Stochastic and Analytic Methods in Mathematical Physics

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Martingale method: applications in the theory of limit theorems and mathematical statistical physics

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The main goals of the talk are:

- to present a new type of limit theorems for random fields with weakly dependent components (particularly, for Gibbs random fields);

- to demonstrate the capabilities of the martingale method for studying some models of the mathematical statistical physics (ground states, critical point).

Martingale–difference random fields

A collection of random variables $(\xi_t) = (\xi_t, t \in \mathbb{Z}^d)$, each of which takes value in X will call a random field defined on $\mathbb{Z}^d$ with phase space X, $X \subset \mathbb{R}$ (in this talk we consider only the case of finite phase space, i.e. $1 < |X| < \infty$).
The distribution P of the random field (ξ_t) is the probability measure on $(X^{\mathbb{Z}^d}, \mathscr{B}^{\mathbb{Z}^d})$ ($\mathscr{B}^{\mathbb{Z}^d}$ is a sigma-algebra generated by the set of cylindric subsets of $X^{\mathbb{Z}^d}$) such that $\mathrm{P}(B) = P(\{\xi_t, t \in \mathbb{Z}^d\} \in B), B \in \mathscr{B}^{\mathbb{Z}^d}$.

A random field (ξ_t) is called a *martingale–difference random field* if $E|\xi_t| < \infty$ for any $t \in \mathbb{Z}^d$ and

$$E(\xi_t/\xi_s, s \in \mathbb{Z}^d \backslash \{t\}) = 0 \text{ (a.s.) for any } t \in \mathbb{Z}^d.$$

Example. Positive random field with symmetric (with respect to zero) phase space and even finite-dimensional probability distribution, i.e. for all $V \in W$ and $x \in X^V$

$$\mathrm{P}_V(\theta_t x_t, t \in V) = \mathrm{P}_V(x_t, t \in V) \qquad \text{for any } \theta_t \in \{1, -1\}.$$

Gibbs random fields

A random field P is called a Gibbs random field if it is positive and for any $x \in X$ and any $t \in \mathbb{Z}^d$ there exist strictly positive uniform (with respect to $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$) limits

$$q_t^{\bar{x}}(x) = \lim_{V \uparrow \mathbb{Z}^d \setminus \{t\}} \frac{\mathsf{P}_{\{t\} \cup V}(x \bar{x}_V)}{\mathsf{P}_V(\bar{x}_V)}.$$

The set $\mathcal{Q}^{(1)} = \left\{ q_t^{\bar{x}}, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}, t \in \mathbb{Z}^d \right\}$ is called a *canonical 1–specification*.

Representation of Gibbs random fields

Theorem 1. *Let* P *be a Gibbs random field. Then corresponding canonical 1–specification* $\mathcal{Q}^{(1)}$ *admits the Gibbs representation with the aid of uniformly convergent potential*

$$\Phi = \{\Phi_V(x), x \in X^V, V \in W\},$$

that is

$$q_t^{\bar{x}}(x) = \frac{\exp\{H_t^{\bar{x}}(x)\}}{\sum\limits_{z \in X} \exp\{H_t^{\bar{x}}(z)\}}, \qquad x \in X,$$

where

$$H_t^{\bar{x}}(x) = \sum_{J \subset W(\mathbb{Z}^d \setminus \{t\})} \Phi_{\{t\} \cup J}(x\bar{x}_J), \qquad x \in X.$$

Martingale–difference Gibbs random fields

Let P be a Gibbs random field with phase space X and let $\Pi = \{X_1, X_2, ..., X_n\}$ be a partition of X such that for any $k = \overline{1, n}$

$$\sum_{x \in X_k} x = 0.$$

• If canonical 1–specification $\mathcal{Q}^{(1)}$ of the Gibbs random field P is such that for any $t \in \mathbb{Z}^d$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$, and $k = \overline{1, n}$

$$q_t^{\bar{x}}(x) = q_t^{\bar{x}}(x'), \qquad x, x' \in X_k,$$

then P is a martingale–difference Gibbs random fields.

• If potential Φ corresponding to the Gibbs random field P takes constant values on the elements of a partition Π of phase space X, i.e. for any $t \in \mathbb{Z}^d$, $V \in W(\mathbb{Z}^d \setminus \{t\})$, $\bar{x} \in X^V$ and $k = \overline{1, n}$

$$\Phi_{\{t\} \cup V}(x_t \bar{x}_V) = \Phi_{\{t\} \cup V}(x'_t \bar{x}_V), \qquad x, x' \in X_k,$$

then P is a martingale–difference Gibbs random fields.

