Multidimensional martingales associated with the Ising model

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Introduction

The problems relating to the multidimensional martingales are now becoming more and more attractive (see, for instance, [1–12]). In the mentioned works the large and interesting classes of such martingales are described and various limit theorems are obtained. It was also demonstrated the applicability of the martingale method in the mathematical statistical physics, particularly in the theory of Gibbs random fields.

In [3], a martingale model associated with the Ising model was constructed. It was shown that for this model the central limit theorem for the total spin (CLTS) is valid for all values of parameters including the critical one. The latter is of special interest as far as according to the generally accepted hypothesis the asymptotic normality of the total spin at a critical point should not occur.

In this paper the connection formulas between the martingale and the Ising models total spins probability distributions is established. We also found the simple form of the total spin characteristic function of the martingale model expressed in terms of the Ising model total spin distribution and vice versa. It allows to obtain the direct proof of CLTS for the martingale model and specify the way of proving CLTS for the Ising model for non critical parameters. Since CLTS for martingale model holds for all values of parameters we presume that above mentioned connection formulas give a possibility to find a limiting law for the Ising model total spin at the critical point.

1 Preliminaries

Here we present the basic concepts and notations used in this paper. Let $\mathbb{Z}^d$, $d \geq 1$ be the d-dimensional integer lattice, $W = \{V \subset \mathbb{Z}^d, |V| < \infty\}$[1] be the set of all finite subsets of $\mathbb{Z}^d$ and $X \subset \mathbb{R}$ be some set.

A collection of random variables $(\xi_t) = (\xi_t, t \in \mathbb{Z}^d)$, each of which takes value in X we will call a random field defined on $\mathbb{Z}^d$ with phase space X.

For any $S \subset \mathbb{Z}^d$, $x_t \in X$ we denote by $X^S = \{(x_t, t \in S)\}$ the space of all configurations on S. If $S = \emptyset$, we assume that the space $X^{\emptyset} = \{\emptyset\}$. For any $S, T \subset \mathbb{Z}^d$ such that $S \cap T = \emptyset$ and any configurations $x \in X^S$ and $y \in X^T$, we denote by xy the concatenation of x and y, that is, the configuration on $T \cup S$ equal

[1] Here and below the symbol $|V|$ is used to denote the power of the finite set V.

to x on S and to y on T. For any $S \subset T$, $x \in X^T$, we denote by x_S the restriction of x on S.

Let $\Im^{\mathbb{Z}^d}$ be the σ-algebra, generated by cylinder subsets of the set $X^{\mathbb{Z}^d}$. The distribution of a random field (ξ_t) is the probability measure P on $\left(X^{\mathbb{Z}^d}, \Im^{\mathbb{Z}^d}\right)$, such, that

$$\Pr\left\{\left(\xi_t, t \in \mathbb{Z}^d\right) \in B\right\} = P\left(B\right), \qquad B \in \Im^{\mathbb{Z}^d}.$$

Let us define a group of transformations τ_a, $a \in \mathbb{Z}^d$ on $X^{\mathbb{Z}^d}$ such that $(\tau_a x)_t = x_{t+a}$ for any $x \in X^{\mathbb{Z}^d}$. Let $\mathscr{L}$ be the σ-algebra of invariant subsets of $X^{\mathbb{Z}^d}$

$$\mathscr{L} = \{A \in \Im^{\mathbb{Z}^d} : \tau_a A = A\}.$$

A random field (ξ_t) with distribution P is called homogeneous random field if for any $A \in \Im^{\mathbb{Z}^d}$ and $a \in \mathbb{Z}^d$

$$P\left(\tau_a A\right) = P\left(A\right),$$

and is called ergodic random field if its distribution P is trivial on $\mathscr{L}$, i.e. $P(A) \in \{0, 1\}$ when $A \in \mathscr{L}$.

A random field (ξ_t) is called a martingale–difference random field (see [1]), if for any $t \in \mathbb{Z}^d$

$$E\left|\xi_t\right| < \infty \qquad \text{and} \qquad E\left(\xi_t / \xi_s, s \in \mathbb{Z}^d \setminus \{t\}\right) = 0 \text{ (a.s.)}.$$

Here $E\left(\xi_t / \xi_s, s \in \mathbb{Z}^d \setminus \{t\}\right)$ is the conditional expectation of ξ_t with respect to σ-algebra generated by random variables ξ_s, $s \in \mathbb{Z}^d \setminus \{t\}$, $t \in \mathbb{Z}^d$.

The main result for the martingale–difference random fields is the CLT ([2]).

