PROBABILITY INEQUALITIES FOR MULTIPLICATIVE SEQUENCES OF RANDOM VARIABLES

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ABSTRACT. We extend some sharp inequalities for martingale-differences to general multiplicative systems of random variables. The key ingredient in the proofs is a technique reducing the general case to the case of Rademacher random variables without change of the constants in inequalities.

1. INTRODUCTION

A sequence of bounded random variables ϕ_n, $n = 1, 2, \ldots$ (finite or infinite) is said to be multiplicative if the equality

$$\mathbf{E}\left[\phi_{n_1}\phi_{n_2}\ldots\phi_{n_\nu}\right] = 0 \tag{1.1}$$

holds for all possible choices of indexes $n_1 < n_2 < \ldots < n_\nu$. Well-known examples of multiplicative sequences are mean zero independent random variables and more general, the martingale-differences, since the condition

$$\mathbf{E}(\phi_n|\phi_1, \ldots, \phi_{n-1}) = 0$$

in the definition of the martingale-difference implies (1.1). The sequences $\{\sin(2^{k+1}\pi x)\}$ and $\{\sin(2n_k\pi x)\}$, where n_k are integers satisfying $n_{k+1} \geq 3n_k$, are known to be non-martingale examples of a multiplicative systems on the unit interval $(0, 1)$ (see [20], chap. 5).

Note that multiplicative systems were introduced by Alexits in his famous monograph [1]. It was proved by Alexits-Sharma [2] that the uniformly bounded multiplicative systems are convergence systems. Recall that an infinite system of random variables $\{\phi_k\}$ is said to be a convergence system if the condition $\sum_k a_k^2 < \infty$ implies almost sure convergence of series $\sum_k a_k\phi_k$. Furthermore, this and other convergence properties of multiplicative type systems were generalized in the papers [4–7, 14, 15].

Let $\mathfrak{M}$ be a family of nonempty subsets of $\mathbb{Z}_n = \{1, 2, \ldots, n\}$, that is $\mathfrak{M} \subset 2^{\mathbb{Z}_n} \setminus \{\varnothing\}$. A system of random variables $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$ is said to be $\mathfrak{M}$-multiplicative if relation (1.1) holds for all $\{n_1, n_2, \ldots, n_\nu\} \in \mathfrak{M}$. If $\mathfrak{M} = 2^{\mathbb{Z}_n} \setminus \{\varnothing\}$, then ϕ turns to be a "full" multiplicative system. Likewise, ϕ is called $\mathfrak{M}$-independent if for any collection

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$\{n_1, n_2, \ldots, n_\nu\} \in \mathfrak{M}$ the members $\phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_\nu}$ are stochastically independent. We will consider systems of bounded random variables $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$ satisfying

$$A_k \le \phi_k \le B_k, \text{ where } A_k < 0 < B_k. \tag{1.2}$$

Setting $C_k = \min\{-A_k, B_k\}$, we define the multiplicative error of ϕ over a family of index subsets $\mathfrak{M} \subset 2^{\mathbb{Z}_n} \setminus \{\varnothing\}$ to be the quantity

$$\mu = \mu(\phi, \mathfrak{M}) = \sum_{\{n_1,n_2,\ldots,n_\nu\}\in\mathfrak{M}} \frac{1}{C_{n_1} C_{n_2} \ldots C_{n_\nu}} \left|\mathbf{E}[\phi_{n_1}\phi_{n_2}\ldots\phi_{n_\nu}]\right|. \tag{1.3}$$

For an integer $l \le n$ denote by $\mathfrak{M}_l$ the family of nonempty subsets of $\mathbb{Z}_n$ having cardinality $\le l$. If $l = n$, then we have $\mathfrak{M}_n = 2^{\mathbb{Z}_n} \setminus \{\varnothing\}$.

The results of the present paper provide a technique that may reduce the study of some properties of bounded multiplicative type systems to the case of Rademacher random variables.