1. Martingale method
in the theory of limit theorems

Classical limit theorems and their refinements

1. The central limit theorem (CLT);
2. Rate of convergence in the CLT;
3. The law of the iterated logarithm;
4. Asymptotical behavior of moments;
5. The local limit theorem (LLT).

Valid for random fields

1. with independent components;
2. with weak dependent components and suitable decrease of correlations between components;
3. Gibbs r.f. under suitable conditions on corresponding canonical 1–specification or potential.

Let (ξ_t) be a random field. Denote

$$S_{V_n} = \sum_{t \in V_n} \xi_t, \qquad \Im_S = \sigma(\xi_t, t \in S),$$
$$V_n = [-n, n]^d,\ n = 1, 2, ..., \quad S \subset \mathbb{Z}^d.$$

Weak dependence conditions

A homogenous random field (ξ_t) is called *ergodic* if for any $I, \Lambda \in W$

$$\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{a\in V_n} P\Big(\{\xi_t = x_t, t \in I\} \cap \{\xi_{s+a} = \bar{x}_s, s \in \Lambda\}\Big) =$$

$$= P(\xi_t = x_t, t \in I) P(\xi_s = \bar{x}_s, s \in \Lambda),$$

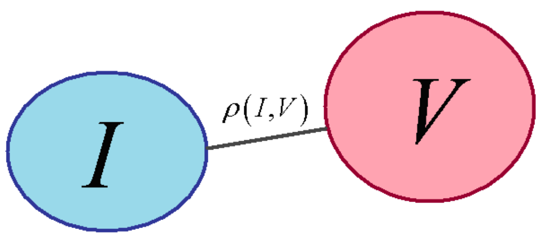
where $x \in X^I$, $\bar{x} \in X^\Lambda$.

1. The weakest condition of weak dependence;
2. If Gibbs random field is unique, then it is ergodic;
3. In the general case it is not enough to obtain classical limit theorems.

A homogenous random field (ξ_t) satisfies *the uniform strong mixing condition with coefficient* φ_I if for any fixed $I \in W$

$$\sup_{A \in \Im_I, B \in \Im_V, P(B)>0} \{|P(A/B) - P(A)|\} \leq \varphi_I(\rho(I,V)),$$

where function $\varphi_I(\rho)$, $\rho \in \mathbb{R}$ is such that $\varphi_I(\rho) \to 0$ as $\rho \to \infty$ and the set I is fixed ($\rho(I,V)$ stands for distance between sets I and V).



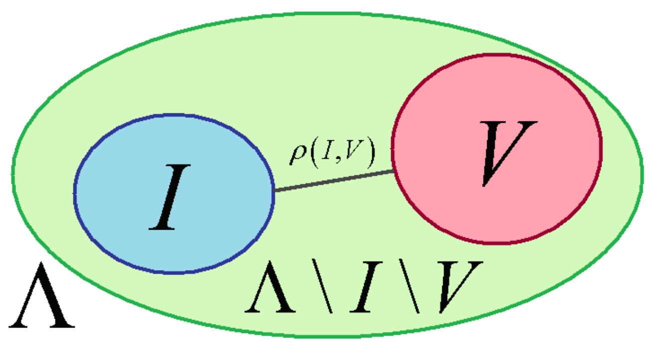
1. Gibbs random fields satisfy this condition;
2. The CLT and logarithmic rate of convergence in it; law of the iterated logarithm;
3. In general case it is not enough to obtain the LLT.

A homogenous random field (ξ_t) is *conditionally independent* with coefficient β_I if for any $I, V, \Lambda \in W$ such that $I \cap V = \varnothing$ and $I, V \subset \Lambda$, and any random variables η_1, η_2 which are $\Im_I$- and $\Im_V$-measurable correspondingly the following relation holds

$$\left| E\left(\eta_1 \cdot \eta_2 / \Im_{\Lambda \setminus \{I \cup V\}}\right) - E\left(\eta_1 / \Im_{\Lambda \setminus \{I \cup V\}}\right) \cdot E\left(\eta_2 / \Im_{\Lambda \setminus \{I \cup V\}}\right) \right| \leq$$

$$\leq \beta_I(\rho(I, V)),$$

where $\beta_I(\rho) \to 0$ as $\rho \to \infty$ (and, hence, $\Lambda \uparrow \mathbb{Z}^d$) and I is fixed.