Theorem 1. *Let (ξ_t) be a homogenous ergodic martingale–difference random field such that $0 < \sigma^2 = E\xi_0^2 < \infty$. Then*

$$\lim_{n\to\infty} P\left(\frac{1}{\sigma\left|V_n\right|}\sum_{t\in V_n}\xi_t < x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du, \qquad x \in \mathbb{R},$$

where V_n is a d-dimensional cube with side length n, $n = 1, 2, \ldots$.

2 The Ising model

Ising ferromagnetic model (further — Ising model) is defined on $\mathbb{Z}^d$ by means of the nearest neighbor pair interaction potential

$$\Phi_{\{t,s\}}\left(x_t x_s\right) = \begin{cases} x_t x_s, & \|t - s\| = 1 \\ 0, & \|t - s\| \neq 1 \end{cases}$$

where $x_t, x_s \in X = \{-1, 1\}$ and $\|t - s\| = \sum\limits_{i=1}^{d}\left|t^{(i)} - s^{(i)}\right|$, $t, s \in \mathbb{Z}^d$. The Hamiltonian of this model is equal

$$H_V^h\left(x/\bar{x}\right) = -\frac{1}{2}\sum_{\substack{t,s\in V:\\ \|t-s\|=1}} x_t x_s - \sum_{\substack{t\in V, s\in\partial V:\\ \|t-s\|=1}} x_t \bar{x}_s - h\sum_{t\in V} x_t,$$

where $x \in X^V$, $\bar{x} \in X^{\mathbb{Z}^d \setminus V}$, $V \in W$ and the parameter $h \in \mathbb{R}$ corresponds to the external field. Let $\beta > 0$ be a parameter proportional to the inverse temperature. The conditional probability distribution of Gibbs random field (ξ_t) corresponding to the Ising model is defined by the following way

$$P_V^{\beta,h}\left(\xi_t = x_t, t \in V / \xi_s = \bar{x}_s, s \in Z^d \setminus V\right) = P_V^{\beta,h}(x/\bar{x}) = \frac{\exp\left\{-\beta H_V^h(x/\bar{x})\right\}}{\sum\limits_{z \in X^V} \exp\left\{-\beta H_V^h(z/\bar{x})\right\}},$$

for any $V \in W$, $x \in X^V$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus V}$.

Denote

$$P_{V,+}^{\beta,h} = P_V^{\beta,h}(x/\bar{x}_s = 1, s \in \partial V) \qquad \text{and} \qquad P_{V,-}^{\beta,h} = P_V^{\beta,h}(x/\bar{x}_s = -1, s \in \partial V),$$

where

$$\partial V = \left\{t \in Z^d \setminus V : \rho(t, V) = 1\right\}, \qquad \rho(t, V) = \min_{s \in V} |t - s|.$$

It is well known that $P_{V,+}^{\beta,h}$ and $P_{V,-}^{\beta,h}$ are weakly converge to some limits $P_+^{\beta,h}$ and $P_-^{\beta,h}$ when $V \uparrow \mathbb{Z}^d$. For $d = 1$ there exist the unique limiting distribution ([13]), i.e.

$$P_+^{\beta,h} = P_-^{\beta,h}$$

for any values of parameters (β, h). In the case $d = 2$ the uniqueness takes place only for $h \neq 0$, $0 < \beta < \infty$ and $h = 0$, $0 < \beta < \beta_{cr}$, where β_{cr} is the critical inverse temperature. But for $h = 0$ and $\beta > \beta_{cr}$ we have

$$P_+^{\beta,h} \neq P_-^{\beta,h}.$$

A point $(\beta_{cr}, 0)$ is the critical point for the Ising model.

3 The martingale model

Consider the model which was introduced in [3]. The pair correlation between spins in this model is equal to zero for any values of (β, h). This model is defined by the following nearest neighbor pair interaction potential

$$\tilde{\Phi}_{\{t,s\}}(y_t y_s) = \begin{cases} |y_t| \cdot |y_s|, & \|t - s\| = 1 \\ 0, & \|t - s\| \neq 1, \end{cases}$$

where $y_t, y_s \in Y = \{-1, 0, 1\}$, $t, s \in \mathbb{Z}^d$. This potential is even, i.e.

$$\Phi_V(x_t, t \in V) = \Phi_V(|x_t|, t \in V), \qquad x \in X^V, V \in W.$$

It was shown in [1] that a Gibbs random field with symmetric (with respect to zero) phase space and even potential is a martingale–difference random field. Hence the Gibbs random field (η_t) corresponding to the potential $\tilde{\Phi}$ is a martingale–difference. The Hamiltonian of this martingale model is given by

$$\tilde{H}_V^h(y/\bar{y}) = -\frac{1}{2} \sum_{\substack{t,s \in V: \\ \|t-s\|=1}} |y_t| \cdot |y_s| - \sum_{\substack{t \in V, s \in \partial V: \\ \|t-s\|=1}} |y_t| \cdot |\bar{y}_s| - h \sum_{t \in V} |y_t|,$$

where $y \in Y^V$, $\bar{y} \in Y^{\mathbb{Z}^d \setminus V}$, $V \in W$.