Theorem 1.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a convex function and $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$ be a system of random variables satisfying (1.2). Then for any integer $l \le n$ and a choice of coefficients $a_1, \ldots, a_n$ it holds the inequality*

$$\mathbf{E}\left[\Phi\left(\sum_{k=1}^n a_k\phi_k\right)\right] \le (1 + \mu(\phi, \mathfrak{M}_l))\mathbf{E}\left[\Phi\left(\sum_{k=1}^n a_k\xi_k\right)\right], \tag{1.4}$$

where ξ_k, $k = 1, 2, \ldots, n$ are $\{A_k, B_k\}$- valued mean zero $\mathfrak{M}_l$-independent random variables.

Notice that if a system ϕ is $\mathfrak{M}_l$-multiplicative, then $\mu(\phi, \mathfrak{M}_l) = 0$. So applying Theorem 1.1 for $\mathfrak{M}_l$-multiplicative systems with the parameters $A_k = -1$ and $B_k = 1$, we immediately obtain the following.

Corollary 1.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a convex function. If $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$, is a system of $\mathfrak{M}_l$-multiplicative ($l \le n$) random variables satisfying $\|\phi_k\|_\infty \le 1$, then for any choice of coefficients $a_1, \ldots, a_n$ we have*

$$\mathbf{E}\left[\Phi\left(\sum_{k=1}^n a_k\phi_k\right)\right] \le \mathbf{E}\left[\Phi\left(\sum_{k=1}^n a_k r_k\right)\right], \tag{1.5}$$

where r_k, $k = 1, 2, \ldots, n$, are Rademacher $\mathfrak{M}_l$-independent random variables.

Komlós [14] and Gaposhkin [7] independently proved that

Theorem A (Komlós-Gaposhkin)**.** *If an infinite sequence of random variables $\phi = \{\phi_n\}$ satisfies condition (1.1) for a fixed even integer $\nu > 2$ and the norms $\|\phi_n\|_\nu$ are uniformly bounded, then $\{\phi_k\}$ is a convergence system.*

Moreover, the papers [7, 14] in fact prove a Khintchin type inequality

$$\left\|\sum_{k=1}^n a_k\phi_k\right\|_\nu \le K(\nu)\left(\sum_{k=1}^n a_k^2\right)^{1/2}, \tag{1.6}$$

that implies Theorem A according to a well-known result due to Stechkin (see [13], chap 9.4). On the other hand none of those papers provide an estimation for the Khintchin constant $K(\nu)$. A careful examination of paper [14] may provide only $K(\nu) \lesssim \nu$ even if the norms $\|\phi_k\|_\infty$ are uniformly bounded. While for many classical examples of multiplicative systems it holds the bound $K(\nu) \lesssim \sqrt{\nu}$. For lacunary trigonometric systems $\sin(2\pi n_k x)$, $n_{k+1} > \lambda n_k$, $\lambda > 1$, such a bound is due to Zygmund (see [20], chap. 5), for the uniformly bounded martingale-differences it follows from the Azuma-Hoeffding inequality [3, 9]. In the case of Rademacher independent random variables the Khintchin inequality holds with the constant

$$K(p) = 2^{1/2} \left(\Gamma((p+1)/2)/\pi)\right)^{1/p}, \quad p > 2, \tag{1.7}$$

which is known to be optimal (see [8], [18], [19]). Using Corollary 1.1, the Khintchin sharp inequality for Rademacher independent random variables can be extended to general uniformly bounded multiplicative systems.

Corollary 1.2. *If a system of random variables $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$ is multiplicative and $\|\phi_k\|_\infty \le 1$, then for any choice of coefficients $a_1, \ldots, a_n$ we have*

$$\left\|\sum_{k=1}^n a_k\phi_k\right\|_p \le K(p)\left(\sum_{k=1}^n a_k^2\right)^{1/2}, \quad p > 2,$$

where $K(p)$ is the optimal constant from (1.7).

It is well known that the classical proof of the Khintchin inequality for even integers p only $\mathfrak{M}_p$-independence of Rademacher functions is used. It is also known that in this case the Khintchin optimal constant is $((p-1)!!)^{1/p}$ (see [18] or [16] chap 2). So once again applying Corollary 1.1, we can extend this result to the following inequality.

