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On Exponential Summability of Rectangular Partial Sums of Double Trigonometric Fourier Series

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Abstract—In this paper, we study the a.e. exponential strong summability problem for the rectangular partial sums of double trigonometric Fourier series of functions in $L\log L$.

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1. INTRODUCTION

We denote the set of all nonnegative integers by $\mathbb{N}$. Let $\mathbb{T} := [-\pi, \pi) = \mathbb{R}/2\pi$, and let $\mathbb{R} := (-\infty, \infty)$. By $L^1(\mathbb{T})$ we denote the class of all 2π-periodic measurable functions f on $\mathbb{R}$ satisfyingn the condition

$$\|f\|_1 := \int_{\mathbb{T}} |f| < \infty.$$

The Fourier series of a function $f \in L^1(\mathbb{T})$ with respect to the trigonometric system is

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{1}$$

where

$$c_n := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx}\, dx$$

are the Fourier coefficients of f. Let $S_n(x, f)$ denote the partial sums of the Fourier series of f, and let

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x, f)$$

be the $(C, 1)$ means of (1). Fejér [1] proved that $\sigma_n(f)$ converges to f uniformly for any 2π-periodic continuous function. Lebesgue [2] established the almost everywhere convergence of $(C, 1)$ means, provided that $f \in L^1(\mathbb{T})$. The strong summability problem, i.e., the convergence of the strong means

$$\frac{1}{n} \sum_{k=0}^{n-1} |S_k(x, f) - f(x)|^p, \qquad x \in \mathbb{T}, \quad p > 0, \tag{2}$$

was first considered by Hardy and Littlewood in [3]. They showed that, for any $f \in L^r(\mathbb{T})$, $1 < r < \infty$, the strong means tend to 0 a.e. as $n \to \infty$. The trigonometric Fourier series of $f \in L^1(\mathbb{T})$ is said to be (H, p)-*summable at* $x \in \mathbb{T}$ if the values (2) converge to 0 as $n \to \infty$. The (H, p)-summability problem

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in $L^1(\mathbb{T})$ was investigated by Marcinkiewicz [4] for $p = 2$ and later by Zygmund [5] for the general case $1 \le p < \infty$.

Let $\Phi\colon [0,\infty) \to [0,\infty)$, and let $\Phi(0) = 0$ be a continuous increasing function. We say that a series with partial sums s_n is strongly Φ-summable to a limit s if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|s_k - s|) = 0.$$

In [6], Oskolkov first considered the a.e. strong Φ-summability problem of Fourier series with exponentially growing Φ. Namely, he proved the a.e. strong Φ-summability of Fourier series under the assumption

$$\ln \Phi(t) = O\biggl(\frac{t}{\ln\ln t}\biggr) \qquad \text{as} \quad t \to \infty.$$

In [7], Rodin proved the following theorem.

Theorem (Rodin). *If a continuous function $\Phi\colon [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, satisfies the condition*

$$\limsup_{t\to+\infty} \frac{\ln \Phi(t)}{t} < \infty,$$

then, for any $f \in L^1(\mathbb{T})$, the relation

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|S_k(x,f) - f(x)|) = 0 \tag{3}$$

holds for a.e. $x \in \mathbb{T}$.

Karagulyan [8], [9] proved that the exponential growth in Rodin's theorem is optimal. Moreover, he also proved the following assertion.

Theorem (Karagulyan). *If a continuous increasing function $\Phi\colon [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, satisfies the condition*

$$\limsup_{t\to+\infty} \frac{\ln \Phi(t)}{t} = \infty,$$

then there exists a function $f \in L^1(\mathbb{T})$ for which the relation

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(|S_k(x,f)|) = \infty$$

holds everywhere on $\mathbb{T}$.

In this paper, we study the exponential summability problem for the rectangular partial sums of double Fourier series. Let $f \in L^1(\mathbb{T}^2)$ be a function with Fourier series

$$\sum_{m,n=-\infty}^{\infty} c_{nm} e^{i(mx+ny)}, \tag{4}$$

where

$$c_{nm} = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(x_1,x_2) e^{-i(mx_1+nx_2)}\, dx_1\, dx_2$$

are the Fourier coefficients of the function f. The rectangular partial sums of (4) are defined by

$$S_{MN}(f) = S_{MN}(x_1,x_2,f) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} c_{nm} e^{i(mx_1+nx_2)}.$$

We use $L\log L(\mathbb{T}^2)$ to denote the class of measurable functions f with

$$\iint_{\mathbb{T}^2} |f|\log^+|f| < \infty,$$

where $\log^+ u := \mathbb{I}_{(1,\infty)}\log u$, $u>0$. For the rectangular partial sums of two-dimensional trigonometric Fourier series, Jessen, Marcinkiewicz, and Zygmund [10] proved that, given any $f\in L\log L(\mathbb{T}^2)$,

$$\lim_{n,m\to\infty}\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(S_{ij}(x_1,x_2,f)-f(x_1,x_2))=0 \qquad \text{a.e.}$$

for $(x_1,x_2)\in\mathbb{T}^2$. They also showed that, for every nonnegative function $\omega:[0,\infty)\to[0,\infty)$ satisfying the conditions $\omega(t)\uparrow\infty$ and $\omega(t)(\log^+ t)^{-1}\to 0$ as $t\to\infty$, there exists a function f such that $|f|\omega(|f|)\in L^1(\mathbb{T}^2)$ and the $(C,1,1)$ means of the double Fourier series of f diverge a.e.