Let us rewrite the Hamiltonian $\tilde{H}_V^h(y/\bar{y})$ in a more convenient form. Using the same idea which was used in [3] one can obtain

$$\tilde{H}_V^h(y/\bar{y}) = -\frac{1}{8}\sum_{\substack{t,s\in V:\\ \|t-s\|=1}} (2|y_t|-1)\cdot(2|y_s|-1) - \frac{1}{4}\sum_{\substack{t\in V, s\in\partial V:\\ \|t-s\|=1}} (2|y_t|-1)\cdot(2|\bar{y}_s|-1) -$$

$$-\frac{h+d}{2}\sum_{t\in V}(2|y_t|-1) - \frac{1}{2}\sum_{\substack{t\in V, s\in\partial V:\\ \|t-s\|=1}} |\bar{y}_s| - f(|V|, |\partial V|) =$$

$$= \frac{1}{4}H_V^{\frac{h+d}{2}}(2|y| - 1/2|\bar{y}| - 1) - \frac{1}{2}\sum_{\substack{t\in V, s\in\partial V:\\ \|t-s\|=1}} |\bar{y}_s| + f(|V|, |\partial V|)$$

where

$$f(|V|, |\partial V|) = \frac{d}{2}|V| + \frac{h}{2}|V| - \frac{1}{8}\sum_{\substack{t,s\in V:\\ \|t-s\|=1}} 1 - \frac{1}{4}\sum_{\substack{t\in V, s\in\partial V:\\ \|t-s\|=1}} 1,$$

and

$$2|y| - 1 = \{2|y_t| - 1, t\in V\}, \qquad 2|\bar{y}| - 1 = \{2|\bar{y}_s| - 1, s\in\partial V\}.$$

The conditional probability distribution of the Gibbs random field (η_t) takes the form

$$Q_V^{\beta,h}\left(\eta_t = y_t, t\in V\big/\eta_s = \bar{y}_s, s\in Z^d\setminus V\right) = Q_V^{\beta,h}(y/\bar{y}) = \frac{\exp\left\{-\beta\tilde{H}_V^h(y/\bar{y})\right\}}{\sum\limits_{z\in Y^V}\exp\left\{-\beta\tilde{H}_V^h(z/\bar{y})\right\}} =$$

$$= \frac{\exp\left\{-\frac{\beta}{4}H_V^{\frac{h+d}{2}}(2|y| - 1/2|\bar{y}| - 1)\right\}}{\sum\limits_{z\in Y^V}\exp\left\{-\frac{\beta}{4}H_V^{\frac{h+d}{2}}(2|z| - 1/2|\bar{y}| - 1)\right\}}.$$

It is not difficult to see that

$$\sum_{z\in Y^V}\exp\left\{-\frac{\beta}{4}H_V^{\frac{h+d}{2}}(2|z| - 1/2|\bar{y}| - 1)\right\} =$$
$$= 2^{|V|/2}\sum_{z\in X^V}\exp\left\{-\frac{\beta}{4}H_V^{\frac{h+d-4(\ln 2)/\beta}{2}}(2|z| - 1/2|\bar{y}| - 1)\right\}.$$

Hence

$$Q_V^{\beta,h}(y/\bar{y}) =$$
$$= 2^{-\sum\limits_{t\in V}|y_t|}\cdot\frac{\exp\left\{-\frac{\beta}{4}H_V^{\frac{h+d-4(\ln 2)/\beta}{2}}(2|y| - 1/2|\bar{y}| - 1)\right\}}{\sum\limits_{z\in Y^V}\exp\left\{-\frac{\beta}{4}H_V^{\frac{h+d-4(\ln 2)/\beta}{2}}(2|z| - 1/2|\bar{y}| - 1)\right\}} =$$

$$= 2^{-\sum\limits_{t\in V}|y_t|}P_V^{\frac{\beta}{4},\frac{h+d-4(\ln 2)/\beta}{2}}(2|y| - 1/2|\bar{y}| - 1).$$

Denote

$$Q_{V,0}^{\beta,h} = Q_V^{\beta,h}(y/\bar{y}_s = 0, s\in\partial V),$$

$$Q_{V,+}^{\beta,h} = Q_V^{\beta,h}\left(y/\bar{y}_s = 1, s \in \partial V\right),$$
$$Q_{V,-}^{\beta,h} = Q_V^{\beta,h}\left(y/\bar{y}_s = -1, s \in \partial V\right).$$

