ON EQUIVALENCY OF MARTINGALES AND RELATED PROBLEMS

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Abstract. We study some almost everywhere convergence problems of martingales. We obtain different equivalency theorems, which show, that in some problems of martingale theory generale martingales can be replaced with Haar martingales. We bring some applications of these results in the theory of differentiation of integrals and in some problems of convergence of Riemann sums.

1. Introduction

The study of almost sure convergence of martingales has a deep connection with different problems in harmonic analysis. Many results in the martingale theory are direct generalizations of some theorems on Fourier-Haar series, differentiation of integrals, maximal functions. A central result in martingale theory is due Doob [4]. It says, that any L^1-bounded martingale converges almost surely. An analogous problem for multiple martingales is considered by R. Cairoli in [6]. He has proved an almost surely convergence of d-multiple martingales which are bounded in $L \log^{d-1} L$.

In this paper we obtain some equivalency theorems for martingales. One-dimensional and multiple martingales are considered. It will be shown that in some problems concerning to almost everywhere convergence of martingales one can consider simply Haar martingales. We shall discuss also some applications of these equivalency theorems in maximal functions and Riemann sums.

We consider the probability space of Lebesgue measureable sets in $[0, 1)$. Let $\mathcal{L}$ be the family of Lebesgue measurable sets in $[0, 1)$. Denote by $\mathcal{S}(E)$ the minimal σ-algebra containing a family of sets $E \subset \mathcal{L}$. We say a sequence of σ-algebras $\{\mathcal{A}_n,\, n = 1, 2, \ldots\}$ in $[0, 1)$ is regular, if

1) $\mathcal{A}_n \subset \mathcal{A}_{n+1}$,
2) $\mathcal{A}_n$ is generated by a finite number of intervals of the form $[a, b)$,
3) the σ-algebra $\mathcal{S}(\cup_n \mathcal{A}_n)$ does not have an atom.

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We shall consider the probability space $(Q_d, \mathcal{L}_d, m_d)$ of Lebesgue measure on d-dimensional unit cube $Q_d = [0,1)^d$. Let $\{\mathcal{A}_n^{(k)},\ n \in \mathbb{N}\}$, $k = 1, 2, \ldots, d$, be sequences of σ-algebras in $[0,1)$. We consider the Cartesian product of these σ-algebras

$$\mathcal{A}_{\mathbf{n}} = \mathcal{A}_{n_1,\ldots,n_d} = \otimes_{k=1}^d \mathcal{A}_{n_k}^{(k)}, \quad \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d. \tag{1}$$

We say the multiple sequence $\{\mathcal{A}_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$ is regular, if each component $\{\mathcal{A}_n^{(k)},\ n \in \mathbb{N}\}$ is regular. For a given function $f(x) \in L^1(Q_d)$ we denote by

$$E^{\mathcal{A}_{\mathbf{n}}} f(x), \quad \mathbf{n} \in \mathbb{N}^d,$$

the conditional expectation of $f(x)$ with respect to σ-algebra $\mathcal{A}_{\mathbf{n}}$. A function $\Theta : \mathcal{L}_d \to \mathcal{L}_d$ is said to be measure preserving transformation (MP-transformation) of Q_d, if

$$\Theta(E \cup F) = \Theta(E) \cup \Theta(F),$$
$$m_d(\Theta(E)) = m_d(E),$$

for any $E, F \in \mathcal{L}_d$. If Θ is an MP-transformation of Q_d, then one can easily check that

$$\begin{aligned} &\text{if } E \cap F = \varnothing, \text{ then } m_d(\Theta(E) \cap \Theta(F)) = 0, \\ &\text{if } E \subset F, \text{ then } m_d(\Theta(F) \setminus \Theta(E)) = 0. \end{aligned} \tag{2}$$

For an arbitrary measurable $f(x)$ we define $\Theta f(x)$ as follows. If $f(x)$ has a form

$$\sum_{i=1}^m \alpha_i \mathbb{I}_{E_i}(x), \tag{3}$$

where $E_i \in \mathcal{L}_d$, $i = 1, 2, \ldots, m$, is a pairwise disjoint family of measurable sets, then

$$\Theta f(x) = \sum_{i=1}^m \alpha_i \mathbb{I}_{\Theta(E_i)}(x).$$

In the generale case the function $f(x)$ is a pointwise limit of a sequence of functions $s_n(x)$ of the form (3). That is

$$f(x) = \lim_{n\to\infty} s_n(x), \quad x \in Q_d. \tag{4}$$

Using (2) and a standard argument one can observe that $\Theta s_n(x)$ converges almost everywhere and the limit function doesn't depend on the representation (4). So we denote this limit by $\Theta f(x)$.
For a given sequence of integers $\mathcal{N} = \{\nu_i \in \mathbb{N} : \nu_1 < \nu_2 < \ldots\}$ we consider the multiple sequence of σ-algebras

$$\tilde{\mathcal{A}}_{\mathbf{n}} = \mathcal{A}_{\nu_{n_1},\ldots,\nu_{n_d}}, \quad \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d. \tag{5}$$

The sequence (5) is said to be a subsequence of (1).

The following two theorems are the main results of the paper.

Theorem 1. *Let $\{\mathcal{A}_{\mathbf{n}},\ \mathbf{n}\in\mathbb{N}^d\}$ and $\{\mathcal{B}_{\mathbf{n}},\ \mathbf{n}\in\mathbb{N}^d\}$ be arbitrary multiple regular sequences of σ-algebras in the unit cube Q_d. Then for any sequence of numbers $\varepsilon_k>0$, $k=1,2,\ldots$, there exist a measure preserving transformation Θ, a sequence of sets $G_{\mathbf{n}}\subset Q_d$ and a subsequence $\{\tilde{\mathcal{B}}_{\mathbf{n}}\}$ of $\{\mathcal{B}_{\mathbf{n}}\}$, such that*

$$G_{\mathbf{n}}\subset G_{\mathbf{n}'},\quad \mathbf{n}\le\mathbf{n}', \tag{6}$$

$$|G_{\mathbf{n}}|>1-\varepsilon_m,\quad \min_{1\le i\le d} n_i\ge m \tag{7}$$

$$\left(\Theta\circ\mathbf{E}^{\mathcal{A}_{\mathbf{n}}}\right)f(x)=\left(\mathbf{E}^{\tilde{\mathcal{B}}_{\mathbf{n}}}\circ\Theta\right)f(x),\quad x\in G_{\mathbf{n}},\quad \mathbf{n}\in\mathbb{N}^d. \tag{8}$$

for any $f\in L^1(Q_d)$, where $\mathbf{n}=(n_1,\ldots,n_d)\le\mathbf{n}'=(n_1',\ldots,n_d')$ means $n_i\le n_i'$, $i=1,2,\ldots,$.

Theorem 2. *If $\{\mathcal{B}_{\mathbf{n}},\ \mathbf{n}\in\mathbb{N}^d\}$ is a subsequence of a regular sequence of multiple σ-algebras $\{\mathcal{A}_{\mathbf{n}},\ \mathbf{n}\in\mathbb{N}^d\}$, then there exist a measure preserving transformation Θ, such that*

$$\left(\Theta\circ\mathbf{E}^{\mathcal{A}_{\mathbf{n}}}\right)f(x)=\left(\mathbf{E}^{\mathcal{B}_{\mathbf{n}}}\circ\Theta\right)f(x)\ \text{a.e.} \tag{9}$$

for any $f\in L^1(Q_d)$.