The two-dimensional a.e. strong rectangular (H,p)-summability, i.e., the relation

$$\lim_{n,m\to\infty}\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{ij}(x_1,x_2,f)-f(x_1,x_2)|^p=0 \qquad \text{a.e.},$$

was proved by Gogoladze [11] for $f\in L\log L(\mathbb{T}^2)$. These results show that, in the two-dimensional case, the optimal classes of functions for $(C,1,1)$-summability and strong summability coincide. They equal the class of functions $L\log L(\mathbb{T}^2)$.

We prove the following theorem.

Theorem 1. *If a continuous increasing function $\Phi\colon[0,\infty)\to[0,\infty)$, $\Phi(0)=0$, satisfies the condition*

$$\limsup_{t\to+\infty}\frac{\ln\Phi(t)}{\sqrt{t/\ln\ln t}}<\infty, \tag{5}$$

then, for any $f\in L\log L(\mathbb{T}^2)$, the relation

$$\lim_{n,m\to\infty}\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Phi(|S_{ij}(x_1,x_2,f)-f(x_1,x_2)|)=0 \tag{6}$$

holds for a.e. $(x_1,x_2)\in\mathbb{T}^2$.

As a corollary of this result, we obtain Gogoladze's theorem [11] on the a.e. H^p-summability of double Fourier series. The theorem of Jessen, Marcinkiewicz, and Zygmund [10] implies that the class $L\log L(\mathbb{T}^2)$ in our theorem is necessary in the context of the strong summability question. That is, the theorem is invalid for classes larger than $L\log L(\mathbb{T}^2)$. Our method of proof does permit obtaining (6) under the weaker condition

$$\limsup_{t\to+\infty}\frac{\ln\Phi(t)}{\sqrt{t}}<\infty. \tag{7}$$

There is a conjecture that (7) is the optimal bound of Φ ensuring the a.e. rectangular strong summability (6) for every function $f\in L\log L(\mathbb{T}^2)$.

The results on strong summability and approximation by trigonometric Fourier series have been extended for some other orthogonal systems; see the papers [12]–[14] by Schipp, [15]–[18] by Leindler, [19]–[22] by Totik, [23] and [24] by Goginava and Gogoladze, [25] by Goginava, Gogoladze, and Karagulyan, [26] and [27] by Gat, Goginava, and Karagulyan, and [28]–[31] by Weisz.

2. AUXILIARY LEMMAS

We use the notation $a \lesssim b$ for

$$a < c \cdot b,$$

where $c > 0$ is an absolute constant. We shall write $a \sim b$ if the relations $a \lesssim b$ and $b \lesssim a$ hold simultaneously. Throughout the paper, $q > 1$ is used as the conjugate of $p > 1$, that is, $1/p + 1/q = 1$. By $[a]$ we denote the integer part of $a \in \mathbb{R}$.

The maximal function of a function $f \in L^1(\mathbb{T})$ is defined by

$$Mf(x) := \sup_{I: x \in I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f(y)|\, dy,$$

where I is an open interval. The following one-dimensional operators introduced by Gabisonia [32] are important tools in the investigations of strong summability problems:

$$G_p^{(n)} f(x) := \biggl(\sum_{k=1}^{[n\pi]} \biggl(\frac{n}{k} \int_{(k-1)/n}^{k/n} |f(x+t)| + |f(x-t)|\, dt \biggr)^q \biggr)^{1/q},$$
$$G_p f(x) := \sup_{n \in \mathbb{N}} G_p^{(n)} f(x).$$

Oskolkov's lemma cited below plays the key role in the proof of our basic lemma.

Lemma 1 ([6]). *For any family of pairwise disjoint intervals $\Delta_k \subset \mathbb{T}$ with midpoints c_k, the inequality*

$$\biggl| \biggl\{ x \in \mathbb{T} : \sup_{p>1} \frac{\sum_j (|\Delta_j|/(|x - c_j| + |\Delta_j|))^q}{p \ln \ln(p+2)} > \lambda \biggr\} \biggr| \lesssim \exp(-c\lambda), \qquad \lambda > 0, \tag{8}$$

holds, where $c > 0$ is an absolute constant.

