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The Fatou Property for General Approximate Identities on Metric Measure Spaces

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Abstract—Abstract approximate identities on metric measure spaces are considered in this paper. We find exact conditions on the geometry of domains for which the convergence of approximate identities occurs almost everywhere for functions from the spaces L^p, $p \ge 1$. The results are illustrated with examples of Poisson kernels and their powers in the unit ball in $\mathbb{R}^n$ or $\mathbb{C}^n$, and also of convolutions with dilatations on $\mathbb{R}^n$. In all these examples, the conditions found are exact.

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1. STATEMENT OF THE PROBLEM

Let us consider an example that was one of the motivations for posing the problem studied in this paper.

The boundary behavior of Poisson integrals

$$Pf(z) = \int_{\mathbb{T}} p(z,\zeta) f(\zeta)\,\frac{d\zeta}{2\pi}, \qquad \text{where} \quad p(z,\theta) = \frac{1-|z|^2}{|z-e^{i\theta}|^2} \tag{1.1}$$

is the Poisson kernel and $\mathbb{T} = [-\pi,\pi]$, in the unit disk of the complex plane is well known (see [1], [2]). For functions $f \in L^p(\mathbb{T})$, $1 \le p < \infty$, the boundary behavior is the same and does not depend on p: almost everywhere there exists a nontangential limit, i.e., a limit along the Fatou domains

$$D(\theta) = \{z \in \mathbb{C} : |z - e^{i\theta}| < a(1-|z|)\}, \qquad \theta \in \mathbb{T}$$

(here and further, $a > 0$ denotes any fixed number).

This result is exact in the sense that it becomes invalid for all domains of the form

$$\{z \in \mathbb{C} : |z-e^{i\theta}| < \Phi(1-|z|)(1-|z|)\},$$

where the function $\Phi\colon (0,1] \mapsto \mathbb{R}_+$ satisfies the condition $\Phi(t) \to +\infty$ as $t \to +0$ [3] (see also [4, Theorem 7.44]).

Let us now consider the boundary behavior of the following normalized convolutions with powers of the Poisson kernel:

$$\mathcal{P}_l f(z) = \frac{P_l f(z)}{P_l 1(z)}\,, \tag{1.2}$$

where

$$P_l f(z) = \int_{\mathbb{T}} [p(z,\xi)]^{l+1/2} f(\xi)\,d\xi, \qquad l \ge 0. \tag{1.3}$$

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If the kernels are not normalized, it is difficult to expect a good boundary behavior of such operators.

A similar question for the operator (1.2), where $l > 0$, is solved in the same way as for $l = 1/2$, because it coincides with the classical Poisson integral (1.1). But if $l = 0$ and $f \in L^p(\mathbb{T})$, $p \geq 1$, then, for almost all $\theta \in \mathbb{T}$ on the boundary, $\mathcal{P}_0 f$ has a limit along the domains

$$\left\{z \in \mathbb{C} : |z - e^{i\theta}| < a(1 - |z|)\left(\log \frac{2}{1 - |z|}\right)^p\right\}, \tag{1.4}$$

which essentially depend on p.

The boundary behavior of operators for functions from $L^p(\mathbb{T})$ (1.2) was studied in [6] for $p = 1$ and [7] for $p > 1$. In these papers, convergence almost everywhere was derived from a weak-type inequality for maximal operators corresponding to the domains (1.4). In [8, Theorem 2], these results were generalized to similar operators on metric measure spaces and, in [9, Theorem 1], strong-type inequalities for maximal operators were proved.

The boundary behavior of convolutions of functions from $L^1(\mathbb{T})$ with kernels φ_t with no special dependence on t on $\mathbb{T}$ were considered in [10], while results from [10] were extended in [11] to functions from $L^p(\mathbb{T})$, $p > 1$.

The following problem naturally arises: Find conditions on the kernels of approximate identities that will take into account the degree of summability of functions and explain why the Fatou domains for functions from L^p depend on p in some cases (as they do for the operators (1.2) for $l = 0$) and do not depend on p in other cases (as for the operators (1.1) or, more generally, for the operators (1.2) for $l > 0$).

We will study this problem for approximate identities in the case of functions on any metric measure space. In such a general situation, it is more natural to consider conditions on kernels that are not related to the concrete form of these kernels.

2. BASIC TERMINOLOGY

Let X be a Hausdorff space with Borel measure μ and quasimetric d (the triangle inequality in the axioms of the metric is replaced by the following condition: there exists a number $a_d \geq 1$ such that the inequality

$$d(x, y) \leq a_d[d(x, z) + d(z, y)]$$

holds for all $x, y, z \in X$), the measure of each ball $B \subset X$ being positive and finite. We will use the following notation:

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

for an open ball of radius $r > 0$ centered at a point $x \in X$ and

$$f_B = \fint_B f\, d\mu = \frac{1}{\mu(B)} \int_B f\, d\mu$$

for the mean value of a function $f \in L^1(B)$ over the ball $B \subset X$.

For $p \in [1, \infty)$, we use the standard notation

$$\|f\|_{L^p(X)} = \|f\|_p := \left(\int_X |f|^p\, d\mu\right)^{1/p}, \qquad 1 \leq p < \infty,$$

and $L^p(X)$ denotes the set (of equivalence classes) of measurable functions for which this value is finite. If $p > 1$, then q will always denote the conjugate exponent, and $1/p + 1/q = 1$.

The usual Chebyshev norm is denoted by

$$\|f\|_{L^\infty(X)} = \|f\|_\infty := \sup_{x \in X} |f(x)|.$$

The family of maximal Hardy–Littlewood functions M_p, $p \geq 1$, is defined by the equality

$$M_p f(x) = \sup_{B \ni x}\left(\fint_B |f|^p\, d\mu\right)^{1/p}, \qquad x \in X,$$

where the supremum is taken over all balls $B \subset X$ containing the point x.