2. Notations and auxiliary lemmas

A set in $[0,1)$ is said to be simple, if it is a finite union of intervals of the form $[\alpha,\beta)$. If a finite family of simple sets $A=\{a_i:\ i=1,2,\ldots,n\}$ satisfy the relations

$$\cup_{i=1}^n a_i=[0,1),\quad a_i\cap a_j=\varnothing,\quad i\ne j,$$

then we say A is a partition of $[0,1)$. Any partition A defines a σ-algebra $\mathcal{A}$ generated by the atoms a_i, $i=1,2,\ldots,n$. We denote the family of such σ-algebras by $\mathfrak{M}$. We shall use the letters A, B and C for partitions and the letters $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ for σ-algebras corresponding to these partitions. For any two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ we denote by

$$\mathcal{A}\wedge\mathcal{B}=\mathcal{S}\{a\cap b:\ a\in A,\ b\in B\}$$

the σ-algebras generated by the sets of the form $a\cap b$, where $a\in A$, $b\in B$. Let a be a simple set. One can easily check, that the function

$$\phi_a(x)=|[0,x)\cap a|$$

defines one to one correspondence from a to $[0,|a|)$, such that

$$|E|=|\phi_a(E)|$$

for any Lebesgue measurable set $E \subset a$. For arbitrary simple sets a and b we denote

$$\phi_{a,b} = \phi_b^{-1} \circ \left(\frac{|b|}{|a|} \phi_a \right).$$

It is easy to observe that $\phi_{a,b}$ defines one to one correspondence from a to b and we have

$$|\phi_{a,b}(E)| = \frac{|b|}{|a|}|E|$$

if $E \subset a$ is measurable. If $\mathcal{A}, \mathcal{B} \in \mathfrak{M}$, then we denote by $[\mathcal{A}, \mathcal{B}]$ an MP-transformation defined as follows. At first we consider the case $\mathcal{A} \subset \mathcal{B}$. Take an arbitrary $x \in [0, 1)$. We have $x \in a$ for some $a \in A$ and $a = \cup_{i=1}^k b_i$, where $b_i \in B$. We let

$$[\mathcal{A}, \mathcal{B}](x) = \{\phi_{a,b_i}(x) : \, i = 1, \ldots, k\}$$

and

$$[\mathcal{A}, \mathcal{B}](E) = \bigcup_{x \in E} [\mathcal{A}, \mathcal{B}](x).$$

If A and B are arbitrary, then $\mathcal{A} \subset \mathcal{A} \wedge \mathcal{B}$ and we define

$$[\mathcal{A}, \mathcal{B}] = [\mathcal{A}, \mathcal{A} \wedge \mathcal{B}].$$

It is easy to observe that $\theta = [\mathcal{A}, \mathcal{B}]$ defines an MP-transformation. We say a function $g(x)$ is a rearrangement of $f(x)$, if

$$P\{g(x) > \lambda\} = P\{f(x) > \lambda\}, \quad \lambda \in \mathbb{R}.$$

We shall use the notation $g \sim f$ for this relation. The following simple properties of MP-transformations and the operator of conditional expectation $\mathbf{E}^{\mathcal{A}}$ can be checked easily.

P1) For any MP-transformation θ and measurable function $f(x)$, defined on Q_d, we have

$$\theta f(x) \sim f(x).$$

P2) For any MP-transformation θ the operator $\theta f(x)$ is linear-bounded in $L^1(Q_d)$ and

$$\|\theta\|_{L_1(Q_d) \to L_1(Q_d)} = 1.$$

P3) For any $\mathcal{A}, \mathcal{B} \in \mathfrak{M}$ we have $\theta(a) = a$, $a \in \mathcal{A}$, where $\theta = [\mathcal{A}, \mathcal{B}]$.

P4) For any σ-algebras $\mathcal{A}, \mathcal{B} \in \mathfrak{M}$ we have

$$\mathbf{E}^{\mathcal{A}} f(x) = \mathbf{E}^{\mathcal{B}} f(x), \quad x \in a, \quad a \in A \cap B.$$

P5) If θ is an MP-transformation, then

$$\theta \circ \mathbf{E}^{\mathcal{A}} = \mathbf{E}^{\theta(\mathcal{A})} \circ \theta.$$

P6) If $\mathcal{A}, \mathcal{B} \in \mathfrak{M}$, $\mathcal{A} \subset \mathcal{B}$ and $\theta = [\mathcal{A}, \mathcal{B}]$ then

$$\mathbf{E}^{\mathcal{A}} \circ \theta = \mathbf{E}^{\mathcal{A}} = \mathbf{E}^{\mathcal{B}} \circ \theta.$$

P7) If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathfrak{M}$, $\mathcal{A} \subset \mathcal{C}$, $\mathcal{A} \subset \mathcal{B}$ and $\theta = [\mathcal{A}, \mathcal{B}]$, then

$$\mathbf{E}^{\theta(\mathcal{C})} \circ \theta = \mathbf{E}^{\theta(\mathcal{C}) \wedge \mathcal{B}} \circ \theta.$$

P8) If $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ and $\theta = [\mathcal{A}, \mathcal{C}]$ then the equality

$$\mathbf{E}^{\mathcal{A}} g(x) = \mathbf{E}^{\mathcal{B}} g(x)$$

implies

$$\mathbf{E}^{\mathcal{B}} g(x) = \left(\mathbf{E}^{\mathcal{B}} \circ \theta\right) g(x).$$

Everywhere below any MP-transformations θ and conditional expectations $\mathbf{E}^{\mathcal{A}}$ will be considered as linear bounded operators from $L^1[0,1)$ to $L^1[0,1)$. Let

$$\{\mathcal{A}_k : k = 1, 2, \ldots\}, \tag{10}$$

$$\{\mathcal{B}_k : k = 1, 2, \ldots\}, \tag{11}$$

be sequences of σ-algebras from $\mathfrak{M}$, satisfying

$$\mathcal{A}_k \subset \mathcal{A}_{k+1}, \quad \mathcal{B}_k \subset \mathcal{B}_{k+1}, \quad k = 1, 2, \ldots,$$
$$\max_{a \in A_n} |a| \to 0, \quad \max_{b \in B_n} |b| \to 0.$$

We note, that if in addition each partition A_n consists of intervals, then (10) turns to be regular. We associate with (10) and (11) a sequence of MP-transformations Θ_k, $k = 1, 2, \ldots$, defined as follows. We take

$$\Theta_1 = [\mathcal{A}_1, \mathcal{B}_1], \quad \Theta_k = [\Theta_{k-1}(\mathcal{A}_k), \mathcal{B}_k] \circ \Theta_{k-1}, \quad k = 2, 3, \ldots.$$

Denoting

$$\theta_1 = \Theta_1, \quad \theta_k = [\Theta_{k-1}(\mathcal{A}_k), \mathcal{B}_k], \quad k = 2, 3, \ldots, \tag{12}$$

we have

$$\Theta_k = \theta_k \circ \Theta_{k-1} = \theta_k \circ \theta_{k-1} \circ \ldots \circ \theta_1. \tag{13}$$

Lemma 1. *For any sequences (10) and (11) the sequence*

$$\mathcal{A}_1, \Theta_1(\mathcal{A}_2), \ldots, \Theta_{n-1}(\mathcal{A}_n) \ldots, \tag{14}$$

is an increasing sequence of σ-algebras from $\mathfrak{M}$.

Proof. It is clear that $\Theta_{k-1}(\mathcal{A}_k) \in \mathfrak{M}$. Such that $\mathcal{A}_k \subset \mathcal{A}_{k+1}$, we get

$$\Theta_k(\mathcal{A}_k) \subset \Theta_k(\mathcal{A}_{k+1}). \tag{15}$$

According to (12) and the property P3), we have $\theta_k(\Theta_{k-1}(\mathcal{A}_k)) = \Theta_{k-1}(\mathcal{A}_k)$. Thus, using (13), we obtain

$$\Theta_k(\mathcal{A}_k) = \theta_k(\Theta_{k-1}(\mathcal{A}_k)) = \Theta_{k-1}(\mathcal{A}_k).$$

This together with (15) implies $\Theta_{k-1}(\mathcal{A}_k) \subset \Theta_k(\mathcal{A}_{k+1})$. The second and third conditions of regularity of (14) clearly follows from the corresponding conditions of $\mathcal{A}_k$ and this completes the proof of lemma. □

Lemma 2. *For any sequence (10) we have*

$$\Theta_l(\mathcal{A}_k) = \Theta_{k-1}(\mathcal{A}_k), \quad l \geq k.$$

Proof. From (12) and property P3), we get

$$\theta_m(a) = a, \quad a \in \Theta_{m-1}(\mathcal{A}_m). \tag{16}$$

By Lemma 1, we have

$$\Theta_{k-1}(\mathcal{A}_k) \subset \Theta_{m-1}(\mathcal{A}_m), \quad m \geq k. \tag{17}$$

From (16) and (17) it follows that

$$\theta_m(a) = a, \quad a \in \Theta_{k-1}(\mathcal{A}_k),$$

and therefore, we get

$$\theta_m(\Theta_{k-1}(\mathcal{A}_k)) = \Theta_{k-1}(\mathcal{A}_k), \quad m \geq k. \tag{18}$$

From (13) we have

$$\Theta_l = \theta_l \circ \theta_{l-1} \circ \ldots \circ \theta_k \circ \Theta_{k-1}.$$

Thus, using (18) for $m = k, k+1, \ldots, l$, we obtain

$$\Theta_l(\mathcal{A}_k) = \Theta_{k-1}(\mathcal{A}_k), \quad l \geq k.$$

□

Lemma 3. *If (10) is regular, then there exists an MP-transformation Θ, such that*

$$\|\Theta_n(f) - \Theta(f)\|_{L_1} \to 0, \quad n \to \infty, \tag{19}$$

for any $f \in L^1[0,1)$.