One can easily check that

$$\sup_{p>1} \frac{\bigl(\sum_j (|\Delta_j|/(|x - c_j| + |\Delta_j|))^q\bigr)^{1/q}}{p \ln \ln(p+2)} \lesssim \biggl\{ 1, \sup_{p>1} \frac{\sum_j (|\Delta_j|/(|x - c_j| + |\Delta_j|))^q}{p \ln \ln(p+2)} \biggr\}.$$

Combining this with (8), we obtain

$$\int_{\mathbb{T}} \sup_{p>1} \frac{(\sum_j (|\Delta_j|/(|x - c_j| + |\Delta_j|))^q)^{1/q}}{p \ln \ln(p+2)} \lesssim 1. \tag{9}$$

Lemma 2. *If $f \in L^1(\mathbb{T})$, then*

$$\biggl| \biggl\{ x \in \mathbb{T} : \sup_{p>1} \frac{G_p f(x)}{p \ln \ln(p+2)} > \lambda \biggr\} \biggr| \lesssim \biggl(\frac{1}{\lambda} \|f\|_1 \biggr)^{1/2}, \qquad \lambda > 0. \tag{10}$$

Proof. It is enough to prove the same estimate for the modified operators

$$G'_p f(x) := \sup_{n \in \mathbb{N}} \biggl(\sum_{k=1}^{[n\pi]} \biggl(\frac{n}{k} \int_{(k-1)/n}^{k/n} |f(x+t)|\, dt \biggr)^q \biggr)^{1/q}. \tag{11}$$

Using the Calderon–Zygmund lemma, we obtain the following relation for the maximal function:

$$R_\lambda := \{ x \in \mathbb{T} : Mf(x) > \sqrt{\lambda} \} = \bigcup_{k=0}^{\infty} \Delta_k, \qquad \lambda > 0, \tag{12}$$

where $\Delta_k \subset \mathbb{T}$ are disjoint open intervals such that

$$\sqrt{\lambda} \le \frac{1}{|\Delta_k|}\int_{\Delta_k} |f(t)|\,dt \le 2\sqrt{\lambda}\,, \tag{13}$$

$$|R_\lambda| \le \frac{1}{\sqrt{\lambda}}\|f\|_1. \tag{14}$$

We set $\delta_k^n := [(k-1)/n, k/n]$ and $\delta_k^n(x) := x + \delta_k^n$. Separating out the terms in the sum (11) for which k satisfies the condition $\delta_k^n(x) \subset R_\lambda$, we obtain

$$\begin{aligned} G'_p f(x) &\le \sup_{n\in\mathbb{N}}\biggl(\sum_{k:\ \delta_k^n(x)\subset R_\lambda}\biggl(\frac{n}{k}\int_{(k-1)/n}^{k/n}|f(x+t)|\,dt\biggr)^q\biggr)^{1/q} \\ &\quad + \sup_{n\in\mathbb{N}}\biggl(\sum_{k:\ \delta_k^n(x)\not\subset R_\lambda}\biggl(\frac{n}{k}\int_{(k-1)/n}^{k/n}|f(x+t)|\,dt\biggr)^q\biggr)^{1/q} \\ &:= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{15}$$

From the definition of R_λ in the case $\delta_k^n(x) \not\subset R_\lambda$, it follows that

$$n\int_{(k-1)/n}^{k/n}|f(x+t)|\,dt \le \sqrt{\lambda}\,.$$

Thus,

$$\mathrm{II} \le \sqrt{\lambda}\biggl(\sum_{k=1}^{\infty}\frac{1}{k^q}\biggr)^{1/q} \lesssim \sqrt{\lambda}\biggl(\frac{1}{q-1}\biggr)^{1/q} \lesssim p\sqrt{\lambda}\,. \tag{16}$$

Given $x \in \mathbb{T}$, we set

$$k_i(x) = \begin{cases} \min\{k : \delta_k^n(x) \subset \Delta_i\} & \text{if } \{k : \delta_k^n(x) \subset \Delta_i\} \ne \varnothing, \\ \infty & \text{if } \{k : \delta_k^n(x) \subset \Delta_i\} = \varnothing. \end{cases}$$

We also set $\widetilde{R}_\lambda := \bigcup_{k=1}^\infty 3\Delta_k$ and take an arbitrary point $x \in \mathbb{T}\setminus\widetilde{R}_\lambda$. One can easily check that if $k_i(x) \ne \infty$, then

$$\Delta_i \ni \frac{k_i(x)}{n} \sim |x - c_i|,$$

where c_i is the center of the interval Δ_i. Thus, for any $x \notin \widetilde{R}_\lambda$, we have