We will assume that the doubling condition holds: there exists a number $a_\mu > 0$, such that

$$\mu(B(x,2r)) \le a_\mu \mu(B(x,r)), \qquad x \in X, \quad r > 0. \tag{2.1}$$

This will provide us with standard L^p-inequalities for $M_p f$ (see, for example, [5, Chap. 2]):

$$\mu\{M_p f > \Lambda\} \lesssim \left(\frac{1}{\Lambda}\|f\|_p\right)^p, \qquad \Lambda > 0.$$

The regularity condition for the measure μ provides the density of the class of continuous functions in the spaces $L^p(X)$, $1 \le p < \infty$.

In addition, we will need the following condition on the measure μ: there exist constants $C_1, C_2 > 1$ such that

$$\mu(B(x,C_1 r)) \ge C_2 \mu(B(x,r)), \qquad x \in X, \quad 0 < r < \frac{\operatorname{diam} X}{C_1}. \tag{2.2}$$

The expression $A \lesssim B$ will always mean that $A \le cB$, where c is a positive constant depending, perhaps, on certain parameters, but these dependencies are insignificant for us. In addition, the the expression $A \asymp B$ will mean that $A \lesssim B$ and $B \lesssim A$.

3. UNIFORM CONVERGENCE

We consider the families of integral operators

$$\mathcal{F}_t f(x) = \int_X \varphi_t(x,z) f(z)\, d\mu(z), \tag{3.1}$$

where the kernels $\varphi_t \colon X \times X \to \mathbb{R}$, $t \in (0,1]$, are measurable on the product $X \times X$ and satisfy the following conditions:

$$\int_X \varphi_t(x,z)\, d\mu(z) = 1 \qquad \text{for all} \quad x \in X, \quad t \in (0,1], \tag{3.2}$$

$$\sup_{t\in(0,1)} \sup_{x\in X} \int_X |\varphi_t(x,z)|\, d\mu(z) < \infty, \tag{3.3}$$

$$\lim_{t\to+0} \sup_{x\in X} \int_{d(x,z)>\delta} |\varphi_t(x,z)|\, d\mu(z) = 0 \qquad \text{for any} \quad \delta > 0. \tag{3.4}$$

Under conditions (3.2)–(3.4), for any bounded uniformly continuous function $f \in C(X)$, the following statement is valid:

$$\mathcal{F}_t f(y) \to f(x) \quad \text{uniformly on} \quad x \in X \qquad \text{if} \quad (y,t) \to (x,0).$$

This fact is well known (see, for example, [2, Chap. 1, Sec. 3]). In this case, the operators $\mathcal{F}_t f$ are said to form the *approximation* of the unit.

4. THE FATOU PROPERTY

Let there be given an increasing function $\lambda \colon (0,1] \to (0,1]$, $\lambda(+0) = 0$, generating the following domains of approach to the "boundary" of the abstract half-space $X \times (0,1]$:

$$D_\lambda(x) = \{(y,t) \in X \times (0,1] : d(x,y) < \lambda(t)\}. \tag{4.1}$$

We will be interested in the conditions on the kernels φ_t under which, for any function $f \in L^p(X)$, $p \ge 1$, the operators $\mathcal{F}_t f$ will have limits almost everywhere on the boundary along such domains, i.e.,

$$D_\lambda(x) - \lim \mathcal{F}_t f = f(x) \tag{4.2}$$

for almost all $x \in X$. In this case, the value of the locally summable function f at the point $x \in X$ is understood as the limit

$$\lim_{r \to +0} f_{B(x,r)} = f^*(x).$$

Under the doubling condition (2.1) on (X, d, μ), for any locally summable function f, this limit exists μ-almost everywhere and f^* is contained in the equivalence class of f (see, for example, [5, Theorem 1.8]).

The existence of the limits (4.2) almost everywhere will be first studied by using weak-type inequalities for the maximal Fatou operator

$$\mathcal{N}_\lambda u(x) := \sup\{|u(y,t)| : (y,t) \in D_\lambda(x)\}, \tag{4.3}$$

corresponding to the domains $D_\lambda(x)$.

Let us introduce the function

$$\varphi_t^*(x,y) := \sup\{|\varphi_t(x,z)| : d(x,y) \le d(x,z)\}, \qquad x, y \in X. \tag{4.4}$$

Its appearance in the study of convergence almost everywhere is not new. The similar function

$$\sup\{|\varphi(y)| : |y| \ge |x|\},$$

(the "radial majorant") appears in the study of the convergence of approximation identities on $\mathbb{R}^d$ generated by the dilatations $t^{-d}\varphi(x/t)$ of a fixed function (see, for example, [12, Chap. 3, Sec. 2]), which was required to be summable on $\mathbb{R}^d$.

We will use a similar requirement and introduce the quantity

$$C_\varphi := \sup_{t \in (0,1]} \sup_{x \in X} \|\varphi_t^*(x, \cdot)\|_{L^1(X)}. \tag{4.5}$$

Note that if $C_\varphi < \infty$, then condition (3.3) holds.

With the function λ that defines the geometry of Fatou domains (4.1) we associate the quantity

$$C_{\lambda,p} := \sup_{t \in (0,1]} \sup_{x \in X} [\mu(B(x,\lambda(t)))]^{1/p} \left(\int_{B(x,\lambda(t))} |\varphi_t(x,z)|^q \, d\mu(z) \right)^{1/q}, \qquad 1 < p < \infty, \tag{4.6}$$

$$C_{\lambda,1} := \sup_{t \in (0,1]} \sup_{x \in X} \mu(B(x,\lambda(t))) \sup_{y,z \in B(x,\lambda(t))} |\varphi_t(y,z)|. \tag{4.7}$$

Theorem 1. *Let*

1) *the space* (X, d, μ) *satisfy conditions* (2.1)*,* (2.2);

2) *the function* φ_t *satisfy*

$$C_\varphi < \infty; \tag{4.8}$$

3) $1 \le p < \infty$ *and the function* λ *be such that*

$$C_{\lambda,p} < \infty. \tag{4.9}$$

Then

$$\mathcal{N}_\lambda \mathcal{F}_t f(x) \lesssim (C_\varphi + C_{\lambda,p}) M_p f(x), \qquad f \in L^p(X). \tag{4.10}$$

Corollary 1. *Let conditions* (3.2)–(3.4) *and the conditions of Theorem* 1 *hold. Then, for any function* $f \in L^p(X)$*,* $1 \le p < \infty$*, for almost all* $x \in X$*, equality* (4.2) *holds.*

This statement follows from Theorem 1 in the standard way and also from the fact that $C(X)$ is dense in $L^p(X)$ and the functions $\mathcal{F}_t f$ converge uniformly for $f \in C(X)$ (see Sec. 3). For this fact, see, for example, [12, Chap. 1, Sec. 1].