1. Gibbs random fields with finite range potentials;
2. The LLT.

The Central Limit Theorem

Theorem 2 (N., 1995). *Let* (ξ_t) *be a homogeneous ergodic (Gibbs) martingale–difference random field such that* $0 < E\xi_0^2 < \infty$. *Then*

$$\lim_{n\to\infty} P\left(\frac{S_{V_n}}{\sqrt{DS_{V_n}}} < x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du, \qquad x \in \mathbb{R}.$$

Rate of convergence in the CLT. Law of the iterated logarithm

Theorem 3 (Kh., N., 2013). *Let* (ξ_t) *be a homogenous (Gibbs) martingale–difference random field with phase space* X*, satisfying the uniform strong mixing condition with coefficient* φ_I *such that*

$$\varphi_I(j) \le |I| \cdot \varphi(j) \qquad \text{and} \qquad \sum_{j=1}^{\infty} j^{d-1}\varphi(j) < \infty,$$

and let $E\xi_0^2 > 0$*. Then*

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S_{V_n}}{\sqrt{DS_{V_n}}} < x\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \right| \le C \cdot n^{-d/8},$$

where positive constant C *does not depend on* n*.*

Theorem 4 (Kh., N., 2013). *Under conditions of Theorem 3*

$$P\left(\limsup_{n\to\infty} \frac{S_{V_n}}{\sqrt{2DS_{V_n} \ln\ln |V_n|}} = 1\right) = 1.$$

Exact asymptotic for the moments of sums of components

Theorem 5 (Kh., N., 2013). *Let* (ξ_t) *be a homogenous (Gibbs) martingale–difference random field with phase space* X*. Then for any* $k = 1, 2, ...$

$$E(S_{V_n})^{2k-1} = C_{2k-1} \cdot |V_n|^{k-1},$$

where constant C_{2k-1} *does not depend on* n*.*
If, moreover, the random field (ξ_t) *satisfies the uniform strong mixing condition with coefficient* φ_I *such that*

$$\varphi_I(j) \leq |I| \cdot \varphi(j) \qquad \text{and} \qquad \sum_{j=1}^{\infty} j^{d-1}\varphi(j) < \infty,$$

then for any $k = 1, 2, ...$

$$E(S_{V_n})^{2k} = (2k-1)!!(E\xi_0^2)^k |V_n|^k + C_{2k} |V_n|^{k-1},$$

where constant C_{2k} *does not depend on* n*.*

Theorem 6 (Kh., N., 2013). *Let* (ξ_t) *be a (Gibbs) martingale–difference random field with phase space* X*, satisfying the uniform strong mixing condition with coefficient* φ_I *such that*

$$\varphi_I(j) \leq |I| \cdot \varphi(j) \qquad \text{and} \qquad \sum_{j=1}^{\infty} j^{d-1}\varphi(j) < \infty,$$

and let $E\xi_0^2 > 0$*. Then for random field* (ξ_t) *the CLT is valid and for any* $k = 1, 2, ...$

$$E\left(\frac{S_{V_n}}{\sqrt{DS_{V_n}}}\right)^k \to E\zeta^k \qquad \text{as } n \to \infty,$$

where ζ *is a random variable with standard normal distribution.*

Local limit theorem

Theorem 7 (Kh., N., 2016)**.** *Let* (ξ_t) *be a (Gibbs) martingale–difference random field with phase space* $X \subset \mathbb{Z}$*, and let there exists* $\gamma > 0$ *such that for any finite* $I \subset V \subset \mathbb{Z}^d$

$$P\left(S_I = y/\Im_{V\setminus I}\right) \geq \gamma \qquad \text{for any possible value } y \text{ of } S_I.$$

If, in addition, (ξ_t) *is a conditionally independent with coefficient* β_I *such that*

$$\beta_I(\rho) \leq |I|\beta(\rho) \qquad \text{and} \qquad \beta(\rho) = \mu(\rho)\cdot\rho^{-3d/2},$$

where $\mu(\rho) \to 0$ *arbitrarily slow as* $\rho \to \infty$*, then for the martingale–difference random field* (ξ_t) *the LLT is valid.*

Associated martingale–difference random fields

For any given random field (ξ_t) by the method of randomization one can construct a martingale–difference random field (η_t) someway associated with (ξ_t).