$$\begin{aligned} I &= \sup_{n\in\mathbb{N}}\biggl(\sum_{i=1}^{\infty}\sum_{k:\ \delta_k^n(x)\subset\Delta_i}\biggl(\frac{n}{k}\int_{\delta_k^n(x)}|f(t)|\,dt\biggr)^q\biggr)^{1/q} \\ &\le \sup_{n\in\mathbb{N}}\biggl(\sum_{i=1}^{\infty}\biggl(\sum_{k:\ \delta_k^n(x)\subset\Delta_i}\frac{n}{k}\int_{\delta_k^n(x)}|f(t)|\,dt\biggr)^q\biggr)^{1/q} \\ &\le \sup_{n\in\mathbb{N}}\biggl(\sum_{i=1}^{\infty}\biggl(\frac{n|\Delta_i|}{k_i(x)}\frac{1}{|\Delta_i|}\int_{\Delta_i}|f(t)|\,dt\biggr)^q\biggr)^{1/q} \\ &\lesssim \sqrt{\lambda}\sup_n\biggl(\sum_{i=1}^{\infty}\biggl(\frac{n|\Delta_i|}{k_i(x)}\biggr)^q\biggr)^{1/q} \\ &\lesssim \sqrt{\lambda}\biggl(\sum_{i=1}^{\infty}\biggl(\frac{|\Delta_i|}{|x-c_i|+|\Delta_i|}\biggr)^q\biggr)^{1/q}, \qquad x \notin \widetilde{R}_\lambda. \end{aligned} \tag{17}$$

Using Chebyshev's inequality and relations (9), (16), and (17), we obtain

$$
\begin{aligned}
&\left|\left\{x \in \mathbb{T}\setminus \widetilde{R}_\lambda : \sup_{p>1} \frac{G'_p f(x)}{p\ln\ln(p+2)} > \lambda\right\}\right| \\
&\qquad \lesssim \left|\left\{x \in \mathbb{T}\setminus \widetilde{R}_\lambda : \sqrt{\lambda}\left(1+\sup_{p>1}\frac{\left(\sum_j(|\Delta_j|/(|x-c_j|+|\Delta_j|))^q\right)^{1/q}}{p\ln\ln(p+2)}\right) \ge c\lambda\right\}\right| \\
&\qquad \lesssim \frac{1}{\sqrt{\lambda}}\int_{\mathbb{T}}\sup_{p>1}\frac{\left(\sum_j(|\Delta_j|/(|x-c_j|+|\Delta_j|))^q\right)^{1/q}}{p\ln\ln(p+2)}\,dx \\
&\qquad \lesssim \frac{1}{\sqrt{\lambda}}
\end{aligned}
$$

for an appropriate absolute constant $c>0$. In view of homogeneity, we have

$$
\left|\left\{x \in \mathbb{T}\setminus \widetilde{R}_\lambda : \sup_{p>1} \frac{G'_p f(x)}{p\ln\ln(p+2)} > \lambda\right\}\right| \lesssim \left(\frac{\|f\|_1}{\lambda}\right)^{1/2}, \qquad \lambda > 0. \tag{18}
$$

Consequently, (14)–(18) imply

$$
\begin{aligned}
&\left|\left\{x \in \mathbb{T} : \sup_{p>1} \frac{G'_p f(x)}{p\ln\ln(p+2)} > \lambda\right\}\right| \\
&\qquad \le \left|\left\{x \in \mathbb{T}\setminus \widetilde{R}_\lambda : \sup_{p>1} \frac{G'_p f(x)}{p\ln\ln(p+2)} > \lambda\right\}\right| + |\widetilde{R}_\lambda| \lesssim \left(\frac{\|f\|_1}{\lambda}\right)^{1/2} + \frac{\|f\|_1}{\sqrt{\lambda}}\,.
\end{aligned}
$$

Again using homogeneity, we obtain (10). □

We will need the following estimates.

Lemma 3 ([32]). *If $p>1$ and $f \in L^1(\mathbb{T})$, then*

$$
\left(\frac{1}{n}\sum_{j=0}^{n-1}|S_j(x,f)|^p\right)^{1/p} \lesssim G_p^{(n)} f(x). \tag{19}
$$

Lemma 4 ([33]). *If $f \in L^1(\mathbb{T})$, then*

$$
\left(\frac{1}{n}\sum_{j=0}^{n-1}|S_j(x,f)|^p\right)^{1/p} \lesssim pG_2 f(x). \tag{20}
$$

Rodin [7] obtained a weak $(1,1)$-type estimate for the operators $G_p f(x)$ with fixed $p>1$. From this estimate one can derive the following assertion in a standard way.

Lemma 5 ([7]). *Let $f \in L\log L(\mathbb{T})$. Then*

$$
\|G_2(f)\|_1 \lesssim 1 + \int_{\mathbb{T}} |f|\log|f|.
$$

For any function $f \in L^1(\mathbb{T}^2)$, we set

$$
\begin{aligned}
G_{p,1}(x_1,x_2;f) &= G_p f_{x_2}(x_1), &\qquad G_{p,2}(x_1,x_2;f) &= G_p f_{x_1}(x_2),\\
G_{p,1}^{(n)}(x_1,x_2;f) &= G_p^{(n)} f_{x_2}(x_1), &\qquad G_{p,2}^{(n)}(x_1,x_2;f) &= G_p^{(n)} f_{x_1}(x_2),
\end{aligned}
$$

where $f_{x_2}(\,\cdot\,) = f(\,\cdot\,,x_2)$ and $f_{x_1}(\,\cdot\,) = f(x_1,\,\cdot\,)$ are considered as functions of x_1 and x_2, respectively. Similarly, we denote the one-dimensional partial sums of $f(x_1,x_2)$ with respect to each variable by

$$
S_{n,1}(x_1,x_2,f) = S_n(x_1,f_{x_2}), \qquad S_{n,2}(x_1,x_2,f) = S_n(x_2,f_{x_1}).
$$