In the following theorem, the set of points where (4.2) holds will be specified in more concrete form.

A point $x \in X$ is said to be a *p-Lebesgue point* for the function $f \in L^p(X)$ if

$$\lim_{h\to+0} ⨍_{B(x,h)} |f(x) - f(z)|^p \, d\mu(z) = 0. \tag{4.11}$$

Under condition (2.1), almost all points possess this property for any function $f \in L^p(X)$ (see, for example, [5, Chap. 2]).

Theorem 2. *Let*

1) *the space* (X, d, μ) *satisf the previous conditions*;

2) *the function* φ_t *is such that* (3.2), (4.8) *hold and*

$$\lim_{t\to+0} \sup_{x\in X} \int_{d(x,z)>\delta} \varphi_t^*(x,z)\, d\mu(z) = 0 \quad \textit{for any} \quad \delta > 0; \tag{4.12}$$

3) $1 \le p < \infty$ *and the function* λ *is such that condition* (4.9) *holds.*

Then, for any function $f \in L^p(X)$*, at each* p*-Lebesgue point* $x \in X$*,* (4.2) *is valid.*

5. PROOF OF THEOREM 1

To estimate the maximal operator (4.3), we take $x \in X$, $t \in (0,1]$, and let $y \in X$ satisfy the condition $d(x,y) < \lambda(t)$. We split the integral in (3.1) into two parts:

$$\mathcal{F}_t f(y) = \int_{d(y,z)<\lambda(t)} \varphi_t(y,z) f(z)\, d\mu(z) + \int_{d(y,z)\ge\lambda(t)} \varphi_t(y,z) f(z)\, d\mu(z) = I_1 + I_2 \tag{5.1}$$

and we will estimate each summand separately, starting from I_1.

If $p > 1$, then I_1 is estimated by using Hölder's inequality as follows:

$$\begin{aligned} |I_1| &\le \int_{B(y,\lambda(t))} |\varphi_t(y,z)| \cdot |f(z)|\, d\mu(z) \\ &\le \biggl(\int_{B(y,\lambda(t))} |\varphi_t(y,z)|^q \, d\mu(z)\biggr)^{1/q} \biggl(\int_{B(y,\lambda(t))} |f(z)|^p\, d\mu(z)\biggr)^{1/p} \\ &\le (\mu(B(y,\lambda(t))))^{1/p} \biggl(\int_{B(y,\lambda(t))} |\varphi_t(y,z)|^q\, d\mu(z)\biggr)^{1/q} M_p f(x) \le C_\lambda^p M_p f(x) \end{aligned}$$

(recall that $x \in B(y,\lambda(t))$).

If $p = 1$, then, for $d(x,y) < \lambda(t)$, the integral I_1 is easily estimated:

$$\begin{aligned} |I_1| &\le \|\varphi_t^*(y,\,\cdot\,)\|_{L^\infty(X)} \frac{\mu(B(y,\lambda(t)))}{\mu(B(y,\lambda(t)))} \int_{B(y,\lambda(t))} |f(z)|\, d\mu(z) \\ &\le \|\varphi_t^*(y,\,\cdot\,)\|_{L^\infty(X)} \mu(B(y,\lambda(t))) M_1 f(x) \le C_\lambda^1 M_1 f(x). \end{aligned}$$

To estimate I_2, we will use additional constructions. Let us introduce the layers

$$S_k(y) = \{z : C_1^{k-1}\lambda(t) \le d(y,z) < C_1^k \lambda(t)\}, \qquad k \in \mathbb{N}_0;$$

then

$$X = \left(\bigcup_{k=1}^{\infty} S_k(y) \right) \cup B(y, \lambda(t)).$$

Now, for each $k \in \mathbb{N}$, using conditions (2.2) and (2.1), we can write

$$\frac{\mu(B(y, C_1^k \lambda(t)))}{\mu(S_{k-1}(y))} = \frac{\mu(B(y, C_1^k \lambda(t)))}{\mu(B(y, C_1^{k-1} \lambda(t)) - \mu(B(y, C_1^{k-2} \lambda(t))} \lesssim 1.$$

Also, it follows from the definition (4.4) of the majorant φ_t^* that, for all $z \in S_{k-1}(y)$,

$$\sup_{z \in S_k(y)} |\varphi_t(y,z)| \le \varphi_t^*(y,z) \le \mu(S_{k-1}(y)) \int_{S_{k-1}(y)} \varphi_t^*(y,z)\, d\mu(z).$$

Thus, for $k \in \mathbb{N}$ the following inequalities hold:

$$\begin{aligned} &\sup_{z \in S_k(y)} |\varphi_t(y,z)| \mu(B(y, C_1^k \lambda(t))) \\ &\qquad \le \frac{\mu(B(y, C_1^k \lambda(t)))}{\mu(S_{k-1}(y))} \int_{S_{k-1}(y)} \varphi_t^*(y,z)\, d\mu(z) \lesssim \int_{S_{k-1}(y)} \varphi_t^*(y,z)\, d\mu(z). \end{aligned} \tag{5.2}$$