We have

$$Q_{V,0}^{\beta,h} = 2^{-\sum\limits_{t\in V}|y_t|} P_{V,-}^{\frac{\beta}{4},\frac{h+d-4(\ln 2)/\beta}{2}}\left(2\,|y|-1\right),$$
$$Q_{V,+}^{\beta,h} = Q_{V,-}^{\beta,h} = 2^{-\sum\limits_{t\in V}|y_t|} P_{V,+}^{\frac{\beta}{4},\frac{h+d-4(\ln 2)/\beta}{2}}\left(2\,|y|-1\right).$$

Further for any $I \subset V$ one can write

$$\left(Q_{V,0}^{\beta,h}\right)_I(y) = 2^{-\sum\limits_{t\in V}|y_t|}\left(P_{V,-}^{\frac{\beta}{4},\frac{h+d-4(\ln 2)/\beta}{2}}\right)_I\left(2\,|y|-1\right) \tag{1}$$

and

$$\left(Q_{V,+}^{\beta,h}\right)_I(y) = \left(Q_{V,-}^{\beta,h}\right)_I(y) = 2^{-\sum\limits_{t\in V}|y_t|}\left(P_{V,+}^{\frac{\beta}{4},\frac{h+d-4(\ln 2)/\beta}{2}}\right)_I\left(2\,|y|-1\right). \tag{2}$$

Since as $V \uparrow \mathbb{Z}^d$ the limits in the right hand sides of (1) and (2) exist, then there also exist limits

$$\left(Q_j^{\beta,h}\right)_I(y) = \lim_{V\uparrow\mathbb{Z}^d}\left(Q_{V,j}^{\beta,h}\right)_I(y), \qquad j \in \{0,+,-\},$$

and

$$\left(Q_0^{\beta,h}\right)_I(y) = 2^{-\sum\limits_{t\in V}|y_t|}\left(P_{-}^{\frac{\beta}{4},\frac{h+d-4(\ln 2)/\beta}{2}}\right)_I\left(2\,|y|-1\right),$$
$$\left(Q_+^{\beta,h}\right)_I(y) = \left(Q_-^{\beta,h}\right)_I(y) = 2^{-\sum\limits_{t\in V}|y_t|}\left(P_{+}^{\frac{\beta}{4},\frac{h+d-4(\ln 2)/\beta}{2}}\right)_I\left(2\,|y|-1\right).$$

From these equations we conclude that the limiting distribution for the martingale model is unique whenever

$$\frac{h+d-4\,(\ln 2)\,/\beta}{2} \neq 0, \qquad \beta > 0$$

and

$$\frac{h+d-4\,(\ln 2)\,/\beta}{2} = 0, \qquad 0 < \frac{\beta}{4} < \beta_{cr}$$

(since for corresponding parameters we have uniqueness in the Ising model), and there is a phase transition on the curve

$$h + d - \frac{4\ln 2}{\beta} = 0, \qquad \beta > 4\beta_{cr}.$$

The critical point (β_{cr}^*, h_{cr}^*) of the martingale model has coordinates

$$\beta_{cr}^* = 4\beta_{cr}, \qquad h_{cr}^* = \frac{4\ln 2}{\beta_{cr}^*} - d,$$

where β_{cr} is the critical inverse temperature for the Ising model. Note that at the critical point the limiting distribution $Q^{\beta_{cr}^*,h_{cr}^*}$ is unique.

4 Connection formulas of total spins probability distributions

Denote by $S_V^\xi = \sum\limits_{t\in V} \xi_t$ and $S_V^\eta = \sum\limits_{t\in V} \eta_t$ the total spins in the volume $V \in W$ for the Ising model and the martingale model respectively. The relation

$$\xi_t = 2\,|\eta_t| - 1, \qquad t \in \mathbb{Z}^d$$

allows to find the connection between probability distributions of total spins S_V^ξ and S_V^η. In [3] the formula expressing the probability distribution of S_V^η by means of the probability distribution of S_V^ξ and the inverse formula expressing the probability distribution of S_V^ξ by means of the probability distribution of S_V^η were established.

Let us note that the inverse formula obtained in [3] is quite complicated and inconvenient for further study. In this paper a more compact form for the connection formula of total spins probability distribution is found.

For the sake of simplicity we will consider the random field (ζ_t) with phase space $Z = \{0,1\}$ instead of the random field (ξ_t) with phase space $X = \{-1,1\}$ corresponding to the Ising model, such that

$$\zeta_t = \frac{\xi_t + 1}{2}, \qquad t \in \mathbb{Z}^d.$$

Then

$$\zeta_t = |\eta_t|\,, \qquad t \in \mathbb{Z}^d.$$

The following two theorems present the connection formulas between probability distributions of total spins S_V^ζ and S_V^η.