Lemma 6. *If $f \in L\log L(\mathbb{T}^2)$, then*

$$\left|\left\{\sup_{p>1}\sup_{n,m\in\mathbb{N}}\frac{\left(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(x_1,x_2,f)|^p\right)^{1/p}}{p^2\ln\ln(p+2)}>\lambda\right\}\right| \lesssim \left(\frac{1}{\lambda}\left(1+\iint_{\mathbb{T}^2}|f|\log^+|f|\right)\right)^{1/2}, \qquad \lambda>0.$$

Proof. Using (19), (20), and the generalized Minkowski inequality, we obtain

$$\begin{aligned}
&\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(x_1,x_2,f)|^p\\
&\quad=\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,1}(x_1,x_2,S_{j,2}(f))|^p\le\frac{1}{m}\sum_{j=0}^{m-1}(G_{p,1}^{(n)}(x_1,x_2,|S_{j,2}(f)|))^p\\
&\quad\le\left(G_{p,1}^{(n)}\left(x_1,x_2,\left(\frac{1}{m}\sum_{j=0}^{m-1}|S_{j,2}(f)|^p\right)^{1/p}\right)\right)^p\\
&\quad\le\left(G_{p,1}\left(x_1,x_2,\left(\frac{1}{m}\sum_{j=0}^{m-1}|S_{j,2}(f)|^p\right)^{1/p}\right)\right)^p\lesssim p^p(G_{p,1}(x_1,x_2,G_{2,2}(f)))^p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Omega&=\left\{(x_1,x_2)\in\mathbb{T}^2:\sup_{p>1}\sup_{n,m\in\mathbb{N}}\frac{\left(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(x_1,x_2,f)|^p\right)^{1/p}}{p^2\ln\ln(p+2)}>\lambda\right\}\\
&\subset\left\{(x_1,x_2)\in\mathbb{T}^2:\sup_{p>1}\frac{G_{p,1}(x_1,x_2,G_{2,2}(f))}{p\ln\ln(p+2)}>\lambda\right\}.
\end{aligned}$$

Then, applying Lemmas 2 and 5, we see that

$$\begin{aligned}
|\Omega|&=\int_{\mathbb{T}^2}\mathbb{I}_\Omega(x_1,x_2)\,dx_1\,dx_2=\int_{\mathbb{T}}dx_2\int_{\mathbb{T}}\mathbb{I}_\Omega(x_1,x_2)\,dx_1\\
&\lesssim\int_{\mathbb{T}}\left(\frac{1}{\lambda}\int_{\mathbb{T}}G_{2,2}(x_1,x_2,f)\,dx_1\right)^{1/2}dx_2\\
&\lesssim\int_{\mathbb{T}}\left[\frac{1}{\lambda}\left(1+\int_{\mathbb{T}}|f(x_1,x_2)|\log^+|f(x_1,x_2)|\,dx_1\right)\right]^{1/2}dx_2\\
&\lesssim\left[\frac{1}{\lambda}\left(1+\int_{\mathbb{T}^2}|f(x_1,x_2)|\log^+|f(x_1,x_2)|\,dx_1\,dx_2\right)\right]^{1/2}.
\end{aligned}$$

This completes the proof of the lemma. □

3. PROOF OF THEOREM 1

Let $L_M := L_M(\mathbb{T}^2)$ be the Orlicz space of functions on $\mathbb{T}^2$ generated by the Young function $M(t)=t\log^+ t$. It is known that L_M is a Banach space with respect to the Luxemburg norm

$$\|f\|_M:=\inf\left\{\lambda:\lambda>0,\ \int_X M\left(\frac{|f|}{\lambda}\right)\le 1\right\}<\infty.$$

According to Theorem 9.5 in [34, Chap. 2], we have

$$\frac{1}{2}\left(1+\int_{\mathbb{T}^2}M(|f|)\right)\le\|f\|_M\le 1+\int_{\mathbb{T}^2}M(|f|)$$

provided that $\|f\|_M = 1$. Hence Lemma 6 implies

$$\left|\left\{\sup_{p>1}\sup_{n,m\in\mathbb{N}} \frac{\left(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(f)|^p\right)^{1/p}}{p^2\log\log(p+2)} > \lambda\right\}\right| \lesssim \left(\frac{\|f\|_M}{\lambda}\right)^{1/2}. \qquad (21)$$

Indeed, first, we derive the inequality in the case $\|f\|_M = 1$ and then, using a homogeneity argument, we prove it in the general case.