Corollary 1.3. *Let $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$ be a $\mathfrak{M}_p$-multiplicative system such that $\|\phi_k\|_\infty \le 1$ and p is an even integer with $2 \le p \le n$. Then we have*

$$\left\|\sum_{k=1}^n a_k\phi_k\right\|_p \le K(p)\cdot\left(\sum_{k=1}^n a_k^2\right)^{1/2},$$

with the optimal constant $K(p) = ((p-1)!!)^{1/p}$.

Applying Theorem 1.1, we also prove the following generalization of a well-known martingale inequality due to Azuma-Hoeffding [3, 9].

Theorem 1.2. *If a system of random variables $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$ satisfies (1.2), then it holds the inequality*

$$\left|\left\{\sum_{k=1}^n \phi_k > \lambda\right\}\right| \le (1 + \mu(\phi, \mathfrak{M}_n)) \exp\left(-\frac{2\lambda^2}{\sum_{k=1}^n (B_k - A_k)^2}\right), \quad \lambda > 0.$$

Corollary 1.4. *If $\phi=\{\phi_k:\ k=1,2,\ldots,n\}$ is multiplicative and satisfies (1.2), then*

$$\left|\left\{\sum_{k=1}^n\phi_k>\lambda\right\}\right|\le\exp\left(-\frac{2\lambda^2}{\sum_{k=1}^n(B_k-A_k)^2}\right),\quad\lambda>0. \tag{1.8}$$

In the proof of the main result we use some arguments used in the papers [10, 11] that is a transformation of a bounded orthogonal system into a system of two-valued functions in the study of certain problems. 1) First, we extend functions ϕ_k up to the interval $[0,1+\mu)$, where the new system becomes $\mathfrak{M}_l$-multiplicative (Lemma 2.2), 2) an approximation procedure applied by Lemma 2.3 reduces our problem to the case of step functions, 3) and those after a certain transformation procedure, giving by Lemma 2.4, produce $\mathfrak{M}_l$-multiplicative system of $\{A_k,B_k\}$-valued functions, 4) a dilation of the interval $[0,1+\mu)$ back to $[0,1)$ finalizes the proof, generating the constant $1+\mu$ in (1.4), since any $\mathfrak{M}_l$-multiplicative system of two-valued functions is $\mathfrak{M}_l$-independent (Lemma 2.5).

The paper is organized as follows. In Section 2 we have collected preliminary lemmas. Section 3 provides the proofs of the main results. In Section 4 we give applications concerning sub-Gausian estimations for lacunary subsequences of trigonometric systems.

2. Preliminary lemmas

A real-valued function f defined on $[a,b)$ is said to be a step function if it can be written as a finite linear combination of indicator functions of intervals $[\alpha,\beta)\subset[a,b)$.

Lemma 2.1. *For any integer $n\ge 2$ and interval $I=[a,b)$ there exists a system of unimodular step functions $f_k(x)$, $k=1,2,\ldots,n$, on I such that*

$$\prod_{k=1}^n f_k(x)=1,\quad x\in I, \tag{2.1}$$

$$\int_I\left(\prod_{k\in U}f_k\right)=0. \tag{2.2}$$

for any nonempty $U\subsetneq\{1,2,\ldots,n\}$.

Proof. Suppose unimodular step functions f_k, $k=1,2,\ldots,n$, satisfy (2.1) and let $V\subsetneq\{1,2,\ldots,n\}$ be nonempty. Chose $m\in\{1,2,\ldots,n\}\setminus V$ and $l\in V$ arbitrarily. Let $J\subset I$ be a maximal constancy interval of functions f_k, J^- and J^+ be the left and right halves of J. We redefine f_l and f_m, changing their signs on I^+. We do so with each interval of constancy. Clearly, at the end of this procedure we will have

$$\int_I\left(\prod_{k\in V}f_k\right)=0.$$

Moreover, one can check if (2.2) is satisfied for some U before the reconstruction of the functions, then so we will also have after the reconstruction. Starting with functions

$f_k(x) = 1$, $k = 1, 2, \ldots$, we apply this procedure for every $U \subsetneq \{1, 2, \ldots, n\}$. In this way we get functions satisfying the conditions of lemma. $\square$