Proof. At first we prove the convergence of $\Theta_n(f)$ in L_1. Since (10) is regular we have

$$\|\mathbf{E}^{\mathcal{A}_n}(f) - f\|_{L_1} \to 0,$$

as $n \to \infty$. So for any $\varepsilon > 0$ one may find a number N, such that

$$\|\mathbf{E}^{\mathcal{A}_n}(f) - f\|_{L_1} < \varepsilon, \quad n > N.$$

Since Θ_k are measure preserving, the last inequality immediately implies

$$\|\Theta_k[\mathbf{E}^{\mathcal{A}_n}(f)] - \Theta_k(f)\|_{L_1} < \varepsilon, \quad n > N, \tag{20}$$

for any $k = 1, 2, \ldots$. From Lemma 2 it follows that

$$\Theta_m[\mathbf{E}^{\mathcal{A}_n}(f)] = \Theta_n[\mathbf{E}^{\mathcal{A}_n}(f)], \quad m > n.$$

Combining this with (20), we get

$$\|\Theta_m(f) - \Theta_n(f)\|_{L_1} < 2\varepsilon, \quad m > n > N,$$

which implies the convergence of $\Theta_k(f)$ in L^1. Denoting the limit function by $U(f)$, we have

$$\|\Theta_n(f) - U(f)\|_{L_1} \to 0, \quad n \to \infty. \tag{21}$$

On the other hand we have

$$m\{\Theta_n(f) > \lambda\} = m\{f > \lambda\}, \quad \lambda \in \mathbb{R}.$$

Thus, using (21), we get

$$m\{U(f) > \lambda\} = m\{f > \lambda\}.$$

Taking $f = \mathbb{I}_F$ for a measurable F we will have $U(f) = \mathbb{I}_G$ for some measurable G with $mG = mF$. So we define MP-transformation Θ, taking $\Theta(F) = G$. To complete the proof of lemma it remains to show that

$$\Theta(f) = U(f), \quad f \in L^1[0,1). \tag{22}$$

From (21) it follows that $U(f)$ is a bounded linear operator from L^1 to L^1, since the same property have the operators Θ_n (see P2)). According to the definition of Θ the equality (22) holds if $f = \mathbb{I}_F$ is a characteristic function. Therefore by linearity the (22) will be satisfied for the functions of the form (3). Thus, using the continuity of the operators $\Theta(f)$ and $U(f)$ we obtain (22) for any integrable functions. The lemma is proved. □

Definition 1. *We say the sequence (11) divides (10), if*

$$\mathcal{A}_1 \subset \mathcal{B}_1, \quad \Theta_{k-1}(\mathcal{A}_k) \subset \mathcal{B}_k, \quad k = 2, 3, \ldots. \tag{23}$$

Lemma 4. *If the σ-algebra (11) divides (10), then we have*

$$\mathbf{E}^{\mathcal{B}_k} \circ \Theta_l = \mathbf{E}^{\Theta_{k-1}(\mathcal{A}_k)} \circ \Theta_l, \quad l \ge k. \tag{24}$$

Proof. The σ-algebras $\mathcal{B} = \mathcal{B}_n$, $\mathcal{A} = \Theta_{n-1}(\mathcal{A}_n)$ and $\mathcal{C} = \Theta_{n-1}(\mathcal{A}_{n+1})$ satisfy the conditions of the property P7). Indeed $\mathcal{A} \subset \mathcal{B}$ follows from the fact, that (11) divides (10), the embedding $\mathcal{A} \subset \mathcal{C}$ follows from the regularity of $\mathcal{A}_n$, that is $\mathcal{A}_n \subset \mathcal{A}_{n+1}$. By (12), we have

$$\theta_n = [\Theta_{n-1}(\mathcal{A}_n), \mathcal{B}_n] = [\mathcal{A}, \mathcal{B}].$$

Consider a function $f \in L^1$. Applying the property P7), we obtain

$$\begin{aligned}\left(\mathbf{E}^{\theta_n(\Theta_{n-1}(\mathcal{A}_{n+1}))} \circ \theta_n\right)(\Theta_{n-1}(f)) \\ = \left(\mathbf{E}^{\theta_n(\Theta_{n-1}(A_{n+1}))\wedge B_n} \circ \theta_n\right)(\Theta_{n-1}(f)).\end{aligned} \tag{25}$$

Such that $\theta_n \circ \Theta_{n-1} = \Theta_n$ we obtain

$$\mathbf{E}^{\Theta_n(\mathcal{A}_{n+1})}(\Theta_n(f)) = \mathbf{E}^{\Theta_n(A_{n+1})\wedge B_n}(\Theta_n(f)).$$

The σ-algebras $\mathcal{A} = \Theta_n(\mathcal{A}_{n+1})$, $\mathcal{B} = \Theta_n(\mathcal{A}_{n+1}) \wedge \mathcal{B}_n$ and $\mathcal{C} = \mathcal{B}_{n+1}$ satisfy the conditions of P8). The condition $\mathcal{A} \subset \mathcal{B}$ is clear. By regularity of $\mathcal{B}_n$ we have $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. Besides, since $\{\mathcal{B}_n\}$ divides $\{\mathcal{A}_n\}$, we get $\Theta_n(\mathcal{A}_{n+1}) \subset \mathcal{B}_{n+1}$. Therefore we get $\mathcal{B} \subset \mathcal{C}$. We have

$$\theta_{n+1} = [\Theta_n(\mathcal{A}_{n+1}), \mathcal{B}_{n+1}] = [\mathcal{A}, \mathcal{C}]$$

Hence, using (25) and the property P8), we obtain

$$\begin{aligned}\mathbf{E}^{\Theta_n(\mathcal{A}_{n+1})\wedge\mathcal{B}_n}&(\Theta_n(f))\\ &= (\mathbf{E}^{\Theta_n(\mathcal{A}_{n+1})\wedge\mathcal{B}_n} \circ \theta_{n+1})(\Theta_n(f))\\ &= \mathbf{E}^{\Theta_n(\mathcal{A}_{n+1})\wedge\mathcal{B}_n}(\Theta_{n+1}(f)).\end{aligned}$$

Thus, such that $\mathcal{B}_k \subset \Theta_n(\mathcal{A}_{n+1}) \wedge \mathcal{B}_n$ if $k \le n$, we get

$$(\mathbf{E}^{\mathcal{B}_k} \circ \Theta_n)(f) = (\mathbf{E}^{\mathcal{B}_k} \circ \Theta_{n+1})(f), \quad k \le n.$$

From this we obtain

$$(\mathbf{E}^{\mathcal{B}_k} \circ \Theta_l)(f) = (\mathbf{E}^{\mathcal{B}_k} \circ \Theta_{l-1})(f) = \ldots = (\mathbf{E}^{\mathcal{B}_k} \circ \Theta_k)(f).$$

Hence we have

$$(\mathbf{E}^{\mathcal{B}_k} \circ \Theta_l)(f) = (\mathbf{E}^{\mathcal{B}_k} \circ \Theta_k)(f) = (\mathbf{E}^{\mathcal{B}_k} \circ \theta_k)(\Theta_{k-1}(f)) \tag{26}$$

and therefore, since

$$\Theta_{k-1}(\mathcal{A}_k) \subset \mathcal{B}_k, \tag{27}$$

we obtain

$$\begin{aligned}(\mathbf{E}^{\Theta_{k-1}(\mathcal{A}_k)} \circ \Theta_l)(f) &= (\mathbf{E}^{\Theta_{k-1}(\mathcal{A}_k)} \circ \Theta_k)(f)\\ &= (\mathbf{E}^{\Theta_{k-1}(\mathcal{A}_k)} \circ \theta_k)(\Theta_{k-1}(f)).\end{aligned} \tag{28}$$

Finally, again taking into account of (27), from the property P6) we deduce

$$(\mathbf{E}^{\mathcal{B}_k} \circ \theta_k)(\Theta_{k-1}(f)) = (\mathbf{E}^{\Theta_{k-1}(\mathcal{A}_k)} \circ \theta_k)(\Theta_{k-1}(f)).$$

Combining this with (26) and (28) we get (24). □

Lemma 5. *If the σ-algebra (11) divides (10) and (10) is regular, then there exists an MP-transformation Θ, such that*

$$\Theta \circ \mathbf{E}^{\mathcal{A}_k} = \mathbf{E}^{\mathcal{B}_k} \circ \Theta, \quad k = 1, 2, \ldots. \tag{29}$$

Proof. Such that Θ_l is measure preserving, from P5) we get

$$\Theta_l \circ \mathbf{E}^{\mathcal{A}_k} = \mathbf{E}^{\Theta_l(\mathcal{A}_k)} \circ \Theta_l. \tag{30}$$