Theorem 2. *The probability distribution of total spin S_V^η by means of probability distribution of total spin S_V^ζ is given by the following formulas*

$$P\left(S_V^\eta = k\right) = \sum_{m=0}^{[(|V|-k)/2]} 2^{-(k+2m)} C_{k+2m}^m P\left(S_V^\zeta = k+2m\right),^{2} \tag{3}$$

$$P\left(S_V^\eta = k\right) = P\left(S_V^\eta = -k\right), \qquad k = 0, 1, ..., |V|\,,$$

where $[\cdot]$ denotes the integer part of number.

Proof. Denote $V^+ = \{t \in V : \eta_t = +1\}$ and $V^- = \{t \in V : \eta_t = -1\}$. Then

$$S_V^\eta = \sum_{t\in V} \eta_t = \left|V^+\right| - \left|V^-\right|.$$

Hence the event $\{\omega : S_V^\eta = k\}$ can be rewrite as follows

$$\left\{\omega : \sum_{t\in V} \eta_t = k\right\} = \left\{\omega : |V^+| = k+m, |V^-| = m, 0 \le m \le \frac{|V|-k}{2}\right\} =$$
$$= \bigcup_m \bigcup_{\substack{I\subset V\\ |I|=k+2m}} \bigcup_{\substack{\tilde I\subset I\\ |\tilde I|=m}} \left\{\omega : \eta_t = -1, t\in \tilde I, \eta_t = +1, t \in I\backslash \tilde I, \eta_t = 0, t\in V\backslash I\right\},$$

[2] Here and below C_n^k stands for the binomial coefficient, i.e. the coefficient of x^k in the expansion of $(1+x)^n$.

where $I = \{t \in V : \eta_t = \pm 1\}$, $\tilde{I} = \{t \in V : \eta_t = -1\}$, $\tilde{I} \subset I \subset V$, $0 \le k \le |V|$. Then

$$P(S_V^\eta = k) = \sum_{m=0}^{[(|V|-k)/2]} \sum_{\substack{I \subset V \\ |I|=k+2m}} \sum_{\substack{\tilde{I} \subset I \\ |\tilde{I}|=m}} P\left(\eta_t = -1, t \in \tilde{I}, \eta_t = +1, t \in I \backslash \tilde{I}, \eta_t = 0, t \in V \backslash I\right) =$$

$$= \sum_{m=0}^{[(|V|-k)/2]} \sum_{\substack{I \subset V \\ |I|=k+2m}} \sum_{\substack{\tilde{I} \subset I \\ |\tilde{I}|=m}} 2^{-(k+2m)} P(\zeta_t = 1, t \in I, \zeta_t = 0, t \in V \backslash I) =$$

$$= \sum_{m=0}^{[(|V|-k)/2]} 2^{-(k+2m)} C_{k+2m}^m \sum_{\substack{I \subset V \\ |I|=k+2m}} P(\zeta_t = 1, t \in I, \zeta_t = 0, t \in V \backslash I) =$$

$$= \sum_{m=0}^{[(|V|-k)/2]} 2^{-(k+2m)} C_{k+2m}^m P\left(S_V^\zeta = k+2m\right).$$

Similarly

$$P(S_V^\eta = -k) = \sum_{m=0}^{[(|V|-k)/2]} 2^{-(k+2m)} C_{k+2m}^m P\left(S_V^\zeta = k+2m\right),$$

i.e. $P(S_V^\eta = -k) = P(S_V^\eta = k)$, $0 \le k \le |V|$.

□

More important is the following theorem.

Theorem 3. *The probability distribution of total spin S_V^ζ by means of probability distribution of total spin S_V^η is given by the following formulas*

$$P\left(S_V^\zeta = 0\right) = P(S_V^\eta = 0) + 2 \sum_{m=1}^{[|V|/2]} (-1)^m P(S_V^\eta = 2m),$$

$$P\left(S_V^\zeta = k\right) = 2^k \sum_{m=0}^{[(|V|-k)/2]} (-1)^m \frac{k+2m}{k+m} C_{k+m}^m P(S_V^\eta = k+2m) \tag{4}$$

for any $k = 1, 2, ..., |V|$.

Proof. Denote

$$b_{k+2m} = 2^{-(k+2m)} P\left(S_V^\zeta = k+2m\right).$$

The relation (3) will be written as follows

$$P(S_V^\eta = k) = \sum_{m=0}^{[(|V|-k)/2]} C_{k+2m}^m b_{k+2m},$$

which is one of mutually inverse relations of Chebyshev type (see, for example, [14]). Then

$$b_k = \sum_{m=0}^{[(|V|-k)/2]} (-1)^m \left(C_{k+m}^m + C_{k+m-1}^{m-1}\right) P(S_V^\eta = k+2m) =$$
$$= \sum_{m=0}^{[(|V|-k)/2]} (-1)^m \frac{k+2m}{k+m} C_{k+m}^m P(S_V^\eta = k+2m),$$

which proves the relation (4). The expression for $P\left(S_V^\zeta = 0\right)$ follows from the requirement

$$\sum_{k=0}^{|V|} P\left(S_V^\zeta = k\right) = 1.$$

□

5 Characteristic functions

Relations obtained in Theorem 2 allow to express the characteristic function of the total spin S_V^η in terms of the probability distribution of the total spin S_V^ζ.