Proof of the Theorem. First, we shall prove that, for any $f \in L\log L(\mathbb{T}^2)$, we have

$$\lim_{n,m\to\infty}\sup_{p>1} \frac{\left(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(f)-f|^p\right)^{1/p}}{p^2\ln\ln(p+2)} = 0 \qquad \text{a.e.} \qquad (22)$$

Observe that (22) trivially holds for the double trigonometric polynomials. Indeed, let T be a trigonometric polynomial of degree (s_1, s_2). Then we have

$$\begin{aligned}
&S_{i,j}(T) - T = 0, && i \ge s_1, && j \ge s_2,\\
&S_{i,j}(T) - T = S_{s_1,j}(T) - T, && i \ge s_1, && 0 \le j < s_2,\\
&S_{i,j}(T) - T = S_{i,s_2}(T) - T, && 0 \le i < s_1, && j \ge s_2.
\end{aligned}$$

Thus, for any integers $n > s_1$ and $m > s_2$, we have

$$\begin{aligned}
&\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(T)-T|^p\\
&\quad= \frac{1}{n}\sum_{i=0}^{s_1-1}\frac{1}{m}\sum_{j=s_2}^{m-1}|S_{i,s_2}(T)-T|^p + \frac{1}{m}\sum_{j=0}^{s_2-1}\frac{1}{n}\sum_{i=s_1}^{n-1}|S_{s_1,j}(T)-T|^p\\
&\qquad+ \frac{1}{nm}\sum_{i=0}^{s_1-1}\sum_{j=0}^{s_2-1}|S_{i,j}(T)-T|^p\\
&\quad\le \frac{1}{n}\sum_{i=0}^{s_1-1}|S_{i,s_2}(T)-T|^p + \frac{1}{m}\sum_{j=0}^{s_2-1}|S_{s_1,j}(T)-T|^p + \frac{1}{nm}\sum_{j=0}^{s_2-1}\sum_{i=0}^{s_2-1}|S_{i,j}(T)-T|^p\\
&\quad\le \frac{c_1}{n} + \frac{c_2}{m},
\end{aligned}$$

where c_1 and c_2 are constants depending on T. Therefore, relation (22) holds for $f = T$. To prove it in the general case, it is enough to show that the set

$$G_\lambda = \left\{\limsup_{n,m\to\infty}\sup_{p>1} \frac{\left(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(f)-f|^p\right)^{1/p}}{p^2\ln\ln(p+2)} > \lambda\right\}$$

has measure zero for any $\lambda > 0$. Since $M(t)$ satisfies the Δ_2-condition, it follows that the function f can be approximated by a trigonometric polynomial T (see [34]), that is, there exists a T for which

$$\|f - T\|_M < \varepsilon, \qquad \|f - T\|_{L^1} < \varepsilon.$$

Since (22) holds for T, applying (21), we obtain

$$\begin{aligned}
|G_\lambda| &= \left|\left\{\limsup_{n,m\to\infty}\sup_{p>1} \frac{\left(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(f-T)-(f-T)|^p\right)^{1/p}}{p^2\ln\ln(p+2)} > \lambda\right\}\right|\\
&\le \left|\left\{\sup_{n,m\in\mathbb{N}}\sup_{p>1} \frac{\left(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(f-T)|^p\right)^{1/p}}{p^2\ln\ln(p+2)} > \frac{\lambda}{2}\right\}\right| + \left|\left\{\sup_{p>1}\frac{|f-T|}{p^2\ln\ln(p+2)} > \frac{\lambda}{2}\right\}\right|\\
&\lesssim \left(\frac{\|f-T\|_M}{\lambda}\right)^{1/2} + \frac{\|f\|_{L^1}}{\lambda} \le \left(\frac{\varepsilon}{\lambda}\right)^{1/2} + \frac{\varepsilon}{\lambda}.
\end{aligned}$$

Since $\varepsilon > 0$ can be taken arbitrarily small, we conclude that $|G_\lambda| = 0$ for any $\lambda > 0$, and hence (22) holds. To prove (6), observe that

$$u(s) = \exp\biggl(\sqrt{\frac{s}{\ln\ln(s+2)}}\biggr) \le v(s) = \sum_{k=1}^{\infty}\biggl(\frac{d}{k}\sqrt{\frac{s}{\ln\ln(k+2)}}\biggr)^k, \qquad s > 1, \tag{23}$$

for some absolute constant d. Indeed, if $s \ge 1$, then one can check that

$$1 < \sqrt{\frac{s}{\ln\ln(s+2)}} < k(s) = \biggl[\sqrt{\frac{s}{\ln\ln(s+2)}}\biggr] + 1 < 2\sqrt{\frac{s}{\ln\ln(s+2)}},$$