Given these inequalities, we now estimate the integral I_2 from (5.1), dividing the integration domain into layers $S_k(y)$:

$$\begin{aligned} |I_2| &\le \sum_{k=1}^{\infty} \int_{S_k(y)} |\varphi_t(y,z)| \cdot |f(z)|\, d\mu(z) \\ &\le \sum_{k=1}^{\infty} \sup_{z \in S_k(y)} |\varphi_t(y,z)| \int_{S_k(y)} |f(z)|\, d\mu(z) \\ &\le \sum_{k=1}^{\infty} \sup_{z \in S_k(y)} |\varphi_t(y,z)| \mu(B(y, C_1^k \lambda(t))) ⨍_{B(y, C_1^k \lambda(t))} |f(z)|\, d\mu(z) \\ &\lesssim M_1 f(x) \sum_{k=1}^{\infty} \int_{S_{k-1}(y)} \varphi_t^*(y,z)\, d\mu(z) \lesssim M_1 f(x) \|\varphi_t^*(y, \cdot\,)\|_{L^1(X)}. \end{aligned}$$

Thus, we have proved the inequality $|I_2| \lesssim C_\varphi M_1 f(x)$.

Combining the estimates for I_1 and I_2, we obtain inequality (4.10).

6. PROOF OF THEOREM 2

Let condition (4.11) hold at the point $x \in X$. Let us take $\varepsilon > 0$, $t \in (0,1]$ and $y \in X$ so that the condition $d(x,y) < \lambda(t)$ holds, i.e., $(y,t) \in D_\lambda(x)$ (see (4.1)).

Consider the difference

$$\begin{aligned} f(x) - \mathcal{F}_t f(y) &= \int_{d(y,z) < \lambda(t)} \varphi_t(y,z)[f(x) - f(z)]\, d\mu(z) \\ &\qquad + \int_{d(y,z) \ge \lambda(t)} \varphi_t(y,z)[f(x) - f(z)]\, d\mu(z) := I_1 + I_2. \end{aligned}$$

Denoting $B_t := B(y, \lambda(t))$ and $B_t^* = B(x, 2a_d \lambda(t))$ for brevity, we estimate I_1; then $B_t \subset B_t^*$.

If $p > 1$, then we use Hölder's inequality and condition (4.9), obtaining

$$|I_1| \le \int_{B_t} |\varphi_t(y,z)| \cdot |f(x) - f(z)|\, d\mu(z)$$

$$\le \biggl(\int_{B_t} |\varphi_t(y,z)|^q\,d\mu(z)\biggr)^{1/q}\biggl(\int_{B_t} |f(x)-f(z)|^p\,d\mu(z)\biggr)^{1/p}$$
$$\le (\mu(B_t))^{1/p}\biggl(\int_{B_t} |\varphi_t(y,z)|^q\,d\mu(z)\biggr)^{1/q}\biggl(\fint_{B_t} |f(x)-f(z)|^p\,d\mu(z)\biggr)^{1/p}$$
$$\le C_\lambda^p\biggl(\fint_{B_t^*} |f(x)-f(z)|^p\,d\mu(z)\biggr)^{1/p}.$$

By virtue of condition (4.11), the last integral is less than ε for sufficiently small t.

If $p=1$, then, for $d(x,y)<\lambda(t)$, the integral I_1 can be estimated as follows:

$$|I_1| \le \|\varphi_t^*(y,\,\cdot\,)\|_{L^\infty(X)}\frac{\mu(B_t)}{\mu(B_t)}\int_{B(y,\lambda(t))} |f(x)-f(z)|\,d\mu(z) \le C_\lambda^1 \fint_{B_t^*} |f(x)-f(z)|\,d\mu(z),$$

and then we again use (4.11).

To estimate the integral I_2, we first argue exactly as we when estimating I_2 in the proof of Theorem 1, arriving at the following inequality (the notation is preserved):

$$\begin{aligned} |I_2| &\le \sum_{k=1}^\infty \int_{S_k(y)} |\varphi_t(y,z)|\cdot|f(x)-f(z)|\,d\mu(z) \\ &\le \sum_{k=1}^\infty \sup_{z\in S_k(y)} |\varphi_t(y,z)| \int_{S_k(y)} |f(x)-f(z)|\,d\mu(z) \\ &\le \sum_{k=1}^\infty \sup_{z\in S_k(y)} |\varphi_t(y,z)|\mu(B(y,C_1^k\lambda(t))) \fint_{B(y,C_1^k\lambda(t))} |f(x)-f(z)|\,d\mu(z) \\ &\lesssim \sum_{k=1}^\infty \int_{S_{k-1}(y)} \varphi_t^*(y,z)\,d\mu(z) \fint_{B(x,2a_dC_1^k\lambda(t))} |f(x)-f(z)|\,d\mu(z) = S_1+S_2 \end{aligned} \tag{6.1}$$

using inequality (5.2)) and the inclusions $B(y,C_1^k\lambda(t))\subset B(x,2a_dC_1^k\lambda(t))$. Here S_1 denotes the sum of terms with numbers not exceeding n and S_2 denotes the sum of terms with numbers starting from $n+1$. The number n will now be determined.

Let us use condition (4.11), choosing a $\delta>0$ so small that

$$\sup_{o<h\le\delta} \fint_{B(x,h)} |f(x)-f(z)|\,d\mu(z) < \varepsilon, \tag{6.2}$$

and choose a number $n\in\mathbb{N}$ so that $2a_dC_1^n\lambda(t)<\delta\le 2a_dC_1^{n+1}\lambda(t)$.