Advantages:

1. The class of random fields with martingale properties is extended.
2. The class of random fields for which classical limits theorems are valid is extended.
3. The connection between a given random field and associated martingale–difference can be used to study the asymptotical behavior of sums of components of the given random field by using the good properties martingale–differences.
4. Random field, associated with a Gibbs random field, is also Gibbsian.

2. Martingale method
in the mathematical statistical physics

Martingale model

$$\tilde{\Phi}_V(y) = \begin{cases} -\beta h \cdot |y_t|, & V = \{t\} \\ -\beta |y_t| \cdot |y_s|, & V = \{t, s\} \text{ and } \|t - s\| = 1 \\ 0, & \text{in other cases} \end{cases}$$

where $\|t - s\| = \sum\limits_{i=1}^{d} \left| t^{(i)} - s^{(i)} \right|, \qquad t, s \in \mathbb{Z}^d$ and

$$y_t, y_s \in Y = \{-1, 0, 1\}$$

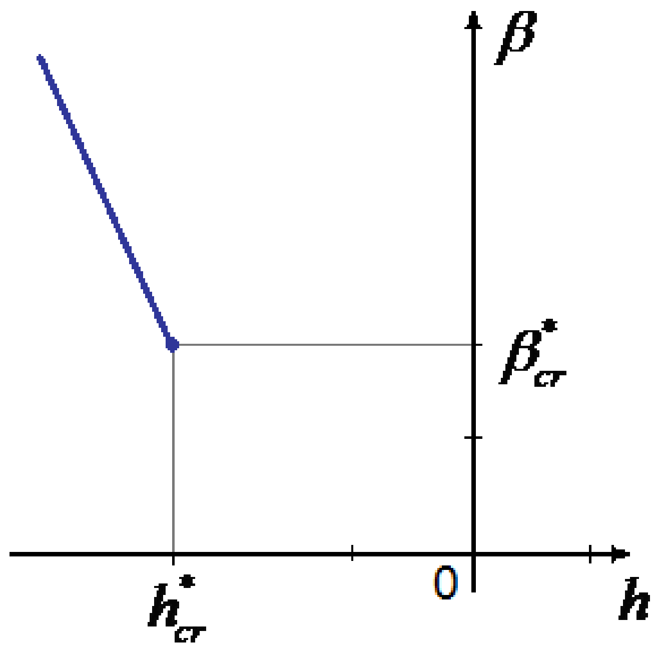
Hamiltonian has a form

$$\widetilde{H}_V^{\beta,h}(y/\bar{y}) = -\frac{\beta}{2} \sum_{\substack{t,s \in V: \\ \|t-s\|=1}} |y_t| \cdot |y_s| - \beta \sum_{\substack{t \in V, s \in \mathbb{Z}^d \setminus V: \\ \|t-s\|=1}} |y_t| \cdot |\bar{y}_s| - h \sum_{t \in V} |y_t|,$$

where $h \in \mathbb{R}$ — the external field,

$\beta > 0$ — the inverse temperature.

Phase diagram for the martingale model



Coordinates of the critical point (h^*_{cr}, β^*_{cr}) for the martingale model

$$\beta^*_{cr} = 2\ln(1+\sqrt{2}), \qquad h^*_{cr} = \ln 2 - 4\ln(1+\sqrt{2}).$$

Ground states

The martingale model has infinitely many ground states, namely all configurations which do not contain zeros

$$y^{\pm} = (|y_t| = 1, t \in \mathbb{Z}^d).$$

For configuration

$$y^0 = (y_t = 0, t \in \mathbb{Z}^d)$$

we have

$$H_t^{y^0_{\mathbb{Z}^d \setminus \{t\}}}(y_t) = 0, \qquad t \in \mathbb{Z}^d.$$

Classical limit theorems for the Gibbs random field corresponding to the martingale model

Outside the critical point

- the CLT;
- power rate of convergence in the CLT;
- the law of iterated logarithm;
- asymptotical behavior of moments of sums of components in finite volumes (after suitable normalization) coincides with the behavior of moments of standard normal distribution;
- the LLT.

At the critical point

- the CLT;
- the LLT.