and therefore for d large enough, we have

$$v(s) \ge \biggl(\frac{d}{k(s)}\sqrt{\frac{s}{\ln\ln(k(s)+2)}}\biggr)^{k(s)} > \biggl(\frac{d}{2}\sqrt{\frac{\ln\ln(s+2)}{\ln\ln(k(s)+2)}}\biggr)^{k(s)} > e^{k(s)} \ge u(s),$$

which implies (23). If the function Φ satisfies (5), then one can check that

$$\Phi(s) \le \exp\biggl(\sqrt{\frac{A\cdot s}{\ln\ln(A\cdot s+2)}}\biggr) = u(As), \qquad s > S,$$

for some positive numbers $A > 1$ and $S > 1$. Consider the functions

$$\varphi_{i,j}(f) = S_{i,j}(f) - f,$$
$$\varphi^*_{i,j}(f) = \begin{cases} \varphi_{i,j}(f) & \text{if } |\varphi_{i,j}(f)| \le S, \\ 0 & \text{if } |\varphi_{i,j}(f)| > S, \end{cases}$$
$$\varphi^{**}_{i,j}(f) = \varphi_{i,j}(f) - \varphi^*_{i,j}(f).$$

It follows from (23) and the definition of Φ that

$$\begin{aligned}
\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Phi(|\varphi_{i,j}(f)|) &= \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Phi(|\varphi^*_{i,j}(f)|) + \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Phi(|\varphi^{**}_{i,j}(f)|) \\
&\le \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Phi(|\varphi^*_{i,j}(f)|) + v(A|\varphi^{**}_{i,j}(f)|) \\
&= \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Phi(|\varphi^*_{i,j}(f)|) + \sum_{k=1}^{\infty}\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\biggl(\frac{d}{k}\sqrt{\frac{A\cdot|\varphi^{**}_{i,j}(f)|}{\ln\ln(k+2)}}\biggr)^k \\
&\le \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Phi(|\varphi^*_{i,j}(f)|) \\
&\quad + \sum_{k=1}^{\infty}(d\sqrt{A})^k\biggl(\sup_{p>1/2}\frac{\bigl(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|\varphi_{i,j}(f)|^p\bigr)^{1/p}}{4p^2\ln\ln(2p+2)}\biggr)^{k/2}.
\end{aligned}$$

The second term of the last expression tends to zero almost everywhere, because, according to (22), we have

$$\begin{aligned}
&\limsup_{n,m\to\infty}\sup_{p>1/2}\frac{\bigl(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|\varphi_{i,j}(f)|^p\bigr)^{1/p}}{4p^2\ln\ln(2p+2)} \\
&\qquad \le \lim_{n,m\to\infty}\sup_{p>1}\frac{\bigl(1/(nm)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}|S_{i,j}(f)-f|^p\bigr)^{1/p}}{p^2\ln\ln(p+2)} = 0 \qquad \text{a.e.}
\end{aligned}$$

Hence, to obtain (6), it is enough to prove the same for the first term. Relation (22) and Chebyshev's inequality imply

$$r_{n,m}(x_1,x_2) = \frac{\#\{i,j \in \mathbb{N} : 0 \le i < n,\, 0 \le j < m,\, \varphi_{i,j}(x_1,x_2) > \varepsilon\}}{nm}$$
$$\le \frac{1}{\varepsilon} \cdot \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\varphi_{i,j}(x_1,x_2,f)| \to 0 \quad \text{a.e.},$$

where $\#C$ denotes the cardinality of a finite set C. Thus, for a.e. $(x_1,x_2) \in \mathbb{T}^2$, we have

$$\limsup_{n,m\to\infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi^*_{i,j}(x_1,x_2,f)|)$$
$$\le \limsup_{n,m\to\infty} (r_{n,m}(x_1,x_2)\Phi(S) + (1 - r_{n,m}(x_1,x_2))\Phi(\varepsilon)) = \Phi(\varepsilon) \qquad \text{a.e.}$$

Since $\varepsilon > 0$ can be taken arbitrarily small, we obtain

$$\lim_{n,m\to\infty} \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Phi(|\varphi^*_{i,j}(x_1,x_2,f)|) = 0 \qquad \text{a.e.},$$