By virtue of (6.2), we have

$$S_1 \le \sup_{0<h\le\delta} \fint_{B(x,h)} |f(x)-f(z)|\,d\mu(z) \sup_{y\in X}\int_X \varphi_t^*(y,z)\,d\mu(z) \lesssim \varepsilon.$$

Further, we estimate S_2:

$$\begin{aligned} S_2 &\le |f(x)| \sum_{k=n+1}^\infty \int_{S_{k-1}(y)} \varphi_t^*(y,z)\,d\mu(z) \\ &\quad + \sum_{k=n+1}^\infty \int_{S_{k-1}(y)} \varphi_t^*(y,z)\,d\mu(z)\biggl(\fint_{B(x,2a_dC_1^k\lambda(t))} |f(z)|^p\,d\mu(z)\biggr)^{1/p} \\ &\lesssim \biggl(|f(x)| + \|f\|_p\biggl[\mu\biggl(B\biggl(x,\frac{\delta}{C_1}\biggr)\biggr)\biggr]^{-1/p}\biggr)\int_{d(y,z)>\delta(2a_dC_1^2)^{-1}} \varphi_t^*(y,z)\,d\mu(z). \end{aligned}$$

The last expression will be less than ε for sufficiently small t uniformly over $y \in X$ by virtue of condition (4.12), and Theorem 2 is proved.

Further, we will consider some important examples of the application of Theorems 1 and 2. We emphasize that, in all these examples, the Fatou domains (4.1) obtained by using condition (4.9) are optimal (see Sec. 10).

We will need the following technical statement.

Lemma 1. *Let, for some* $\gamma > 0$*, the following condition hold*:

$$\mu(B(x,r)) \asymp r^{\gamma} \qquad at \quad x \in X, \quad 0 < r < \operatorname{diam} X, \tag{6.3}$$

and let $0 < t < T < \operatorname{diam} X$.

Then

$$\int_{t<d(x,y)<T} \frac{d\mu(y)}{[d(x,y)]^{\alpha}} \asymp \begin{cases} T^{\gamma-\alpha} & \text{if } \gamma > \alpha, \\ \log \dfrac{T}{t} & \text{if } \gamma = \alpha, \\ t^{\gamma-\alpha} & \text{if } \gamma < \alpha. \end{cases}$$

Proof. Let $N = [\log_2 T/t] + 1$. Then

$$\begin{aligned} \int_{t<d(x,y)<T} \frac{d\mu(y)}{[d(x,y)]^{\alpha}} &\asymp \sum_{k=0}^{N} \int_{2^k t<d(x,y)<2^{k+1}t} \frac{d\mu(y)}{[d(x,y)]^{\alpha}} \\ &\asymp \sum_{k=0}^{N} (2^k t)^{-\alpha} \mu(B(x,2^{k+1}t)) \asymp t^{\gamma-\alpha} \sum_{k=0}^{N} 2^{k(\gamma-\alpha)}. \end{aligned}$$

This yields the required estimates in all cases. □

7. THE POISSON INTEGRAL IN THE UNIT BALL OF $\mathbb{R}^n$

Let $X = S := \{\theta \in \mathbb{R}^n : |\theta| = 1\}$ be the unit sphere in $\mathbb{R}^n$, $n \ge 2$, let $\mu = \sigma$ be the Lebesgue surface measure on S normalized by the condition $\sigma(S) = 1$, and let $d(\theta,\xi) = |\theta - \xi|$ be the Euclidean metric on S.

The multidimensional analogue of the operator (1.2) is described by

$$\mathcal{P}_l f(x) = \frac{P_l f(x)}{P_l 1(x)}, \qquad \text{where} \quad P_l f(x) = \int_S [p(x,\xi)]^{l+(n-1)/n} f(\xi)\, d\sigma(\xi), \qquad |x| < 1; \tag{7.1}$$

here

$$p(x,\xi) = \frac{1-|x|^2}{|x-\xi|^n} \asymp \frac{t}{[t+d(\theta,\xi)]^n}, \qquad \text{where} \quad t = 1-|x|,$$

and $\theta = x/|x|$ is the Poisson kernel for the unit ball in $\mathbb{R}^n$; see, for example, [13, Chap. 1, Sec. 1], [14, Sec. 3.3.10].

7.1. *The Case* $l = 0$

It follows from Lemma 1 with $\gamma = n-1$ that the operators (7.1) can be rewritten as

$$\mathcal{P}_l f(x) = \int_S \varphi_t(\theta,\xi) f(\xi)\, d\sigma(\xi), \qquad \varphi_t(\theta,\xi) \asymp \frac{1}{\log 2/t [t+d(\theta,\xi)]^{n-1}},$$

where $t = 1-|x|$ and $\theta = x/|x|$ for $x \ne 0$.

Using Lemma 1 again , we will verify condition (4.9) for this operator. For $p > 1$, we obtain

$$[\mu(B(\theta,\lambda(t)))]^{1/p} \biggl(\int_{B(\theta,\lambda(t))} |\varphi_t(\theta,\xi)|^q \, d\sigma(\xi)\biggr)^{1/q}$$

$$\asymp [\lambda(t)]^{(n-1)/p}\biggl(\int_{B(\theta,t)}|\varphi_t(\theta,\xi)|^q\,d\sigma(\xi)+\int_{t\le d(\theta,\xi)<\lambda(t)}|\varphi_t(\theta,\xi)|^q\,d\sigma(\xi)\biggr)^{1/q}$$

$$\asymp \frac{[\lambda(t)]^{(n-1)/p}}{\log 2/t}\biggl(\int_{B(\theta,t)}t^{-q(n-1)}\,d\sigma(z)+t^{n-1-q(n-1)}\biggr)^{1/q}$$

$$\asymp \biggl[\frac{\lambda(t)}{t}\biggr]^{(n-1)/p}\biggl(\log\frac{2}{t}\biggr)^{-1}.$$

Hence the quantities (4.6) are estimated as follows:

$$C_{\lambda,p}\asymp \sup_{t\in(0,1]}\biggl[\frac{\lambda(t)}{t}\biggr]^{(n-1)/p}\biggl(\log\frac{2}{t}\biggr)^{-1}\qquad \text{at}\quad p\ge 1.$$

Direct calculations show that the same is true for $p=1$. Hence condition (4.9) will hold if

$$\lambda(t)=at\biggl(\log\frac{2}{t}\biggr)^{p/(n-1)},\qquad p\ge 1.$$

Therefore, the Fatou domains for functions from $L^p(S)$, $p\ge 1$, have the form

$$\biggl\{x\in\mathbb{R}^n: |x-\theta|<a(1-|x|)\biggl(\log\frac{2}{1-|x|}\biggr)^{p/(n-1)}\biggr\},\qquad |\theta|=1. \tag{7.2}$$

This result was obtained in [6] for $n=2$, $p=1$, in [7], for $n=2$, $p>1$, and in [8], [9] for the general case. Moreover, a strong-type inequality was proved in [9, Sec. 3.3] for the corresponding maximal Fatou operator (4.3).