Lemma 2.2. *Let $\phi = \{\phi_k : k = 1, 2, \ldots, n\}$ be a system of measurable functions on $[0, 1)$ satisfying (1.2) and $\mathfrak{M}$ be an arbitrary family of subsets of $\mathbb{Z}_n$. Then the functions ϕ_k can be extended up to the interval $[0, 1 + \mu)$ such that*

$$A_k \leq \phi_k(x) \leq B_k \text{ if } x \in [0, 1 + \mu), \tag{2.3}$$

$$\int_0^{1+\mu} \phi_{n_1}\phi_{n_2} \ldots \phi_{n_\nu} = 0 \text{ for all } \{n_1, \ldots, n_\nu\} \in \mathfrak{M}, \tag{2.4}$$

where $\mu = \mu(\phi, \mathfrak{M})$ is the multiplicative error (1.3). Moreover, each ϕ_k is a step function on $[1, 1 + \mu)$.

Proof. Set

$$\delta_{n_1,n_2,\ldots,n_\nu} = \frac{1}{C_{n_1}C_{n_2}\ldots C_{n_\nu}} \left| \int_0^1 \phi_{n_1}\phi_{n_2} \ldots \phi_{n_\nu} \right|. \tag{2.5}$$

Divide $[1, 1+\mu)$ into intervals $I_{n_1,n_2,\ldots,n_\nu}$ of lengths $\delta_{n_1,n_2,\ldots,n_\nu}$ considering only the collections $\{n_1, n_2, \ldots, n_\nu\} \in \mathfrak{M}$. We define functions ϕ_m on such an interval $I = I_{n_1,n_2,\ldots,n_\nu}$ as follows. If $m \notin \{n_1, n_2, \ldots, n_\nu\}$, then we let $\phi_n = 0$ on I. Applying Lemma 2.1, one can define the functions $\phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_\nu}$ such that $|\phi_{n_j}(x)| = C_{n_j}$, $x \in I$, $\quad j = 1, 2, \ldots, \nu$, and

$$\prod_{j=1}^{\nu} \phi_{n_j}(x) = -C_{n_1} \ldots C_{n_\nu} \cdot \operatorname{sign} \left(\int_0^1 \phi_{n_1}\phi_{n_2} \ldots \phi_{n_\nu} \right), \quad x \in I, \tag{2.6}$$

$$\int_I \left(\prod_{j \in U} \phi_j \right) = 0, \quad U \subsetneq \{n_1, n_2, \ldots, n_\nu\}. \tag{2.7}$$

Obviously, this correctly determines the functions ϕ_k on $[1, 1+\mu)$ and those satisfy $A_k \leq \phi_k(t) \leq B_k$ for all $t \in [0, 1 + \mu)$. From (2.5) and (2.6) it follows that

$$\int_I \phi_{n_1}\phi_{n_2} \ldots \phi_{n_\nu} = -\int_0^1 \phi_{n_1}\phi_{n_2} \ldots \phi_{n_\nu}. \tag{2.8}$$

One can also check that

$$\int_{I_{m_1,\ldots,m_l}} \phi_{n_1}\phi_{n_2} \ldots \phi_{n_\nu} = 0, \text{ if } \{m_1, \ldots, m_l\} \neq \{n_1, n_2, \ldots, n_\nu\}. \tag{2.9}$$

Indeed, if there is a $n_k \notin \{m_1, \ldots, m_l\}$, then by definition $\phi_{n_k} = 0$ on $I_{m_1,\ldots,m_l}$ and (2.9) follows. Otherwise we will have $\{n_1, n_2, \ldots, n_\nu\} \subsetneq \{m_1, \ldots, m_l\}$ and (2.9) follows from (2.7). By (2.9) we get

$$\int_{[1,1+\mu)\setminus I} \phi_{n_1}\phi_{n_2} \ldots \phi_{n_\nu} = 0. \tag{2.10}$$