which implies (6). □

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REFERENCES

1. L. Fejér, "Untersuchungen über Fouriersche Reihen," Math. Ann. **58**, 51–69 (1904).
2. H. Lebesgue, "Recherches sur la sommabilité forte des series de Fourier," Math. Ann. **61**, 251–280 (1905).
3. G. H. Hardy and J. E. Littlewood, "Sur la séries de Fourier d'une fonction à carre sommable," Comptes Rendus (Paris) **156**, 1307–1309 (1913).
4. J. Marcinkiewicz, "Sur une methode remarquable de sommation des séries doublées de Fourier," Ann. Scuola Norm. Sup. Pisa **8**, 149–160 (1939).
5. A. Zygmund, *Trigonometric Series* (Cambridge Univ. Press, Cambridge, 1959).
6. K. I. Oskolkov, "On strong summability of Fourier series," in *Trudy Mat. Inst. Steklov* Vol. 172: *Studies in the Theory of Functions of Several Real Variables and Approximation of Functions*, Collection of papers (Nauka, Moscow, 1985), pp. 280–290 [in Russian] [Proc. Steklov Inst. Math. **172**, 303–314 (1987)].
7. V. A. Rodin, "The space BMO and strong means of Fourier series," Anal. Math. **16** (4), 291–302 (1990).
8. G. A. Karagulyan, "On the divergence of strong Φ-means of Fourier series," J. Contemp. Math. Anal. **26** (2), 66–69 (1991).
9. G. A. Karagulyan, "Everywhere divergent Φ-means of Fourier series," Mat. Zametki **80** (1), 50–59 (2006) [Math. Notes **80** (1), 47–56 (2006)].
10. B. Jessen, J. Marcinkiewicz, and A. Zygmund, "Note on the differentiability of multiple integrals," Fund. Math. **25**, 217–234 (1935).
11. L. D. Gogoladze, "On strong summability almost everywhere," Mat. Sb. **135 (177)** (2), 158–168 (1988) [Math. USSR-Sb. **63** (1), 153–164 (1989)].
12. F. Schipp, "Über die starke Summation von Walsh–Fourier Reihen," Acta Sci. Math. (Szeged) **30**, 77–87 (1969).
13. F. Schipp, "On strong approximation of Walsh–Fourier series," Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **19**, 101–111 (1969).
14. F. Schipp and N. X. Ky, "On strong summability of polynomial expansions," Anal. Math. **12** (2), 115–127 (1986).
15. L. Leindler, "On the strong approximation of Fourier series," Acta Sci. Math. (Szeged) **38**, 317–324 (1976).
16. L. Leindler, "Strong approximation and classes of functions," Mitt. Math. Sem. Giessen **132**, 29–38 (1978).
17. L. Leindler, *Strong Approximation by Fourier Series* (Akademiai Kiado, Budapest, 1985).
18. L. Leindler, "Über die Approximation im starken Sinne," Acta Math. Acad. Sci. Hungar. **16**, 255–262 (1965).

19. V. Totik, "On the strong approximation of Fourier series," Acta Math. Acad. Sci. Hungar. **35** (1-2), 151–172 (1980).
20. V. Totik, "On the strong approximation of Fourier series," Acta Math. Acad. Sci. Hungar. **35** (1-2), 151–172 (1980).
21. V. Totik, "On the generalization of Fejér's summation theorem," in *Functions, Series, Operators*, *Colloq. Math. Soc. János Bolyai* (North Holland, Amsterdam–Oxford–New York, 1983), Vol. 35, pp. 1195–1199.
22. V. Totik, "Notes on Fourier series: strong approximation," J. Approx. Theory **43**, 105–111 (1985).
23. U. Goginava and L. Gogoladze, "Strong approximation by Marcinkiewicz means of double Walsh–Fourier series," Constr. Approx. **35** (1), 1–19 (2012).
24. U. Goginava and L. Gogoladze, "Strong approximation of double Walsh–Fourier series," Studia Sci. Math. Hungar. **49** (2), 170–188 (2012).
25. U. Goginava, L. Gogoladze, and G. Karagulyan, "BMO-estimation and almost everywhere exponential summability of quadratic partial sums of double Fourier series," Constr. Approx. **40** (1), 105–120 (2014).
26. G. Gát, U. Goginava, and G. Karagulyan, "Almost everywhere strong summability of Marcinkiewicz means of double Walsh–Fourier series," Anal. Math. **40** (4), 243–266 (2014).
27. G. Gát, U. Goginava, and G. Karagulyan, "On everywhere divergence of the strong Φ-means of Walsh–Fourier series," J. Math. Anal. Appl. **421** (1), 206–214 (2015).
28. F. Weisz, "Strong summability of more-dimensional Ciesielski–Fourier series," East J. Approx. **10** (3), 333–354 (2004).
29. F. Weisz, "Lebesgue points of double Fourier series and strong summability," J. Math. Anal. Appl. **432** (1), 441–462 (2015).
30. F. Weisz, "Lebesgue points of two-dimensional Fourier transforms and strong summability," J. Fourier Anal. Appl. **21** (4), 885–914 (2015).
31. F. Weisz, "Strong summability of Fourier transforms at Lebesgue points and Wiener amalgam spaces," J. Funct. Spaces, No. 420750 (2015).
32. O. D. Gabisoniya, "Points of strong summability of Fourier series," Mat. Zametki **14** (5), 615–626 (1973) [Math. Notes **14** (5), 913–918 (1973)].
33. F. Schipp, "On the strong summability of Walsh series," Publ. Math. Debrecen **52** (3-4), 611–633 (1998).
34. M. A. Krasnosel'skii and Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces* (Fizmatgiz, Moscow, 1958) [in Russian].