Theorem 4. *The characteristic function $f_{S_V^\eta}(t)$ of the total spin S_V^η has the following form*

$$f_{S_V^\eta}(t) = \sum_{k=0}^{|V|} (\cos t)^k P\left(S_V^\zeta = k\right).$$

Proof. For the sake of simplicity let us denote $|V| = n$,

$$P\left(S_V^\zeta = j\right) = a_j^{(n)}, \qquad j = \overline{0, n},$$

$$P\left(S_V^\eta = j\right) = b_j^{(n)}, \qquad j = \overline{-n, n}.$$

By virtue of Theorem 2 one have

$$b_{2k}^{(n)} = \sum_{j=k}^{[n/2]} 2^{-2j} C_{2j}^{j-k} a_{2j}^{(n)}, \qquad k = 0, 1, ..., [n/2],$$

$$b_{2k-1}^{(n)} = \sum_{j=k}^{[n/2]} 2^{-2j+1} C_{2j-1}^{j-k} a_{2j-1}^{(n)}, \qquad k = 1, 2, ..., [n/2],$$

$$b_k = b_{-k}, \qquad k = \overline{-n, n}.$$

Then

$$f_{S_V^\eta}(t) = Ee^{itS_V^\eta} = b_0^{(n)} + \sum_{k=1}^{[n/2]} \left(e^{it2k} + e^{-it2k}\right) b_{2k}^{(n)} + \sum_{k=1}^{[n/2]} \left(e^{it(2k-1)} + e^{-it(2k-1)}\right) \cdot b_{2k-1}^{(n)} =$$
$$= b_0^{(n)} + \sum_{j=1}^{[n/2]} 2^{-2j} a_{2j}^{(n)} \sum_{k=1}^{j} C_{2j}^{j-k} \left(e^{it2k} + e^{-it2k}\right) +$$
$$+ \sum_{j=1}^{[n/2]} 2^{-2j+1} a_{2j-1}^{(n)} \sum_{k=1}^{j} C_{2j-1}^{j-k} \left(e^{it(2k-1)} + e^{-it(2k-1)}\right).$$

It is not difficult to show, that

$$\left(1 + e^{2it}\right)^{2j} = e^{2it} \left[C_{2j}^{j} + \sum_{k=1}^{j} C_{2j}^{j-k} \left(e^{it2k} + e^{-it2k}\right)\right].$$

Hence

$$\sum_{k=1}^{j} C_{2j}^{j-s} \left(e^{it2k} + e^{-it2k}\right) = e^{-2itj} \left(1 + e^{2it}\right)^{2j} - C_{2j}^{j} = \left(e^{-it} + e^{it}\right)^{2j} - C_{2j}^{j}.$$

Similarly, from

$$\left(1+e^{2it}\right)^{2j-1}=e^{it(2j-1)}\sum_{k=1}^{j}C_{2j-1}^{j-k}\left(e^{-it(2k-1)}+e^{it(2k-1)}\right)$$

we have

$$\sum_{k=1}^{j}C_{2j-1}^{j-k}\left(e^{-it(2k-1)}+e^{it(2k-1)}\right)=\left(e^{-it}+e^{it}\right)^{2j-1}.$$

Taking into account that $b_0^{(n)}=\sum\limits_{j=0}^{[n/2]}2^{-2j}C_{2j}^{j}a_{2j}^{(n)}$, we finally obtain

$$f_{S_V^\eta}(t)=\sum_{j=0}^{n}\left(\frac{e^{-it}+e^{it}}{2}\right)^j a_j^{(n)}=\sum_{j=0}^{n}(\cos t)^j\, a_j^{(n)}.$$

□

The next theorem establishes the expression for the characteristic function of the total spin S_V^ζ in terms of probability distribution of the total spin S_V^η. It is interesting that in obtained expression the coefficients are the known Chebyshev polynomials of the first kind.