7.2. The Case $l>0$

By Lemma 1 with $\gamma=n-1$, for the operators (7.1), we have

$$\mathcal{P}_l f(x)=\int_S \varphi_t(\theta,\xi)f(\xi)\,d\sigma(\xi),\qquad \varphi_t(\theta,\xi)\asymp\frac{t^{nl}}{[t+|\theta-\xi|]^{nl+n-1}},$$

where $t=1-|x|$, $\theta=x/|x|$.

Again, we estimate the quantities (4.6) and (4.7) using Lemma 1, obtaining

$$C_{\lambda,p}\asymp \sup_{t\in(0,1]}\biggl[\frac{\lambda(t)}{t}\biggr]^{(n-1)/p}\qquad \text{at}\quad p\ge 1.$$

It is readily seen that only the functions $\lambda(t)=at$, $a>0$, will fulfill condition (4.9). Therefore, the Fatou domains for functions from $L^p(S)$ for all $p\ge 1$ will have the same form

$$\{x=r\xi\in\mathbb{R}^n: |\theta-\xi|<a(1-r)\},\qquad |\theta|=1$$

(these are nontangential domains). In particular, this is true in the classical case $l=1/n$ for Fatou domains that are independent of p.

7.3. The Case $l<0$

Using Lemma 1 for the operators (7.1), we obtain

$$\mathcal{P}_l f(x)=\int_S \varphi_t(\theta,\xi)f(\xi)\,d\sigma(\xi),\qquad \varphi_t(\theta,\xi)\asymp\frac{1}{[t+|\theta-\xi|]^{nl+n-1}}, \tag{7.3}$$

where $t=1-|x|$, $\theta=x/|x|$.

First, let $0 > l > -(n-1)/(np)$; then $q(nl+n-1) > n-1$. Therefore, using Lemma 1 for $\gamma = n-1 < \alpha = q(nl+n-1)$ to check condition (4.9), we obtain

$$C_{\lambda,p} \asymp \sup_{t\in(0,1]} \left[\frac{\lambda(t)}{t}\right]^{(n-1)/p} t^{-nl}.$$

This brings us to the function $\lambda(t) = at^{1+lnp/(n-1)}$ and the corresponding Fatou domains

$$\{x \in \mathbb{R}^n : |x-\theta| < a(1-|x|)^{1+lpn/(n-1)}\}, \qquad |\theta| = 1.$$

This is a special case of a result from [8], where the boundary behavior of operators of the form

$$\int_X \frac{f(y)\,d\mu(y)}{[t+d(x,y)]^{\gamma-\alpha}} \tag{7.4}$$

was considered on general spaces (X, d, μ) with measure satisfying the condition $\mu(B(x,r)) \asymp r^\gamma$. Note that a strong-type inequality for the maximal operator (4.3) was proved in [8].

Now let $l = -(n-1)/(np)$, $p > 1$. Then, by Lemma 1 for $\gamma = n-1 = q(nl+n-1)$, we have

$$C_{\lambda,p} \asymp \sup_{t\in(0,1]} [\lambda(t)]^{(n-1)/p} \left(\log\frac{2}{t}\right)^{1/q},$$

whence we find the function $\lambda(t) = a(\log 2/t)^{-(p-1)/(n-1)}$ and the corresponding Fatou domains

$$\left\{x \in \mathbb{R}^n : /x-\theta| < a\left(\log\frac{2}{1-|x|}\right)^{-(p-1)/(n-1)}\right\}, \qquad |\theta| = 1.$$

The cases $l = -(n-1)/n$, $p = 1$, and $l < -(n-1)/(np)$, $p \ge 1$, are now of no interest (see (7.3)).

8. INVARIANT POISSON INTEGRAL IN A BALL OF $\mathbb{C}^n$

The following example is also a generalization of the operators (1.3) to the multidimensional case.

Let $X = S := \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$ be the unit sphere in $\mathbb{C}^n$, let $\mu = \sigma$ be the Lebesgue surface measure on S normalized by the condition $\sigma(S) = 1$, and let $d(\zeta,\xi) = |1 - \langle\zeta,\xi\rangle|$ be nonisotropic quasimetrics, where $\langle\cdot,\cdot\rangle$ is the complex scalar product in $\mathbb{C}^n$ [14, Sec. 5.1]. Let condition (6.3) hold with $\gamma = n$ [14, Sec. 3.3].

Now it is natural to also consider the invariant Poisson kernel [14, Sec. 3.3]

$$p(z,\xi) = \frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}} \asymp \frac{t^n}{[t+d(\zeta,\xi)]^{2n}}, \qquad \text{where} \quad t = 1-|z|, \quad \zeta = \frac{z}{|z|}.$$

The following operator is also a multidimensional analogue for (1.2):

$$\mathcal{P}_l f(z) = \frac{P_l f(z)}{P_l 1(z)}, \qquad \text{where} \quad P_l f(z) = \int_S [p(z,\xi)]^{l+1/2} f(\xi)\,d\sigma(\xi). \tag{8.1}$$

8.1. The Case $l = 0$

For the operator (8.1), using Lemma 1, in which we must take $\gamma = n$, we have

$$\mathcal{P}_0 f(z) = \int_S \varphi_t(\zeta,\xi) f(\xi)\,d\sigma(\xi), \qquad \varphi_t(\zeta,\xi) \asymp \frac{1}{[t+d(\zeta,\xi)]^n \log(2/t)},$$

where $t = 1-|z|$ and $\zeta = z/|z|$ for $z \ne 0$.