From (2.8) and (2.10) we obtain

$$\int_0^{1+\mu} \phi_{n_1}\phi_{n_2}\dots\phi_{n_\nu} = 0, \quad \{n_1, n_2, \dots, n_\nu\} \in \mathfrak{M},$$

which completes the proof of lemma. □

Lemma 2.3. *Let measurable functions ϕ_k, $k = 1, 2, \dots, n$, defined on $[a, b)$, satisfy (1.2). Then for any $\delta > 0$ one can find step functions f_k, $k = 1, 2, \dots, n$, on $[a, b)$ with $A_k \le f_k \le B_k$ such that*

$$|\{|\phi_k - f_k| > \delta\}| < \delta, \quad k = 1, 2, \dots, n, \tag{2.11}$$

and

$$\int_a^b f_{n_1} f_{n_2} \dots f_{n_\nu} = 0 \text{ as } \{n_1, n_2, \dots, n_\nu\} \in \mathfrak{M}, \tag{2.12}$$

where $\mathfrak{M}$ is the family of collections $\{n_1, n_2, \dots, n_\nu\}$ satisfying (1.1).

Proof. Without loss of generality we can suppose that $[a, b) = [0, 1)$. For any $\varepsilon > 0$ one can find step functions $u = \{u_k\}$, $A_k \le u_k \le B_k$, such that

$$\int_0^1 |\phi_k - u_k| < \varepsilon, \tag{2.13}$$

$$\left|\int_0^1 u_{n_1} u_{n_2} \dots u_{n_\nu}\right| < \varepsilon \text{ as } \{n_1, n_2, \dots, n_\nu\} \in \mathfrak{M}. \tag{2.14}$$

Let $\mu = \mu(u, \mathfrak{M})$ be the multiplicative error of the system u. Applying Lemma 2.2, the functions u_k can be extend to the step functions on $[0, 1+\mu)$ such that $A_k \le u_k(t) \le B_k$, $t \in [0, 1 + \mu)$, and

$$\int_0^{1+\mu} u_{n_1} u_{n_2} \dots u_{n_\nu} = 0 \text{ for all } \{n_1, \dots, n_\nu\} \in \mathfrak{M}.$$

Set

$$f_k(x) = u_k\left((1 + \mu)x\right), \quad x \in [0, 1).$$

Obviously, (2.12) will be satisfied. Then, using (2.14) one can get a small μ by choosing a small enough ε. So by (2.13) we can write

$$\begin{aligned}\int_0^1 |f_k(x) - \phi_k(x)|dx &\le \int_0^1 |f_k(x) - u_k(x)|dx + \int_0^1 |\phi_k(x) - u_k(x)|dx \\ &< \int_0^1 |u_k\left((1 + \mu)x\right) - u_k(x)|dx + \varepsilon < \delta^2.\end{aligned}$$

Applying Chebyshev's inequality, we get (2.11). Lemma is proved. □

Lemma 2.4. *Let $f_1, f_2, \dots, f_n$ be real-valued step functions on $[a,b)$ satisfying $A_k \le f_k(x) \le B_k$. Then there are $\{A_k, B_k\}$-valued step functions $g_1, g_2, \dots, g_n$ on $[a,b)$ such that*

$$\int_a^b g_{n_1} g_{n_2} \dots g_{n_\nu} = \int_a^b f_{n_1} f_{n_2} \dots f_{n_\nu}, \tag{2.15}$$

for any choice of $\{n_1, n_2, \dots, n_\nu\} \in \mathbb{Z}_n$, and for any convex function $\Phi : \mathbb{R} \to \mathbb{R}^+$ it holds the inequality

$$\int_a^b \Phi\left(\sum_{k=1}^n a_k f_k\right) \le \int_a^b \Phi\left(\sum_{k=1}^n a_k g_k\right) \tag{2.16}$$

for any coefficients a_k.