Thus, using Lemma 2 and then Lemma 4 for $l \geq k$, we obtain

$$\mathbf{E}^{\Theta_l(\mathcal{A}_k)} \circ \Theta_l = \mathbf{E}^{\Theta_{k-1}(\mathcal{A}_k)} \circ \Theta_l = \mathbf{E}^{\mathcal{B}_k} \circ \Theta_l. \tag{31}$$

Combination of (30) and (31) implies

$$\Theta_l \circ \mathbf{E}^{\mathcal{A}_k} = \mathbf{E}^{\mathcal{B}_k} \circ \Theta_l, \quad l > k.$$

Since (10) consists of interval σ-algebras, we may define Θ, satisfying the relation (19) of Lemma 3. Then, using the boundedness of operators $\mathbf{E}^{\mathcal{A}_k}$ and $\mathbf{E}^{\mathcal{B}_k}$ in L^1 we get (29). $\square$

For arbitrary σ-algebras $\mathcal{A}, \mathcal{B} \in \mathfrak{M}$, generated by partitions A and B, we denote

$$\rho(\mathcal{A}, \mathcal{B}) = \left| \bigcup_{b \in B \setminus A} b \right| = \left| \bigcup_{a \in A \setminus B} a \right|.$$

Lemma 6. *If $\mathcal{A} \in \mathfrak{M}$ and (11) is regular, then*

$$\lim_{n \to \infty} \rho(\mathcal{A} \wedge \mathcal{B}_n, \mathcal{B}_n) = 0.$$

Proof. One can easily observe, that the number of elements in

$$(A \wedge B_n) \setminus B_n$$

does not exceed a number l depended only on A. Thus, since $\{B_n\}$ is regular, we get

$$\rho(\mathcal{A}, \mathcal{B}_n) = \left| \bigcup_{a \in (A \wedge B_n) \setminus B_n} a \right| \leq l \max_{b \in B_n} |b| \to 0, \quad n \to \infty.$$

$\square$

3. Proof of main theorems

Proof of Theorem 1: One-dimensional case: Let $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ be regular sequences σ-algebras in $[0, 1)$ and $\varepsilon_k > 0$, $k = 1, 2, \ldots$, is a sequence of numbers. Suppose $l_1 < l_2 < \ldots < l_n < \ldots$ are some integers and $\{\tilde{\mathcal{B}}_n = \mathcal{B}_{l_n}, n \in \mathbb{N}\}$ is a subsequence of $\{\mathcal{B}_k\}$. We introduce a sequence

$\mathcal{C}_k \in \mathfrak{M}$ and measure preserving transformations Θ_k as follows:

$$\begin{aligned}
\mathcal{C}_1 =& \mathcal{A}_1 \wedge \mathcal{B}_{l_1}, \quad \Theta_1 = [\mathcal{A}_1, \mathcal{C}_1],\\
\mathcal{C}_2 =& \Theta_1(\mathcal{A}_2) \wedge \mathcal{B}_{l_2}, \quad \Theta_2 = [\Theta_1(\mathcal{A}_2), \mathcal{C}_2] \circ \Theta_1,\\
& \dots\\
\mathcal{C}_n =& \Theta_{n-1}(\mathcal{A}_n) \wedge \mathcal{B}_{l_n}, \quad \Theta_n = [\Theta_{n-1}(\mathcal{A}_n), \mathcal{C}_n] \circ \Theta_{n-1},\\
& \dots
\end{aligned}$$

It is clear that the sequence $\{\mathcal{C}_k\}$ divides $\{\mathcal{A}_k\}$ (see definition in (23)). Therefore, by Lemma 5, there exists an MP-transformation Θ such that

$$\Theta \circ \mathbf{E}^{\mathcal{A}_k} = \mathbf{E}^{\mathcal{C}_k} \circ \Theta, \quad k = 1, 2, \dots. \tag{32}$$

Using Lemma 6, the numbers $\{l_n\}$ can be chosen satisfying

$$\rho(\mathcal{B}_{l_k}, \mathcal{C}_k) < \varepsilon_k.$$

Denoting

$$G_k = \bigcup_{a \in B_{l_k} \cap C_k} a, \tag{33}$$

from the definition of ρ, we will have

$$|G_k| = 1 - \rho(B_{l_k}, C_k) > 1 - \varepsilon_k.$$

On the other hand, by the property P4), we get

$$\left(\mathbf{E}^{\mathcal{C}_k} \circ \Theta\right) f(x) = \left(\mathbf{E}^{\mathcal{B}_{l_k}} \circ \Theta\right) f(x), \quad x \in G_k. \tag{34}$$

From (32) and (34) we immediately obtain

$$\left(\Theta \circ \mathbf{E}^{\mathcal{A}_k}\right) f(x) = \left(\mathbf{E}^{\mathcal{B}_{l_k}} \circ \Theta\right) f(x), \quad x \in G_k,$$

which completes the proof of Theorem 1 in the case of $d = 1$. □

If T is a bounded linear operator in $L^1[0,1)$, then we denote by T_k, $1 \le k \le d$, the operator

$$T_k : L^1(Q_d) \to L^1(Q_d)$$

defined by

$$T_k f(x_1, \dots, x_d) = T f(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_d).$$

It is clear, that the operator T_k is bounded in $L^1(Q_d)$.

Lemma 7. *If T and P are bounded linear operators in $L^1[0,1)$, then*

$$P_n \circ T_m = T_m \circ P_n, \quad n \ne m, \quad 1 \le n, m \le d.$$

Proof. We have to prove that

$$(P_n \circ T_m)f(\mathbf{x}) = (T_m \circ P_n)f(\mathbf{x}). \tag{35}$$

for any $f \in L^1(Q_d)$. If

$$f(\mathbf{x}) = \prod_{k=1}^{d} \mathbb{I}_{E_k}(x_k) \tag{36}$$

where $E_k \subset [0,1)$, $k = 1, 2, \ldots, d$ are measurable sets, then, using the linearity, we have

$$P_n f(\mathbf{x}) = \prod_{k \neq n} \mathbb{I}_{E_k}(x_k) \cdot P(\mathbb{I}_{E_n})(x_n)$$

and then

$$(T_m \circ P_n)f(\mathbf{x}) = P(\mathbb{I}_{E_n})(x_n) \cdot T(\mathbb{I}_{E_m})(x_m) \cdot \prod_{k \neq n,m} \mathbb{I}_{E_k}(x_k).$$

The same we will have for $(P_n \circ T_m)f(\mathbf{x})$. Hence (35) holds for the functions of the form (36) and their linear combinations, i.e. it holds for the all simple functions. If $f(\mathbf{x}) \in L^1(Q_d)$, then we find a sequence of simple functions $s_n(x,y)$ of the form (36), such that $\|s_n - f\|_{L^1} \to 0$. Using the continuity of mappings T_m and P_n, as well as the equality (35) for the functions s_n, we immediately get (35) in general case. $\square$

Proof of Theorem 1: The generale case: Using the above notations, we have

$$\mathbf{E}^{\mathcal{A}_{\mathbf{n}}} = \left(\mathbf{E}^{\mathcal{A}^{(1)}_{n_1}}\right)_1 \circ \ldots \circ \left(\mathbf{E}^{\mathcal{A}^{(d)}_{n_d}}\right)_d, \quad \mathbf{n} = (n_1, \ldots, n_d).$$

Without loss of generality, in the formulation of the theorem one can assume that $\varepsilon_k > \varepsilon_{k+1}$. Applying one-dimensional case of the theorem for the sequences of σ-algebras $\mathcal{A}^{(k)}_n$ and $\mathcal{B}^{(k)}_n$, we may define an MP -transformations Θ_k, $k = 1, 2, \ldots, d$, a sequences of measurable sets $\{G^{(k)}_i\}$, and a subsequences of σ-algebras $\tilde{\mathcal{B}}^{(k)}_i$, such that

$$|G^{(k)}_i| > 1 - \frac{\varepsilon_i}{d},$$

$$\left(\Theta_k \circ \mathbf{E}^{\mathcal{A}^{(k)}_i}\right) f(x) = \left(\mathbf{E}^{\tilde{\mathcal{B}}^{(k)}_i} \circ \Theta_k\right) f(x), \quad x \in G^{(k)}_i, \quad i = 1, 2, \ldots.$$

Denote

$$\Theta = (\Theta_1)_1 \circ \ldots \circ (\Theta_d)_d,$$

$$G_{\mathbf{n}} = \left\{\mathbf{x} \in Q_d : \, x_k \in G^{(k)}_{n_k}\right\}, \quad \mathbf{n} = (n_1, \ldots, n_d).$$