Theorem 5. *The characteristic function $f_{S_V^\zeta}(t)$ of the total spin S_V^ζ has the following form*

$$f_{S_V^\zeta}(t)=\sum_{k=-|V|}^{|V|}\cos\left(k\arccos e^{it}\right)P\left(S_V^\eta=k\right).$$

Proof. Here we will use the notations introduced in the proof of Theorem 4. By virtue of the Theorem 3 one have

$$a_0^{(n)}=b_0^{(n)}+2\sum_{m=0}^{[n/2]}(-1)^m\, b_{2m}^{(n)},$$

$$a_{2k}^{(n)}=(-1)^k\,2^{2k}\sum_{m=k}^{[n/2]}(-1)^k\,\frac{2m}{k+m}C_{m+k}^{m-k}b_{2m}^{(n)},$$

$$a_{2k-1}^{(n)}=(-1)^k\,2^{2k-1}\sum_{m=k}^{[n/2]}(-1)^m\,\frac{2m-1}{m+k-1}C_{m+k-1}^{m-k}b_{2m-1}^{(n)},$$

for any $k=1,2,...,n/2$. Then

$$\begin{aligned}f_{S_V^\zeta}(t)&=Ee^{itS_V^\zeta}=a_0^{(n)}+\sum_{j=1}^{[n/2]}e^{it2j}a_{2j}^{(n)}+\sum_{j=1}^{[n/2]}e^{it(2j-1)}a_{2j-1}^{(n)}=\\&=b_0^{(n)}+\sum_{k=1}^{[n/2]}(-1)^k\,b_{2k}^{(n)}\sum_{j=0}^{k}\frac{2k}{k+j}C_{k+j}^{k-j}\left(-4e^{2it}\right)^j+\\&+\sum_{k=1}^{[n/2]}\frac{(-1)^k\,b_{2k-1}^{(n)}}{2e^{it}}\sum_{j=1}^{k}\frac{2k-1}{k+j-1}C_{k+j-1}^{k-j}\left(-4e^{2it}\right)^j.\end{aligned}$$

Denote $k-j=s$. One have

$$\sum_{j=0}^{k}\frac{2k}{k+j}C_{k+j}^{k-j}\left(-4e^{2it}\right)^{j}=\left(-4e^{2it}\right)^{k}\sum_{s=0}^{k}\frac{2k}{2k-s}C_{2k-s}^{s}\left(-4e^{2it}\right)^{-s}$$

and

$$\sum_{j=1}^{k}\frac{2k-1}{k+j-1}C_{k+j-1}^{k-j}\left(-4e^{2it}\right)^{j}=\left(-4e^{2it}\right)^{k}\sum_{s=0}^{k}\frac{2k-1}{2k-1-s}C_{2k-1-s}^{s}\left(-4e^{2it}\right)^{-s}$$

Hence

$$f_{S_V^{\varsigma}}(t)=b_0^{(n)}+\sum_{k=1}^{n}2^k e^{itk}b_k^{(n)}\sum_{s=0}^{[k/2]}\frac{k}{k-s}C_{k-s}^{s}\left(-4e^{2it}\right)^{-s}.$$

Let us transform the known expression of the Chebyshev polynomials (see, for example, [15]). We obtain

$$\cos\left(k\arccos x\right)=\sum_{s=0}^{[k/2]}(-1)^s\frac{k}{k-s}C_{k-s}^{s}2^{k-2s-1}x^{k-2s}=2^{k-1}x^k\sum_{s=0}^{[k/2]}\frac{k}{k-s}C_{k-s}^{s}\cdot\left(-4x^2\right)^{-s}.$$

Then

$$f_{S_V^{\varsigma}}(t)=b_0^{(n)}+2\sum_{k=1}^{n}\cos\left(k\arccos e^{it}\right)b_k^{(n)}.$$

Since $b_k^{(n)}=b_{-k}^{(n)}$, we finally obtain

$$f_{S_V^{\varsigma}}(t)=\sum_{k=-n}^{n}\cos\left(k\arccos e^{it}\right)b_k^{(n)}.$$

$\square$

6 On the limit theorems

As it was mentioned above the limiting distribution $Q^{\beta_{cr}^*,h_{cr}^*}$ of the martingale model at the critical point is unique. Hence it is ergodic ([16]). Since the potential $\tilde{\Phi}$ is translation invariant the corresponding to this model martingale–difference random field (η_t) is homogenous. Hence the random field (η_t) satisfies the conditions of Theorem 1, and therefore CLTS for the martingale model holds also at the critical point (β_{cr}^*,h_{cr}^*).

However the result of the Theorem 4 allows to easily obtain the direct proof of the asymptotic normality of the total spin S_V^{η}.