On the investigation of the Ising model

The Ising ferromagnetic model is defined on $\mathbb{Z}^d$ by means of the nearest neighbor pair interaction potential

$$\Phi_V(x) = \begin{cases} -\beta h \cdot x_t, & V = \{t\} \\ -\beta \cdot x_t x_s, & V = \{t, s\} \text{ and } \|t - s\| = 1 \\ 0, & \text{in other cases} \end{cases}$$

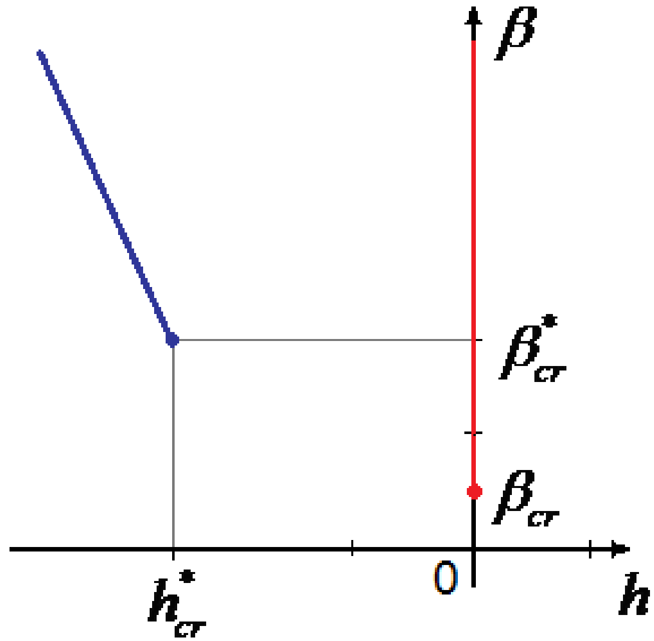
Hamiltonian has a form

$$H_V^{\bar{x},\beta,h}(x) = H_V^{\beta,h}(x/\bar{x}) = -\frac{\beta}{2} \sum_{\substack{t,s \in V: \\ \|t-s\|=1}} x_t x_s - \beta \sum_{\substack{t \in V, s \in \mathbb{Z} \setminus V: \\ \|t-s\|=1}} x_t \bar{x}_s - h \sum_{t \in V} x_t,$$

Connection with martingale model:

$$x = 2|y| - 1.$$

Phase diagram for the Ising model



$(0, \beta_{cr})$ — critical point for the Ising model

(h^*_{cr}, β^*_{cr}) — critical point for the martingale model

$$\beta^*_{cr} = 4\beta_{cr}, \qquad h^*_{cr} = -\frac{\ln 2}{\beta^*_{cr}} - d.$$

Ground states

The Ising model has two ground states

$$x^{+} = (x_t = +1, t \in \mathbb{Z}^d) \qquad \text{and} \qquad x^{-} = (x_t = -1, t \in \mathbb{Z}^d).$$

The ground state x^{+} corresponds with all (infinitely many) ground states $y^{\pm}$ of the martingale model.

The ground state x^{-} corresponds with the state y^{0} which is not the ground state of the martingale model.

Theorem 8. *The probability distribution of total spin S_V^ξ of the Ising model by means of probability distribution of total spin S_V^η of the martingale model is given by the following formula*

$$P\left(S_V^\xi = k\right) =$$

$$= 2^{\frac{k+|V|}{2}} \sum_{j=0}^{\left[\frac{n-|V|}{4}\right]} (-1)^j \frac{k+|V|+4j}{k+|V|+2j} C^j_{\frac{k+|V|}{2}+j} P\left(S_V^\eta = \frac{k+|V|}{2} + 2j\right),$$

for any $-|V| \le k \le |V|$.

The characteristic function $f_{S_V^\xi}(t)$ of the total spin S_V^ξ has the following form

$$f_{S_V^\xi}(t) = e^{-it|V|} \sum_{k=-|V|}^{|V|} \cos\left(k \arccos e^{2it}\right) P\left(S_V^\eta = k\right).$$

Classical limit theorems for the Gibbs random field corresponding to the Ising model

Outside the critical point for the Gibbs random field corresponding to the Ising model the CLT and the LLT are valid. But at the critical point the situation remains unclear.

Since for the martingale model both the CLT and the LLT hold at the critical point (h^*_{cr}, β^*_{cr}) and the connection formulas between martingale and Ising models are known, the martingale method can be applied for discovering a limit distribution of the Ising model total spin at its critical point $(0, \beta_{cr})$.

Thank you

for your attention

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