We have

$$|G_{\mathbf{n}}| > 1 - \sum_{k=1}^{d} \frac{\varepsilon_{n_k}}{d} \geq 1 - \varepsilon_{\min n_k},$$

as well as

$$\otimes_{k=1}^{d} \left(\Theta_k \circ \mathbf{E}^{\mathcal{A}_{n_k}^{(k)}}\right)_k f(\mathbf{x}) = \otimes_{k=1}^{d} \left(\mathbf{E}^{\tilde{\mathcal{B}}_{n_k}^{(k)}} \circ \Theta_k\right)_k f(\mathbf{x}),\ \mathbf{x} \in G_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d. \tag{37}$$

Applying Lemma 7 several times, we conclude

$$\otimes_{k=1}^{d} \left(\Theta_k \circ \mathbf{E}^{\mathcal{A}_{n_k}^{(k)}}\right)_k = \left(\otimes_{k=1}^{d} (\Theta_k)_k\right) \circ \left(\otimes_{k=1}^{d} \left(\mathbf{E}^{\mathcal{A}_{n_k}^{(k)}}\right)_k\right) = \Theta \circ \mathbf{E}^{\mathcal{A}_{\mathbf{n}}},$$

and similarly

$$\otimes_{k=1}^{d} \left(\mathbf{E}^{\tilde{\mathcal{B}}_{i}^{(k)}} \circ \Theta_k\right)_k = \mathbf{E}^{\tilde{\mathcal{B}}_{\mathbf{n}}} \circ \Theta. \tag{38}$$

So from (37) and (38) we get (8). □

Proof of Theorem 2. The proof of Theorem 2 is a direct repetition of the proof of Theorem 1 with a little change. That is, if $\mathcal{B}_{\mathbf{n}}$ is a subsequence of $\mathcal{A}_{\mathbf{n}}$, then we will take $\tilde{\mathcal{B}}_n = \mathcal{B}_n$. Then all the sets G_k in (33) coincide with $[0, 1)$ and therefore in the proof of general case of Theorem 1 we will have $|G_{\mathbf{n}}| = 1$ for each $\mathbf{n} \in \mathbb{N}$. So the equality (9) will be satisfied almost everywhere in the generale case. □

4. Maximal functions

We consider the family of dyadic rectangles

$$\left\{\mathbf{x} = (x_1, \ldots, x_d) \in Q_d : \frac{t_i - 1}{2^{n_i}} \leq x_i < \frac{t_i}{2^{n_i}},\ i = 1, 2, \ldots, d\right\}, 1 \leq t_i \leq 2^{n_i},\ n_i \in \mathbb{N}. \tag{39}$$

in the unit cube Q_d. Denote by $\mathcal{R}_{\mathbf{n}}$ the subfamily of these rectangles corresponding to a fixed multiindex $\mathbf{n} = (n_1, n_2, \ldots, n_d)$. It is clear, that $\mathcal{R}_{\mathbf{n}}$ consists of mutually disjoint rectangles, which union is Q_d. For an infinite set of integers $\mathcal{N} \subseteq \mathbb{N}$ we consider the family of rectangles

$$\mathfrak{R}_{\mathcal{N}} = \bigcup_{\mathbf{n} \in \mathcal{N}^d} \mathcal{R}_{\mathbf{n}}$$

If $\mathcal{N} = \mathbb{N}$, then instead of $\mathfrak{R}_\mathcal{N}$ it will be used also the notation $\mathfrak{R}$. It is said, that a basis $\mathfrak{R}_\mathcal{N}$ differentiates the integral of a function $f \in L^1(Q_d)$, if

$$D_\mathcal{N} f(\mathbf{x}) = \lim_{\operatorname{diam}(R)\to 0,\, \mathbf{x}\in R\in\mathfrak{R}_\mathcal{N}} \frac{1}{|R|}\int_R f(\mathbf{t})d\mathbf{t} = f(\mathbf{x}) \text{ a.e. }.$$

We note, that if $\mathfrak{R}_\mathcal{N} \subset \mathfrak{R}_{\mathcal{N}'}$ and $\mathfrak{R}_\mathcal{N}$ differentiates the integral of $f \in L^1(Q_d)$, then so we will have for the basis $\mathfrak{R}_{\mathcal{N}'}$. By $L\log^{d-1} L(Q_d)$ we denote the class of functions satisfying the bound

$$\int_{Q_d} |f|(\log^+|f|)^{d-1} < \infty.$$

The classical Jessen-Marcinkiewicz-Zygmund theorem [12] states, that the integral of any function from the class $L\log^{d-1} L(Q_d)$ is differentiable with respect to the basis of all dyadic rectangles $\mathfrak{R}$. Consider the maximal function

$$M_\mathcal{N} f(\mathbf{x}) = \sup_{R\colon \mathbf{x}\in\mathfrak{R}_\mathcal{N}} \frac{1}{|R|}\int_R |f(\mathbf{t})|d\mathbf{t}, \tag{40}$$

corresponding to the basis $\mathfrak{R}_\mathcal{N}$. If $\mathcal{N} = \mathbb{N}$, then we shall use $Mf(\mathbf{x})$ instead of $M_\mathcal{N} f(\mathbf{x})$. In the theory of differentiation of integrals it is well known that the differentiation properties of a basis $\mathfrak{R}_\mathcal{N}$ closely connected with some week type estimates of maximal function (40). In the case $\mathcal{N} = \mathbb{N}$, a most general estimate for (40) is due M. de Guzman [7], [8]. That is the week type inequality

$$m_d\{\mathbf{x}\in Q_d : Mf(\mathbf{x}) > \lambda\} \lesssim \int_{Q_d} \frac{|f(\mathbf{t})|}{\lambda}\left(1+\log^+\frac{|f(\mathbf{t})|}{\lambda}\right)^{d-1} d\mathbf{t}, \quad \lambda > 0,$$

In fact, $L\log^{d-1} L$ is the widest Orlicz class of functions, which integrals are differentiable with respect to the basis $\mathfrak{R}$, and it is proved by Saks's in [17] and [18]. Analogous problem as well as some other problems concerning general basis $\mathfrak{R}_\mathcal{N}$ are considered in the papers by K. Hare, A. Stokolos [10], P. Hagelstein [9] and A. Stokolos [19]. Notice that, if $\mathcal{N} = \mathbb{N}$, then $\mathfrak{R}_\mathcal{N}$ coincides with $\mathfrak{R}$, but in general $\mathfrak{R}_\mathcal{N}$ is strongly included in $\mathfrak{R}$ and so we have

$$M_\mathcal{N} f(\mathbf{x}) \le Mf(\mathbf{x}).$$

In spite of this, the basis $\mathfrak{R}_\mathcal{N}$ doesn't have better differentiation property than $\mathfrak{R}$, which is proved by A. Stokolos in [19]. Moreover, he proved, that for any sequence $\mathcal{N}$ and a number $0<\lambda<1$ there exists a measurable set $E\subset Q_d$, such that

$$
\begin{aligned}
(41)\quad & m_d\{\mathbf{x}\in Q_d:\ M_{\mathcal{N}}\mathbb{I}_E(\mathbf{x})>\lambda\}\\
&\gtrsim \int_{Q_d}\frac{|\mathbb{I}_E(\mathbf{t})|}{\lambda}\left(1+\log^+\frac{|\mathbb{I}_E(\mathbf{t})|}{\lambda}\right)^{d-1}d\mathbf{t}.
\end{aligned}
$$

Using this estimate one can easily show that the class $L\log^{d-1}L$ is again the optimal Orlicz class of functions, having differentiable integrals with respect to the basis $\mathfrak{R}_{\mathcal{N}}$. The construction of the set E in [19] satisfying (41) is based on a modified Bohr lader construction (see [8], p. 89), which is used in the Saks theorem too. These problems for differentiation of integrals can be discussed in view of martingale theory. Indeed, consider the sequences of σ-algebras $\mathcal{A}_n^{(k)}$, $k=1,2,\ldots d$, in Q_d generated by the dyadic d-dimensional rectangles

$$
\left\{\mathbf{x}=(x_1,\ldots,x_d)\in Q_d:\ \frac{i-1}{2^n}\le x_k<\frac{i}{2^n}\right\},\quad i=1,2,\ldots,2^n.
$$

It is easy to observe that σ-algebra $\mathcal{A}_{\mathbf{n}}$, defined by (1), coincides with the σ-algebra generated by the rectangles $\mathcal{R}_{\mathbf{n}}$. Moreover, we have

$$
E^{\mathcal{A}_{\mathbf{n}}}f(\mathbf{x})=\frac{1}{|R(\mathbf{x})|}\int_{R(\mathbf{x})}f(\mathbf{t})d\mathbf{t},
$$

where $R(\mathbf{x})$ denotes the dyadic rectangle of the form $\mathcal{R}_{\mathbf{n}}$, containing the point $\mathbf{x}$. So taking $\mathcal{A}_{\mathbf{n}}=\mathcal{S}(\mathcal{R}_{\mathbf{n}})$ in Theorem 2 we will get some equivalencies between the basis $\mathfrak{R}_{\mathcal{N}}$ and $\mathfrak{R}$ and the Stokolos theorem follows from Theorem 4 bellow. So from Theorem 2 we obtain