Proof. Let Δ_j, $j = 1, 2, \dots, m$ be the intervals of constancy of functions f_k. Let $\Delta = [\alpha, \beta)$ be one of those intervals. Observe that the point

$$c = \frac{B_1\alpha - A_1\beta}{B_1 - A_1} + \frac{1}{B_1 - A_1} \cdot \int_\alpha^\beta f_1(t)dt$$

is in the closure of the interval Δ. Then we define g_1 on Δ as

$$g_1(x) = B_1 \cdot \mathbf{1}_{[\alpha,c)}(x) + A_1 \cdot \mathbf{1}_{[c,\beta)}(x), \quad x \in \Delta.$$

Applying this to each Δ_j, we will have g_1 defined on entire $[a,b)$ and one can check

$$\int_{\Delta_j} g_1(t)dt = \int_{\Delta_j} f_1(t)dt, \quad j = 1, 2, \dots, m. \tag{2.17}$$

Since each f_k is constant on the intervals Δ_j, $j = 1, 2, \dots, m$, from (2.17) we conclude that

$$\int_0^1 g_1 f_{n_2} \dots f_{n_\nu} = \int_0^1 f_1 f_{n_2} \dots f_{n_\nu},$$

for any collection $1 < n_2 < \dots < n_\nu$. We also claim that

$$\int_a^b \Phi\left(\sum_{k=1}^n a_k f_k\right) \le \int_a^b \Phi\left(a_1 g_1 + \sum_{k=2}^n a_k f_k\right). \tag{2.18}$$

Fix an interval Δ_j and suppose that $f_k(t) = c_k$ on Δ_j. Applying (2.17) and the Jessen inequality, we get

$$\begin{aligned}\int_{\Delta_j} \Phi\left(\sum_{k=1}^n a_k f_k(t)\right) dt &= \Phi\left(\sum_{k=1}^n a_k c_k\right) |\Delta_j| \\ &= \Phi\left(\frac{1}{|\Delta_j|}\int_{\Delta_j} a_1 g_1(t)dt + \sum_{k=2}^n a_k c_k\right)|\Delta_j| \\ &= \Phi\left(\frac{1}{|\Delta_j|}\int_{\Delta_j}\left(a_1 g_1(t) + \sum_{k=2}^n a_k c_k\right)dt\right)|\Delta_j| \\ &\le \int_{\Delta_j}\Phi\left(a_1 g_1 + \sum_{k=2}^n a_k f_k\right) dt,\end{aligned}$$

then the summation over j implies (2.18). Applying the same procedure to the new system $g_1, f_2, \ldots, f_n$ we can similarly replace f_2 by g_2. Continuing this procedure we will replace all functions f_k to g_k ensuring the conditions of lemma. □

Lemma 2.5. *If g_k, $k = 1, 2, \ldots, n$, is a $\mathfrak{M}_l$-multiplicative system of nonzero random variables such that each g_k takes two values, then g_k are $\mathfrak{M}_l$-independent.*

Proof. Suppose that g_k takes values A_k and B_k. Since $\mathbf{E}(g_k) = 0$, we can say $A_k < 0 < B_k$ and

$$|\{g_k(x) = A_k\}| = \frac{B_k}{B_k - A_k}, \quad |\{g_k(x) = B_k\}| = \frac{A_k}{A_k - B_k}. \tag{2.19}$$

Let C_k be a sequence that randomly equal either A_k or B_k. We need to prove

$$|\{g_j = C_j : j \in M\}| = \prod_{j\in M} |\{g_j = C_j\}|$$

for any $M \in \mathfrak{M}_l$. Without loss of generality we can suppose that $C_j = A_j$ for all $j \in M$. Then, using the multiplicative condition and (2.19), we obtain

$$\begin{aligned}|\{g_j(x) = A_k : j \in M\}| &= \mathbf{E}\left[\prod_{j\in M}\left(\frac{B_j}{B_j - A_j} - \frac{g_j}{B_j - A_j}\right)\right] \\ &= \prod_{j\in M}\frac{B_j}{B_j - A_j} \\ &\quad + \sum_{D\subsetneq M}(-1)^{\mathrm{card}(D)}\Bigg[\prod_{j\in M\setminus D}\frac{B_j}{B_j - A_j} \\ &\quad \times \prod_{j\in D}\frac{1}{B_j - A_j}\cdot\mathbf{E}\left(\prod_{j\in D} g_j\right)\Bigg] \\ &= \prod_{j\in M}|\{g_j(x) = A_j\}|,\end{aligned}$$

completing the proof of lemma. □

3. Proof of Theorems

Proof of Theorem 1.1. