) for random fields, particularly for Gibbs random fields. By this martingale approach one can give an other proof of the asymptotical normality of total spin for Ising model outside the critical point.

Nahapetian, Petrosian, 1992

Martingale-difference random fields

A collection of random variables $(\eta_t) = \left(\eta_t, t \in \mathbb{Z}^d\right)$, each of which takes value in Y will call a random field defined on $\mathbb{Z}^d$ with phase space Y.

A random field (η_t) is called a ***martingale-difference random field*** if for any $t \in \mathbb{Z}^d$

$$E\,|\eta_t| < \infty \qquad \text{and} \qquad E\left(\eta_t \big/ \sigma\left(\eta_s, s \in \mathbb{Z}^d \setminus \{t\}\right)\right) = 0 \text{ a.s.}$$

Nahapetian, 1995

The Central Limit Theorem for martingale-difference random fields

Theorem. Let (η_t) be a homogeneous ergodic martingale-difference random field such that $0 < \sigma^2 = E\eta_0^2 < \infty$. Then

$$\lim_{n\to\infty} P\left(\frac{1}{\sigma \cdot n^{d/2}} \sum_{t\in V_n} \eta_t < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \qquad x \in \mathbb{R}^1,$$

where V_n is a d-dimensional cube with side length n, $n = 1, 2, ...$

Martingale-difference Gibbs random fields

Nahapetian, Petrosian, 1992

Let (η_t) be a Gibbs random field corresponding to a potential Φ and let components of this random field take values in some symmetric (with respect to zero) set Y. If the potential Φ is even, i.e.

$$\Phi_V\left(\theta_t y_t, t \in V\right) = \Phi_V\left(y_t, t \in V\right), \quad y_t \in Y, V \in W,$$

for any $\theta_t \in \{1, -1\}$, then the Gibbs random field (η_t) is a martingale-difference.

The Central Limit Theorem for Gibbs random fields

Corollary 1. Let Φ be an even potential, such that the corresponding Gibbs random field (η_t) is homogenous ergodic random field, and $E\eta_0^2 > 0$. Then for this random field the central limit theorem is valid.

The Local Limit Theorem for Gibbs random fields

Dobrushin, Tirozzi, 1977

Theorem. If the potential Φ is bounded then the local central limit theorem for the corresponding Gibbs field follows from the integral central limit theorem.

Corollary 2. Let Φ be an even potential with finite range, such that corresponding Gibbs random field (η_t) is homogenous ergodic random field and $E\eta_0^2 > 0$. Then for this random field LLT is valid.

Application of martingale method in statistical physics

In classical statistical physics it is usually assumed that the central limit theorem is not valid at the critical points for $d = 2, 3$ dimension lattice models.

Ising ferromagnetic model

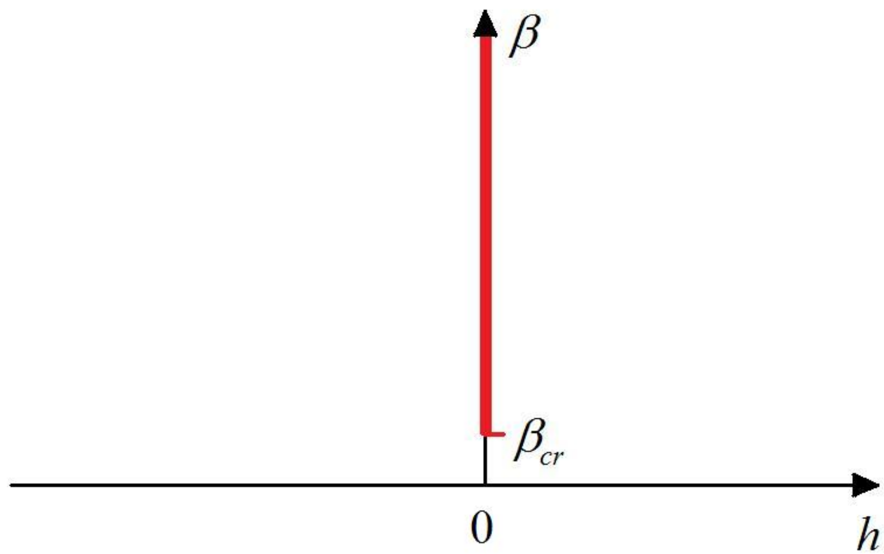
Ising ferromagnetic model is defined on the integer lattice $\mathbb{Z}^d$, $d \geq 1$ by means of the nearest neighbour potential

$$\Phi_{\{t,s\}}(x_t, x_s) = \begin{cases} x_t \cdot x_s, & |t-s| = 1, \\ 0, & |t-s| \neq 1, \end{cases}$$

where $x_t, x_s \in X = \{-1, 1\}$ and $|t-s| = \max\limits_{1 \leq i \leq d} \left| t^{(i)} - s^{(i)} \right|$.

Phase diagram for the Ising model

$h \in \mathbb{R}^1$ — the external field, $\beta \in \mathbb{R}^1_+$ — the inverse temperature



$(0, \beta_{cr})$ — the critical point for the Ising model

There is an opinion that the central limit theorem is not valid at the critical point for multidimensional models.

Nahapetian, 1997

Model with even potential

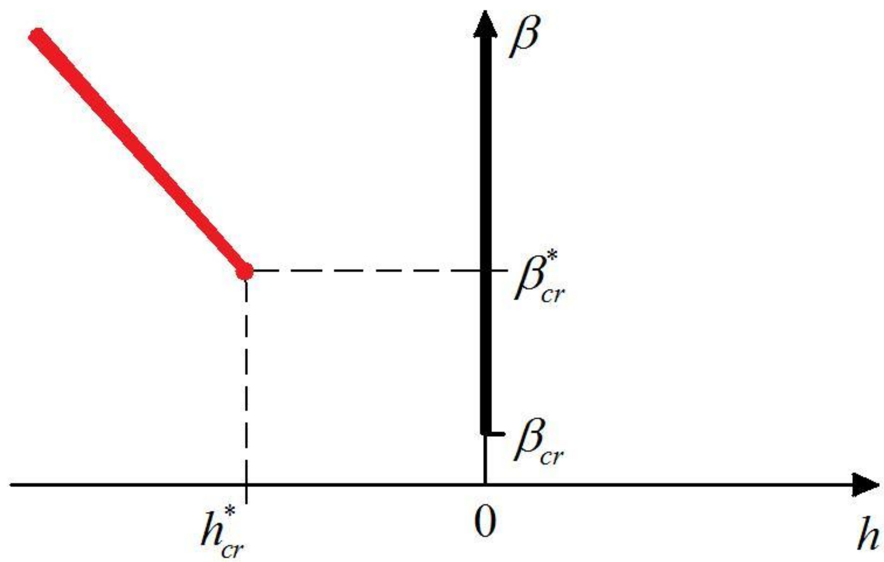
$$\tilde{\Phi}_{\{t,s\}}(y_t, y_s) = \begin{cases} |y_t| \cdot |y_s|, & |t-s| = 1, \\ 0, & |t-s| \neq 1, \end{cases}$$

where $y_t, y_s \in Y = \{-1, 0, 1\}$, $t, s \in \mathbb{Z}^d$.

$$x = 2|y| - 1$$

Phase diagram for the model with even potential

There is a phase transition on the line $h + \beta d - \ln 2 = 0$, $0 < \beta < \infty$



Coordinates of critical point for model with even potential

$$\beta^*_{cr} = 4\beta_{cr} \qquad \text{and} \qquad h^*_{cr} = -\beta^*_{cr} d + \ln 2.$$

The connection formulas of total spins probability distribution

Denote for finite $V \subset \mathbb{Z}^d$

S_V^{Is} — total spin of the Ising model

S_V^{ev} — total spin of the model with even potential

$$P\left(S_V^{ev} = k\right) = \sum_{j=0}^{\frac{|V|-k}{2}} 2^{-(k+2j)} \left(\begin{array}{c} k+2j \\ 2j \end{array} \right) P\left(S_V^{Is} = 2k+4j-n\right),$$

$$P\left(S_V^{ev} = k\right) = P\left(S_V^{ev} = -k\right)$$

$k = 0, 1, ..., |V|$

$$P\left(S_V^{Is}=k\right)=$$

$$2^{\frac{k+|V|}{2}}\sum_{j=0}^{\frac{|V|-k}{4}}(-1)^j\frac{k+|V|+4j}{k+|V|+2j}\begin{pmatrix}\frac{k+|V|}{2}+j\\ j\end{pmatrix}P\left(S_V^{ev}=\frac{k+|V|+4j}{2}\right)$$

$$k=0,1,...,|V|$$

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Associated martingale-difference random fields

Let (ξ_t) be a random field with finite phase space X. Then there exists an associated random field (η_t) with finite phase space Y, which is martingale-difference. For such random fields one has

$$P\left(\xi_t = x_t, t \in V\right) = P\left(\varphi\left(\eta_t\right) = x_t, t \in V\right), \quad x_t \in X$$

where $\varphi : Y \to X$ is a surjective map.

There is a formula expressing finite dimensional probability distributions of the associated random field by means of finite dimensional probability distributions of the given random field.

Properties of associated random fields

1. If (ξ_t) is a homogenous ergodic random field, then associated random field (η_t) is also a homogenous and ergodic.

2. If random field (ξ_t) satisfies mixing condition with some coefficient γ, then associated random field (η_t) satisfies this mixing condition at least with same coefficient γ.

3. If (ξ_t) is a Gibbs random field, then associated random field (η_t) is also a Gibbsian.

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Limit theorems for martingale-difference (Gibbs) random fields

Theorem 1. Let (ξ_t) be a homogenous ergodic (Gibbs) random field. Then there exists a martingale-difference (Gibbs) random field (η_t), associated with random field (ξ_t), for which the central limit theorem is valid.