Theorem 3. *For any $\mathcal{N}\subset\mathbb{N}$ and $f\in L^1(Q_d)$ there exists a function $g\sim f$ such that*

$$
m_d\{\mathbf{x}\in Q_d:\ D_{\mathcal{N}}g(\mathbf{x})=g(\mathbf{x})\}=m_d\{\mathbf{x}\in Q_d:\ Df(\mathbf{x})=f(\mathbf{x})\}.
$$

Theorem 4. *For any $\mathcal{N}\subset\mathbb{N}$ and $f\in L^1(Q_d)$ there exists a function $g\sim f$ such that*

$$
m_d\{\mathbf{x}\in Q_d:\ M_{\mathcal{N}}g(x)>\lambda\}=m_d\{\mathbf{x}\in Q_d:\ Mf(x)>\lambda\}.
$$

It is well known that $E^{\mathcal{A}_{\mathbf{n}}}f(\mathbf{x})$ almost everywhere coincides with the $\mathbf{n}$-th rectangular partial sum of Fourier-Haar series of function $f(\mathbf{x})$. So these theorems can be discussed in view of Haar series too.

5. Some martingale equivalency theorems

In this section we consider some applications of Theorem 1 in some martingale convergence problems. We will prove some theorems, which show that in the problems of a.s. convergence of generale martingales one can consider simply Haar martingales.

Theorem 5. *Let* $\{\mathcal{A}_{\mathbf{n}},\, \mathbf{n}\in\mathbb{N}^d\}$ *be arbitrary sequence and* $\{\mathcal{B}_{\mathbf{n}},\, \mathbf{n}\in\mathbb{N}^d\}$ *be regular sequence of multiple* σ*-algebras in* Q_d*. Then for any* $f\in L^1(Q_d)$ *there exists a function* $g(x)\sim f(x)$*, such that*

$$m_d\{\mathbf{E}^{\mathcal{A}_n}f(\mathbf{x}) \text{ converges }\}\ge m_d\{\mathbf{E}^{\mathcal{B}_n}g(\mathbf{x}) \text{ converges }\}.$$

Let us do a remark about this theorem before starting the proof. Let $\{\mathcal{A}_n,\, n\in\mathbb{N}\}$ is an increasing sequence of σ-algebras in $\mathcal{F}$. Doob in [4] proved that the sequence

$$\mathbf{E}^{\mathcal{A}_n}f(x),\quad n=1,2,\dots,$$

converges a.s. for any $f\in L^1([0,1))$. A generalization of Doob's theorem for multiple sequences of σ-algebras $\{\mathcal{A}_{\mathbf{n}},\, \mathbf{n}\in\mathbb{N}^d\}$ obtained by R. Cairoli in [6]. He proves that the multiple martingale

$$\mathbf{E}^{\mathcal{A}_{\mathbf{n}}}f(\mathbf{x}),\quad \mathbf{n}\in\mathbb{N}^d,$$

converge a.s. for any function $f\in L\log^{d-1}L(Q_d)$, that is

$$\int_{Q_d}|f|(\log^+|f|)^{d-1}<\infty.$$

The simplest martingales are the Haar martingales, which correspond to the σ-algebras $\mathfrak{R}_{\mathbf{n}}$ of d-dimensional dyadic rectangles (or dyadic intervals in the one-dimensional case) defined in (39). Doob's and Cairoli's theorems for Haar martingales are a consequences of Lebesgue and Jessen-Marcinkewicz-Zygmund differentiation theorems correspondingly. So, using Theorem 5, Doob's and Cairoli's theorems, can be also deduced from Jessen-Marcinkewicz-Zygmund differentiation theorem.

To prove Theorem 5 we need the following

Lemma 8. *If* $\{\mathcal{A}_{\mathbf{n}},\, \mathbf{n}\in\mathbb{N}^d\}$ *is an arbitrary sequence of multiple* σ*-algebras and* $f\in L^1(Q_d)$*, then there exists a regular sequence* $\{\mathcal{B}_{\mathbf{n}},\, \mathbf{n}\in\mathbb{N}^d\}$ *such that*

$$m_d\{\mathbf{E}^{\mathcal{A}_{\mathbf{n}}}f(\mathbf{x}) \text{ converges }\}\ge m_d\{\mathbf{E}^{\mathcal{B}_{\mathbf{n}}}f(\mathbf{x}) \text{ converges }\}.$$

Proof. Denote

$$G=\{\mathbf{E}^{\mathcal{A}_{\mathbf{n}}}f(\mathbf{x}) \text{ diverges }\},$$

$$G(\delta)=\left\{\limsup_{\mathbf{n}\to\infty}\mathbf{E}^{\mathcal{A}_{\mathbf{n}}}f(\mathbf{x})-\liminf_{\mathbf{n}\to\infty}\mathbf{E}^{\mathcal{A}_{\mathbf{n}}}f(\mathbf{x})>\delta\right\}.$$

We have

$$G = G(0) = \cup_{\delta>0} G(\delta).$$

Thus for any $\varepsilon > 0$ there exists a number $\delta > 0$, such that

$$m_d\left(G(\delta)\right) > m_d(G) - \varepsilon.$$

Consider a sequence of multiindexes

$$\mathbf{n^{(k)}} = (n_1^{(k)}, \dots, n_d^{(k)}), \quad k = 1, 2, \dots, \tag{42}$$

satisfying the relation $\mathbf{n^{(k)}} \le \mathbf{n^{(k+1)}}$. Consider the sets

$$G^{(k)}(\delta) = \left\{ \sup_{\mathbf{n^{(k)}} \le \mathbf{n} \le \mathbf{n^{(k+1)}}} \mathbf{E}^{\mathcal{A}_\mathbf{n}} f(\mathbf{x}) - \inf_{\mathbf{n^{(k)}} \le \mathbf{n} \le \mathbf{n^{(k+1)}}} \mathbf{E}^{\mathcal{A}_\mathbf{n}} f(\mathbf{x}) > \delta \right\}.$$

Using the definition of $G(\delta)$, it is easy to observe, that for a suitable sequence (42) we can ensure

$$m_d\left(G^{(k)}(\delta)\right) > m_d\left(G(\delta)\right) - \frac{\varepsilon}{2^k}, \quad k = 1, 2, \dots.$$

Then we can find a regular sequence of multiple σ-algebras $\{\mathcal{B}_\mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$, such that

$$m_d\left\{ \sup_{\mathbf{n^{(k)}} \le \mathbf{n} \le \mathbf{n^{(k+1)}}} |\mathbf{E}^{\mathcal{A}_\mathbf{n}} f(\mathbf{x}) - \mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x})| < \frac{\delta}{3} \right\} > 1 - \frac{\varepsilon}{2^k}, \quad k = 1, 2, \dots.$$

Thus we get

$$\begin{aligned} & m_d\left\{ \sup_{\mathbf{n^{(k)}} \le \mathbf{n} \le \underline{n}^{(k+1)}} \mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x}) - \inf_{\mathbf{n^{(k)}} \le \mathbf{n} \le \mathbf{n^{(k+1)}}} \mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x}) > \frac{\delta}{3} \right\} \\ & \ge m_d G^{(k)}(\delta) \cap \left\{ \sup_{\mathbf{n^{(k)}} \le \mathbf{n} \le \mathbf{n^{(k+1)}}} |\mathbf{E}^{\mathcal{A}_\mathbf{n}} f(\mathbf{x}) - \mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x})| < \frac{\delta}{3} \right\} \\ & \ge m_d\left(G(\delta)\right) - \frac{2\varepsilon}{2^k} \ge |G| - \frac{\varepsilon}{2^{k-1}} - \varepsilon \end{aligned}$$

and therefore

$$\begin{aligned} & m_d\{\mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x}) \text{ diverges }\} \\ & \qquad \ge m_d\left(\limsup_{k\to\infty} \left\{ \sup_{\mathbf{n^{(k)}} \le \mathbf{n} \le \mathbf{n^{(k+1)}}} \mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x}) \right.\right. \\ & \qquad\qquad \left.\left. - \inf_{\mathbf{n^{(k)}} \le \mathbf{n} \le \mathbf{n^{(k+1)}}} \mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x}) > \frac{\delta}{3} \right\} \right) \\ & \qquad \ge m_d(G) - \varepsilon. \end{aligned}$$