Applying Lemma 1 to check condition (4.9), we obtain

$$C_{\lambda,p} \asymp \sup_{t\in(0,1]} \left[\frac{\gamma(t)}{t}\right]^{n/p} \left(\log\frac{2}{t}\right)^{-1} \qquad \text{for all} \quad p \ge 1.$$

This shows that (4.9) holds for functions of the form

$$\lambda(t) = at\left(\log\frac{2}{t}\right)^{p/n},$$

to which correspond the Fatou domains

$$\left\{z \in \mathbb{C}^n : |1-\langle z,\zeta\rangle| < a(1-|z|)\left(\log\frac{2}{1-|z|}\right)^{p/n}\right\}, \qquad |\zeta| = 1.$$

8.2. The Case $l > 0$

Calculations using Lemma 1 give us the following estimates for the operator (8.1):

$$\mathcal{P}_l f(z) = \int_S \varphi_t(\zeta,\xi) f(\xi)\,d\sigma(\xi), \qquad \varphi_t(\zeta,\xi) \asymp \frac{t^{2nl}}{[t+d(\zeta,\xi)]^{n(2l+1)}},$$

where $t = 1-|z|$, $\zeta = z/|z|$.

Again, by Lemma 1, we find that

$$C^p_\lambda \asymp \left[\frac{\lambda(t)}{t}\right]^{n/p};$$

therefore, $\lambda(t) = at$ and the standard Korányi domains

$$\{z \in \mathbb{C}^n : |1-\langle z,\zeta\rangle| < a(1-|z|)\}, \qquad |\zeta| = 1,$$

are Fatou domains. In particular, this is true for the classical case $l = 1/2$ [15] (see also [14, Sec. 5.4]).

8.3. The Case $l < 0$

Now, for the operator (8.1), we have the representations

$$\mathcal{P}_l f(z) = \int_S \varphi_t(\zeta,\xi) f(\xi)\,d\sigma(\xi), \qquad \varphi_t(\zeta,\xi) \asymp \frac{1}{[t+d(\zeta,\xi)]^{n(2l+1)}},$$

where $t = 1-|z|$, $\zeta = z/|z|$.

First, we assume that $0 > l > -1/(2p)$. Then

$$C_{\lambda,p} \asymp \left[\frac{\lambda(t)}{t}\right]^{n/p} t^{-2nl};$$

therefore, condition (4.9) holds for the functions $\lambda(t) = t^{1+2pl}$, which correspond to the Fatou domains

$$\{z : |1-\langle z,\zeta\rangle| < a(1-|z|)^{1+2lp}\}, \qquad |\zeta| = 1.$$

This result was obtained in [8, Theorem 2]. A strong-type inequality for the corresponding maximal Fatou operator was also proved in [9, Theorem 1].

If $l = -1/(2p)$, $p > 1$, then $q(n+2nl) = n$, and hence the calculations show that

$$C_{\lambda,p} \asymp [\lambda(t)]^{n/p}\left(\log\frac{2}{t}\right)^{1/q};$$

further, condition (4.9) holds for the functions

$$\lambda(t) = a\left(\log\frac{2}{t}\right)^{-(p-1)/n}$$

and the corresponding Fatou domains are

$$\left\{z : |1-\langle z,\zeta\rangle| < a\left(\log\frac{2}{1-|z|}\right)^{-(p-1)/n}\right\}, \qquad |\zeta| = 1.$$

This result was obtained in [8]. It was also proved in [16] that the maximal operator corresponding to these domains satisfies a strong-type inequality.

The case $l < -1/(2p)$, $p > 1$, is not of interest now, because, in this case, for any function $f \in L^p(S)$, the operator $\mathcal{P}_l f$ can be continuously extended to the boundary of the ball.

9. CONVOLUTIONS WITH EXTENSIONS IN $\mathbb{R}^n$

In conclusion, we consider the following approximate identities on Euclidean spaces $\mathbb{R}^n$ generated by convolutions with dilations of a fixed function $\varphi \in L^\infty \cap L^1(\mathbb{R}^n)$:

$$\mathcal{F}_t f(x) = t^{-n} \int_{\mathbb{R}^n} f(x-y)\varphi\left(\frac{y}{t}\right) dy.$$

We will require that the function φ satisfy the normalization condition

$$\int_{\mathbb{R}^n} \varphi(y)\, dy = 1$$

and the summability condition for a radial majorant $\varphi^* \in L^1(\mathbb{R}^n)$, where

$$\varphi^*(x) = \sup\{|\varphi(y)| : |y| \geq |x|\}.$$

Under these conditions, for each function $f \in L^p(\mathbb{R}^n)$, the Fatou domains are the nontangential cones

$$\{(y,t) : |x-y| < at\}, \qquad x \in \mathbb{R}^n$$

(see [12, Sec. 3.2]).

Let us check condition (4.9) for such approximate identities:

$$[\mu(B(x,\lambda(t)))]^{1/p}\left(\int_{B(0,\lambda(t))} t^{-nq}\left|\varphi\left(\frac{y}{t}\right)\right|^q dy\right)^{1/q} \asymp \left[\frac{\lambda(t)}{t}\right]^{n/p}\left(\int_{B(0,\lambda(t)/t)} |\varphi(y)|^q\, dy\right)^{1/q}.$$

It follows that condition (4.9) only holds for the function $\lambda(t) = at$. This explains why the the Fatou property for such approximate identities does not depend on the degree of summability $p \geq 1$.

Important examples are the Poisson and Gauss—Weierstrass kernels

$$P(x) = c_n(1+|x|^2)^{-(n+1)/2}, \qquad W(x) = c_n \exp\left(-\frac{|x|^2}{4}\right)$$

(the c_n being normalization constants).