Let a Gibbs random field (η_t) with the phase space Y and the potential Φ be given. Assume there exists a partition $\Pi = \{Y_1, Y_2, ..., Y_n\}$ of Y

$$Y = \bigcup_{k=1}^{n} Y_k, \qquad Y_i \cap Y_j = \varnothing, \quad i \neq j$$

such that

$$\sum_{y \in Y_k} y = 0, \qquad k = \overline{1, n}. \tag{1}$$

If the potential Φ takes constant values on elements of partition Π, i.e. for any $V \in W$ and $t \notin V$

$$\Phi_{V \cup \{t\}}(y_t y_V) = \Phi_k(y_V), \qquad y_t \in Y_k, k = \overline{1, n}, \tag{2}$$

then the Gibbs random field (η_t) is a martingale-difference.

Theorem 2. Let (η_t) be a homogeneous ergodic Gibbs random field with the phase space Y and the potential Φ such that $E\eta_0^2 > 0$. If there exist a partition Π of Y satisfying (1) and the potential Φ satisfies (2) then for this Gibbs random field the central limit theorem is valid.

Theorem 3. Let (η_t) be a homogeneous ergodic Gibbs random field with phase space Y and the finite range potential Φ such that $E\eta_0^2 > 0$. If there exist a partition Π of Y satisfying (1) and the potential Φ satisfies (2) then the local limit theorem is valid.

Example of applying the martingale method to the Ising gas model

Let (ξ_t) be the homogenous Gibbs random field with phase space $X = \{0,1\}$, (η_t) be the associated random field with phase space $Y = \{-1,0,1\}$ such that

$$\xi_t = \eta_t^2, \qquad t \in \mathbb{Z}^d$$

and for any $t \in \mathbb{Z}^d$

$$P(\eta_t = 1) = P(\eta_t = -1) = \frac{1}{2} P(\xi_t = 1), \qquad P(\eta_t = 0) = P(\xi_t = 0).$$

The random field (η_t) is a martingale-difference

Connection formula for finite-dimensional probability distributions

$$P(\eta_t = y_t, t \in V) = 2^{-\sum\limits_{t \in V} x_t} P(\xi_t = x_t, t \in V),$$

$y_t \in Y$, $x_t \in X$, $t \in V$, $V \in W$.

The connection formulas of total spins probability distributions

For any finite $V \subset \mathbb{Z}^d$

$$P\left(S_V^{\eta} = k\right) = \sum_{j=0}^{\frac{|V|-k}{2}} 2^{-(k+2j)} \left(\begin{array}{c} k+2j \\ 2j \end{array} \right) P\left(S_V^{\xi} = k+2j\right),$$

$$P\left(S_V^{\eta} = -k\right) = P\left(S_V^{\eta} = k\right),$$

$k = 1, 2, ..., |V|.$

For any finite $V \subset \mathbb{Z}^d$

$$P\left(S_V^{\xi} = k\right) = 2^k \sum_{j=0}^{\frac{|V|-j}{2}} (-1)^j \frac{k+2j}{k+j} \left(\begin{array}{c} k+j \\ j \end{array} \right) P\left(S_V^{\eta} = k+2j\right),$$

$k = 0, 1, ..., |V|$

Characteristic function of total spin of given r.f. by means of total spin probabilities of associated r.f.

For any finite $V \subset \mathbb{Z}^d$

$$f_{S_V^\xi}(t) = \sum_{j=-|V|}^{|V|} \cos\left(j \cdot \operatorname{arccos} e^{it}\right) P\left(S_V^\eta = j\right)$$

The connection formula of moments of total spins

For any $k = 1, 2, ...$

$$E\left(S_V^{\xi}\right)^k = \sum \left(\frac{k!}{m_1! m_2! \cdot ... \cdot m_k! \, (1!)^{m_1} (2!)^{m_2} \cdot ... \cdot (k!)^{m_k}} \cdot \right.$$

$$\left. \cdot \frac{1}{(2m-1)!!} \sum_{i=1}^{m} a_{m,i} E\left(S_V^{\eta}\right)^{2i} \right)$$

where sum is taken by all integers $m_1, m_2, ..., m_k \geq 0$ such that

$$1 \cdot m_1 + 2 \cdot m_2 + ... + k \cdot m_k = k,$$

$m = \sum_{i=1}^{k} m_i$, and coefficients $a_{m,i}$ are defined by the following relation

$$\prod_{s=0}^{m-1} \left(x^2 - s^2\right) = \sum_{i=1}^{m} a_{m,i} x^{2i}$$

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For moments of total spin of associated martingale-difference random field (η_t) the following relations hold

$$E\left(S_{V_n}^{\eta}\right)^{2k-1} = O\left(|V_n|^{k-1}\right),$$

$$E\left(S_{V_n}^{\eta}\right)^{2k} = (2k-1)!!p^k\,|V_n|^k\left(1 + O\left(|V_n|^{-1}\right)\right),$$

where V_n is a d-dimensional cube with side length n, $E\eta_0^2 = p$ and $k = 1, 2, \ldots$.

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Thank you for your attention!