Theorem 6. *Let $f_{S_V^{\eta}/\sqrt{DS_V^{\eta}}}(t)$ be a characteristic function of the normalized total spin $\dfrac{S_V^{\eta}}{\sqrt{DS_V^{\eta}}}$. Then*

$$f_{S_V^{\eta}/\sqrt{DS_V^{\eta}}}(t)\to e^{-t^2/2}\qquad \text{as } V\uparrow\mathbb{Z}^d.$$

Proof. Since random field (ζ_t) is homogenous let us denote $P(\zeta_t = 1) = p$, $t \in \mathbb{Z}^d$. It is clear that $MS_V^{\zeta} = p|V|$. Since (η_t) is a martingale–difference random field we have $MS_V^{\eta} = 0$ and

$$DS_V^{\eta} = E\left(S_V^{\eta}\right)^2 = \sum_{t\in V} E\eta_t^2 = \sum_{t\in V} P\left(\eta_t^2 = 1\right) = \sum_{t\in V} P\left(\zeta_t = 1\right) = p\,|V|\,.$$

Here we used the relation $\zeta_t = |\eta_t| = \eta_t^2$, $t \in \mathbb{Z}^d$.

By virtue of the Theorem 4 one can write

$$f_{S_V^{\eta}/\sqrt{DS_V^{\eta}}}(t) = E\exp\left\{it\frac{S_V^{\eta}}{\sqrt{p\,|V|}}\right\} = \sum_{j=0}^{|V|}\left(\cos\frac{t}{\sqrt{p\,|V|}}\right)^j P\left(S_V^{\zeta} = j\right).$$

Further sequentially applying the Maclaurin series to functions

$$\cos x = 1 - \frac{x^2}{2} + o\left(x^4\right)$$

and

$$(1+x)^j = 1 + jx + o\left(x^2\right),$$

we obtain

$$\left(\cos\frac{t}{\sqrt{p\,|V|}}\right)^j = 1 - j\cdot\frac{t^2}{2p\,|V|} + j\cdot o\left(|V|^{-2}\right) + o\left(|V|^{-4}\right).$$

Then

$$f_{S_V^{\eta}/\sqrt{DS_V^{\eta}}}(t) = \sum_{j=0}^{n}\left(1 - j\cdot\frac{t^2}{2p\,|V|} + j\cdot o\left(|V|^{-2}\right) + o\left(|V|^{-4}\right)\right) P\left(S_V^{\zeta} = j\right) =$$
$$= 1 - \frac{t^2}{2p\,|V|}ES_V^{\zeta} + o\left(|V|^{-2}\right)ES_V^{\zeta} + o\left(|V|^{-4}\right) = 1 - \frac{t^2}{2} + o\left(|V|^{-1}\right),$$

and the assertion of the theorem follows. □

In [17] it was shown that for Gibbs random fields corresponding to bounded finite range potential the local limit theorem for the total spin (LLTS) follows from the central limit theorem. Thereby for the martingale model LLTS is also valid.

Now we return to the consideration of the Ising model. Since $\xi_t = 2\zeta_t - 1$ for any $t \in \mathbb{Z}^d$ we have $S_V^{\xi} = 2S_V^{\zeta} - |V|$ for each $V \in W$. This observation leads to the following results.

Theorem 7. *The probability distribution of total spin S_V^{ξ} by means of probability distribution of total spin S_V^{η} is given by the following formula*

$$P\left(S_V^{\xi} = k\right) = 2^{(k+|V|)/2}\sum_{j=0}^{[(n-|V|)/4]}(-1)^j\frac{k+|V|+4j}{k+|V|+2j}C^j_{\frac{k+|V|}{2}+j}P\left(S_V^{\eta} = \frac{k+|V|}{2} + 2j\right),$$

for any $-|V| \le k \le |V|$.

Theorem 8. *The characteristic function $f_{S_V^{\xi}}(t)$ of the total spin S_V^{ξ} has the following form*

$$f_{S_V^{\xi}}(t) = e^{-it|V|}\sum_{k=-|V|}^{|V|}\cos\left(k\arccos e^{2it}\right)P\left(S_V^{\eta} = k\right).$$

It is known that CLTS for the Ising model holds for non critical values of (β, h) (see, for example [18]). Let us note that the results of Theorems 7 – 8 also give a possibility to prove same result for the Ising model by virtue of their validity for the martingale model.

It is known that when $h = 0$ and $\beta \uparrow \beta_{cr}$ the correlation between components of random field (ξ_t) corresponding to the Ising model grows and one can not consider the components of (ξ_t) as a weakly dependent random variables. Hence it is assumed that CLTS can not be valid at the critical point for the Ising model. In the same time for the martingale model CLTS and LLTS hold for any values of parameters. This fact gives us a hope to find by proposed method a limiting law for the Ising model total spin at the critical point.

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Abstract

A martingale model associated with the Ising model is considered. The characteristic function of the martingale model total spin was expressed in terms of the Ising model total spin distribution and vice versa. The direct proof of central limit theorem for the martingale model total spin was obtained. The possible way of proving the central limit theorem for the Ising model total spin was specified.