10. OPTIMALITY OF THE FATOU DOMAINS

Let us now consider the sharpness of our main results, namely, determine whether the functions λ defining the Fatou domains by equalities (4.1) are optimal.

We will show that all the Fatou domains defined in Secs. 7 and 8 are the best and cannot be replaced by any domains of the form

$$\{(y,t) \in X \times (0,1] : d(x,y) < \Phi(t)\lambda(t)\},$$

where $\Phi(t) \to \infty$ as $t \to +0$.

Let us consider in detail only the operator $\mathcal{P}_l$ from Sec. 8, keeping the notation used there.

We start with the case $l = 0$. Let us set

$$f_\delta = \chi_{B(\zeta_0,\delta)} \ln\frac{2}{\delta}$$

(χ_E is the characteristic function of the set E). Then direct calculations show that

$$\|f_\delta\|_p \asymp \delta^{n/p} \ln\frac{2}{\delta}\,. \tag{10.1}$$

If $z_0 := (1-\delta)\zeta_0$, then

$$\mathcal{P}_0 f_\delta(z_0) \asymp \int_S \frac{f_\delta(\xi)\, d\sigma(\xi)}{[\delta + d(\zeta_0,\xi)]^n \ln 2/\delta} \asymp \int_{B(\zeta_0,\delta)} \frac{d\sigma}{\delta^n} \asymp 1. \tag{10.2}$$

The function λ corresponding to the case $l = 0$ has the form $\lambda(t) = at(\log(2/t))^{p/n}$. For any decreasing function $\Phi\colon (0,1] \to (0,+\infty)$ with the property

$$\lim_{t\to+0} \Phi(t) = +\infty,$$

we denote

$$\lambda_\Phi(t) := \Phi(t)t\biggl(\ln\frac{2}{t}\biggr)^{p/n}, \qquad t \in (0,1];$$

also note that

$$\sigma(B(\zeta_0, \lambda_\Phi(\delta))) \asymp \Phi^n(\delta)\delta^n\biggl(\ln\frac{2}{\delta}\biggr)^p \asymp \Phi^n(\delta)\|f_\delta\|_p^p. \tag{10.3}$$

Let us now define the sequence $\delta_k \downarrow 0$ so that

$$\Phi^n(\delta_k) \ge k^2, \qquad A_k := \|f_{\delta_k}\|_p^{-1}k^{-2/p} \uparrow \infty.$$

Then the following relations are satisfied:

$$\sum_{k=1}^{\infty} A_k^p\|f_{\delta_k}\|_p^p < \infty, \tag{10.4}$$

$$\sum_{k=1}^{\infty} \Phi^n(\delta_k)A_k^p\|f_{\delta_k}\|_p^p = \infty. \tag{10.5}$$

It follows from conditions (10.3) and (10.5) that

$$\sum_{k=1}^{\infty} \sigma(B(\zeta_0), \lambda_\Phi(\delta_k))) = \infty.$$

Hence Lemma 1 from [17] implies the existence of a sequence U_k of unitary transforms of $\mathbb{C}^n$ such that

$$\sigma\Bigl(\varlimsup_{k\to\infty} U_k(B(\zeta_0, \lambda_\Phi(\delta_k)))\Bigr) = \mu(S). \tag{10.6}$$

Finally, we define the function

$$f_0 = \biggl(\sum_{k=1}^{\infty}[A_k(f_{\delta_k} \circ U_k^{-1})]^p\biggr)^{1/p}.$$

Condition (10.4) implies that $f_0 \in L^p(S)$.

Let us show that $\mathcal{P}_0 f_0$ does not have a $D_{\lambda_\Phi}(\zeta)$-limit for σ-almost all $\zeta \in S$.

In fact, by virtue of (10.6), σ-almost every point $\zeta \in S$ for infinitely many k belongs to the surface balls $U_k(B(\zeta_0, \lambda_\Phi(\delta_k))) = B(U_k(\zeta_0), \lambda_\Phi(\delta_k)) \subset S$.

Take any such k and denote $z_k = (1-\delta_k)U_k(\zeta_0)$. Then the inclusion $\zeta \in B(U_k(\zeta_0), \lambda_\Phi(\delta_k))$ implies $z_k \in D_{\lambda_\Phi}(\zeta)$.

In addition, the kernel of the operator $\mathcal{P}_0$ is invariant with respect to the unitary transforms of $\mathbb{C}^n$; therefore, by virtue of condition (10.2), we obtain

$$\mathcal{P}_0 f_0(z_k) \ge A_k\mathcal{P}_0(f_{\delta_k} \circ U_k^{-1})(z_k) = A_k\mathcal{P}_0 f_{\delta_k}(z_0) \asymp A_k.$$

Hence $D_{\lambda_\Phi}(\zeta) - \varlimsup \mathcal{P}_0 f_0 = +\infty$.

Similar arguments apply to the other cases discussed in Sec. 8. The changes only involve the selection of the test function f_δ. If $-1/(2p) < l < 0$, then we can take $f_\delta = \delta^{2ln}\chi_{B(\zeta_0,\delta)}$. In the case $l = -1/(2p)$, the following test function can be taken:

$$f_\delta(\xi) = [\delta + d(\zeta_0, \xi)]^{-n/p}\biggl(\ln\frac{2}{\delta}\biggr)^{-1}.$$

Similarly, it can be shown that all the Fatou domains defined in Sec. 7 are optimal.

In conclusion, we note that, for all the examples discussed above in the case $p > 1$, the maximal operator (4.3) satisfies the inequality

$$\|\mathcal{N}_\lambda \mathcal{F}_t f\|_p \lesssim \|f\|_p. \tag{10.7}$$

It would be of interest to find conditions under which this is true for the approximate identities (3.1). In connection with this problem, we note the papers [18]–[20], in which estimates of the form (10.7) for maximal Fatou operators were proved using various methods.

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