Since ε is arbitrary we get

$$m_d\{\mathbf{E}^{\mathcal{B}_\mathbf{n}} f(\mathbf{x}) \text{ diverges }\} \geq m_d\{\mathbf{E}^{\mathcal{A}_\mathbf{n}} f(\mathbf{x}) \text{ diverges }\},$$

which completes the proof of lemma. □

Proof of Theorem 5. Using Lemma 8, in Theorem 5 we may suppose, that $\{\mathcal{A}_n\}$ is regular too. Thus, applying Theorem 1, we define an MP-transformation Θ corresponding to $\varepsilon_i = i^{-1}$ with the properties (6)-(8). We take $g(\mathbf{x}) = \Theta f(\mathbf{x})$. From (6) and (7) we obtain that for the set

$$G = \bigcup_{\mathbf{n}\in\mathbb{N}^d} G_\mathbf{n}$$

we have $m_d(G) = 1$. According to (8), for any $\mathbf{x} \in G$, we can find an index $\mathbf{n}(\mathbf{x})$, such that

$$\left(\Theta \circ \mathbf{E}^{\mathcal{A}_\mathbf{n}}\right) f(\mathbf{x}) = \left(\mathbf{E}^{\tilde{\mathcal{B}}_\mathbf{n}} \circ \Theta\right) f(\mathbf{x}), \text{ if } \mathbf{n} \geq \mathbf{n}(\mathbf{x}).$$

This means that the sequences

$$\left(\Theta \circ \mathbf{E}^{\mathcal{A}_\mathbf{n}}\right) f(\mathbf{x}) \text{ and } \left(\mathbf{E}^{\tilde{\mathcal{B}}_\mathbf{n}} \circ \Theta\right) f(\mathbf{x}) = \mathbf{E}^{\tilde{\mathcal{B}}_\mathbf{n}} g(\mathbf{x})$$

converge simultaneously. Therefore, since $m_d(G) = 1$ and $\tilde{\mathcal{B}}_\mathbf{n}$ is a subsequence of $\mathcal{B}_\mathbf{n}$, we get

$$\begin{aligned} m_d\{\mathbf{E}^{\mathcal{B}_n} g(\mathbf{x}) \text{ converges }\} & \\ &\leq m_d\{\mathbf{E}^{\tilde{\mathcal{B}}_n} g(\mathbf{x}) \text{ converges }\} \\ &= m_d\{\left(\Theta \circ \mathbf{E}^{\mathcal{A}_\mathbf{n}}\right) f(\mathbf{x}) \text{ converges }\} \\ &= m_d\{\mathbf{E}^{\mathcal{A}_n} f(\mathbf{x}) \text{ converges }\}. \end{aligned}$$

□

For a given multiple sequence of σ-algebras $\mathcal{A} = \{\mathcal{A}_\mathbf{n} : \mathbf{n} \in \mathbb{N}^d\}$ we consider the maximal function

$$\mathcal{M}_\mathcal{A} f(x) = \sup_{\mathbf{n}\in\mathbb{N}^d} E^{\mathcal{A}_\mathbf{n}} |f(x)|$$

These maximal functions play significant role in the martingale convergence theorems. We prove

Theorem 6. *For any two different families $\mathcal{A} = \{\mathcal{A}_\mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$ and $\mathcal{B} = \{\mathcal{B}_\mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$ of regular sequences of multiple σ-algebras and for any function $f \in L^1(X)$ we have*

$$\sup_{g:\, g\sim f} m_d\{x \in X : \mathcal{M}_\mathcal{A} g(x) > \lambda\} = \sup_{g:\, g\sim f} m_d\{x \in X : \mathcal{M}_\mathcal{B} g(x) > \lambda\}.$$

Proof. Using Theorem 1, we define an MP-transformation Θ corresponding to $\varepsilon_i = \varepsilon/i$ with the properties (6)-(8). Let $f(\mathbf{x}) \in L^1(Q_d)$ and $g(\mathbf{x}) = \Theta f(\mathbf{x})$. Such that

$$G_{\mathbf{n}} \supset G_{1,1,\ldots,1}, \quad \mathbf{n} \in \mathbb{N}^d$$

using (8), we get

$$\mathcal{M}_{\mathcal{A}} f(\mathbf{x}) = \mathcal{M}_{\tilde{\mathcal{B}}} g(\mathbf{x}) \le \mathcal{M}_{\mathcal{B}} g(\mathbf{x}), \quad \mathbf{x} \in G_{1,1,\ldots,1}. \tag{43}$$

Therefore, since $P(G_{1,1,\ldots,1}) > 1 - \varepsilon$, we obtain

$$\begin{aligned} m_d\{\mathbf{x} \in Q_d : \mathcal{M}_{\mathcal{A}} f(\mathbf{x}) > \lambda\} \\ &\le m_d\{\mathbf{x} \in Q_d : \mathcal{M}_{\mathcal{B}} g(\mathbf{x}) > \lambda\} + 1 - m_d(G_{1,1,\ldots,1}) \\ &\le \sup_{g:\, g\sim f} m_d\{\mathbf{x} \in Q_d : \mathcal{M}_{\mathcal{B}} g(x) > \lambda\} + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this implies

$$\sup_{g:\, g\sim f} m_d\{\, \mathcal{M}_{\mathcal{A}} g(\mathbf{x}) > \lambda\} \le \sup_{g:\, g\sim f} m_d\{\, \mathcal{M}_{\mathcal{B}} g(x) > \lambda\}$$

Similarly we can prove the converse inequality and the theorem is proved. □

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Denote by $L^\Phi(Q_d)$ the class of functions f on Q_d with $\Phi(|f|) \in L^1(Q_d)$. If Φ satisfies Δ_2-condition $\Phi(2x) \le k\Phi(x)$ then L^Φ is Banach space with the norm $\|f\|_{L^\Phi} = \|f\|_\Phi$ to be the least $c > 0$ for which the inequality

$$\int_{\mathbb{T}} \Phi\left(\frac{|f|}{c}\right) \le 1 \tag{44}$$

holds.

Theorem 7. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Then for any two families $\mathcal{A} = \{\mathcal{A}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ and $\mathcal{B} = \{\mathcal{B}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ of regular sequences of multiple σ-algebras we have*

$$\sup_{g:\, g\sim f} \|\mathcal{M}_{\mathcal{A}} g(x)\|_\Phi = \sup_{g:\, g\sim f} \|\mathcal{M}_{\mathcal{B}} g(x)\|_\Phi.$$

Proof. From the inequality (43), obtained in the proof of Theorem 6, we have

$$\mathcal{M}_{\mathcal{A}} f(\mathbf{x}) \le \mathcal{M}_{\mathcal{B}} g(\mathbf{x}), \quad \mathbf{x} \in G = G_{1,1,\ldots,1}.$$

Therefore

$$\|\mathcal{M}_{\mathcal{A}} f(\mathbf{x})\|_\Phi \le \sup_{g:\, g\sim f} \|\mathcal{M}_{\mathcal{B}} g(\mathbf{x})\|_\Phi + \|\mathcal{M}_{\mathcal{B}} f(\mathbf{x}) \cdot \mathbb{I}_G(\mathbf{x})\|_\Phi. \tag{45}$$

Such that the norm $\|\cdot\|_\Phi$ is absolute continuous the last term in (45) can be sufficiently small. Hence we get

$$\sup_{g:\, g\sim f} \|\mathcal{M}_{\mathcal{A}} g(\mathbf{x})\|_\Phi \le \sup_{g:\, g\sim f} \|\mathcal{M}_{\mathcal{B}} g(\mathbf{x})\|_\Phi$$

and so the converse inequality. This completes the proof of Theorem 7. □

Theorem 8. *Let $\{\mathcal{A}_{\mathbf{n}}, \mathbf{n}\in\mathbb{N}^d\}$ be an arbitrary sequence of multiple σ-algebras and $f\in L^1(Q_d)$. Then there exists a subsequence $\{\mathcal{B}_{\mathbf{n}}, \mathbf{n}\in\mathbb{N}^d\}$ of $\mathcal{A}_{\mathbf{n}}$, such that $\mathbf{E}^{\mathcal{B}_n} f(\mathbf{x})$ converges a.s.*

We say the multiple sequence of functions $f_{\mathbf{n}}(x)$, $x\in Q_d$, $\mathbf{n}\in\mathbb{N}^{\mathbf{d}}$, strongly converges to $f(x)$ in measure, if

$$\lim_{\min_{i\in A_1} n_i\to\infty}\ \lim_{\min_{i\in A_2} n_i\to\infty}\cdots \lim_{\min_{i\in A_k} n_i\to\infty} f_{\mathbf{n}}(x) = f(x) \tag{46}$$

for any partition $A_1, A_2, \ldots, A_k$ of the set $\{1,2,\ldots,d\}$, where the iterated limits are considered in the sense of measure. It is well known that for any $f\in L^1(Q_d)$ and for an arbitrary sequence $\{\mathcal{A}_{\mathbf{n}}, \mathbf{n}\in\mathbb{N}^d\}$ the multiple sequence $\mathbf{E}^{\mathcal{A}_n} f(\mathbf{x})$ strongly converges in measure. So the theorem immediately follows from the following

Lemma 9. *If the multiple sequence of functions $f_{\mathbf{n}}(x)$, $x\in Q_d$, $\mathbf{n}\in\mathbb{N}^{\mathbf{d}}$, strongly converges to $f(x)$ in measure, then there exists a set $\mathcal{N}\subset\mathbb{N}$ such that the subsequence $f_{\mathbf{n}}(x)$, $\mathbf{n}\in\mathcal{N}^d$, converges almost everywhere to $f(x)$.*

Proof. To avoid huge notations and for better understanding we prefer to prove the lemma in the case of $d=2$. So we consider a sequence $f_{n_1,n_2}(x)$ which strongly converges in measure to $f(x)$. According to (46) we have

$$\begin{aligned}
&\lim_{n_1\to\infty} f_{n_1,n_2}(x) = f_{\infty,n_2}(x),\\
&\lim_{n_2\to\infty} f_{n_1,n_2}(x) = f_{n_1,\infty}(x),\\
&\lim_{n_2\to\infty} f_{\infty,n_2}(x) = f(x),\\
&\lim_{n_1\to\infty} f_{n_1,\infty}(x) = f(x),\\
&\lim_{n_1,n_2\to\infty} f_{n_1,n_2}(x) = f(x).
\end{aligned}\tag{47}$$

where all the limits are considered in measure sense. We suppose that the sequence $\mathcal{N}=\{l_1, l_2,\ldots,l_k,\ldots\}$ have been already chosen. Then we denote

$$
\begin{aligned}
G_k = &\\
&\{|f_{l_k,l_i}(x) - f_{\infty,l_i}(x)| < 2^{-k},\ i = 1,2,\ldots,k-1\}\\
&\cap \{|f_{l_i,l_k}(x) - f_{l_i,\infty}(x)| < 2^{-k},\ i = 1,2,\ldots,k-1\}\\
&\cap \{|f_{l_k,\infty}(x) - f(x)| < 2^{-k}\}\\
&\cap \{|f_{\infty,l_k}(x) - f(x)| < 2^{-k}\}\\
&\cap \{|f_{l_k,l_k}(x) - f(x)| < 2^{-k}\}.
\end{aligned}
$$

Using (47), we can chose the sequence $\mathcal{N}$, such that

$$m_d(G_k) > 1 - \frac{1}{2^k}.$$

Putting

$$G = \cup_{n\in\mathbb{N}} \cap_{k\ge n} G_k,$$

we will have $m_d(G) = 1$. Hence, it is enough to prove the convergence of $f_{l_n,l_m}(x)$ for any $x \in G$, as $n, m \to \infty$. If $x \in G$, then

$$x \in G_k, \quad k > k(x).$$

Chose arbitrary integers $n, m > k(x)$. If $n = m$, then, according to the definition G_k, we have

$$|f_{l_n,l_m}(x) - f(x)| = |f_{l_n,l_n}(x) - f(x)| < 2^{-n}.$$

If $n < m$, then we get

$$
\begin{aligned}
|f_{l_n,l_m}(x) - f_{\infty,l_m}(x)| < 2^{-n},\\
|f_{\infty,l_m}(x) - f(x)| < 2^{-n}
\end{aligned}
$$

and therefore we obtain

$$|f_{l_n,l_m}(x) - f(x)| < 2 \cdot 2^{-n}.$$

The lemma is proved. □

6. Riemann sums

In this section we display an application of martingale theorems in Riemann sums, defined by

$$R_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right), \quad x \in \mathbb{T}, \tag{48}$$

where $f(x)$ is an integrable function on the torus $\mathbb{T} = [0,1] = \mathbb{R}/\mathbb{Z}$. It will be an important complement to the paper [13] by author. It is not hard to see, that if f is continuous then these sums converge to the integral of f uniformly and they converge in $L^1(\mathbb{T})$ while f

is Lebesgue integrable. We consider almost everywhere convergence problems of Riemann sums or subsequences of (48). B. Jessen in [11] proved a.e. convergence of subsequence $R_{2^k}f(x)$ for any $f \in L^1(\mathbb{T})$. W. Rudin [16] constructed an example of a function $L^\infty(\mathbb{T})$, such that $R_n f(x)$ everywhere diverges. These are two fundamental theorems in the theory of Riemann sums. Almost everywhere convergence problem of $R_{l_k}f(x)$ for a given sequence of integers $D = \{1 \le 1_1 < l_2 < \ldots\}$ is investigated by many other authors([3], [5], [1], [2], [14], [15]). As it is shown by Rudin [16] convergence properties of $R_{l_k}f(x)$ strongly depend on arithmetic properties of D. L. E. Dubins and J. Pitman in [5] proved, if

$$D = \{n \in \mathbb{N} : \ n = p_1^{k_1} p_2^{k_2} \ldots p_d^{k_d}, \quad k_1, k_2, \ldots, k_d \in \mathbb{N}\} \tag{49}$$

where $p_1, p_2, \ldots, p_d$, are some fixed primes, then for any $f \in L\log^{d-1} L$ the subsequence $R_{l_k}f(x)$ corresponding to (49) converges a.e.. Then Y. Bugeaud and M. Weber in [3] proved that the class $L\log^{d-1} L$ is nearly sharp.

Theorem A (Bugeaud, Weber)**.** *If the sequence $D = \{l_k, \ k = 1, 2, \ldots\}$ is defined by (49) and $0 < \varepsilon < 1$, then there exists a function $f \in L\log^{d-1-\varepsilon} L(\mathbb{T})$ such that $R_{l_k}f(x)$ is almost everywhere divergent.*

The proof of this theorem is based on the method of R. C. Baker [1], where author has proved a weaker version of this theorem. A final correction in the last theorem is made by author in [13].

Theorem B (Karagulyan)**.** *Let l_k be a sequence in (49) and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function satisfying the condition*

$$\lim_{x\to\infty} \frac{\phi(x)}{x \ln^{d-1} x} = 0.$$

Then there exists a function $f(x) \in L^\phi$, that is

$$\int_0^1 \phi\left(|f(x)|\right) dx < \infty,$$

such that the sequence $R_{l_k}f(x)$ is everywhere divergent.

We prove this theorem, establishing a direct connection between Riemann maximal functions and ordinary maximal functions in $\mathbb{R}^d$. We associate with the set of integers (49) the family of d-dimensional rectangles

$$\left\{x \in Q_d : \ \frac{t_i - 1}{p_i^{s_i}} \le x_k < \frac{t_i}{p_i^{s_i}}, \ k = 1, 2, \ldots, d\right\},$$
$$0 \le t_i < p_i^{s_i}, \quad s_i = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots, d$$

and denote it by $\mathfrak{R}_D$. Consider two maximal functions

$$\mathcal{R}_D g(x) = \sup_{n\in D} R_n|g(x)|, \quad x \in \mathbb{T},$$

and

$$M_D f(\mathbf{x}) = \sup_{R:\, \mathbf{x}\in\mathfrak{R}_D} \frac{1}{|R|}\int_R |f(\mathbf{t})|d\mathbf{t}, \quad \mathbf{x} \in Q_d.$$

In [13] we proved, that

$$(50) \quad \sup_{\|g\|_\Phi\le 1} m_d\Big\{x\in\mathbb{T} :\ \mathcal{R}_D g(x) > \lambda\Big\} = \sup_{\|f\|_\Phi\le 1} m_d\Big\{x\in Q_d :\ M_D f(x) > \lambda\Big\},$$

for any $\lambda > 0$, where $\|\cdot\|_\Phi$ is Orlicz norm defined in (44). Using Theorem 6 we have

$$(51) \quad \sup_{f\sim g} m_d\Big\{x\in Q_d :\ M_D f(x) > \lambda\Big\} = \sup_{f\sim g} m_d\Big\{x\in Q_d :\ Mf(x) > \lambda\Big\}.$$

Since the norm $\|\cdot\|_\Phi$ is rearrangement invariant, using (50) and (51), we obtain

Theorem 9. *If D is the set of indexes from (49), then*

$$\sup_{\|g\|_\Phi\le 1} m_d\Big\{x\in\mathbb{T} :\ \mathcal{R}_D g(x) > \lambda\Big\} = \sup_{\|f\|_\Phi\le 1} m_d\Big\{x\in Q_d :\ Mf(x) > \lambda\Big\},$$

where $Mf(x)$ is the maximal function with respect to all dyadic rectangles defined in Section 5.

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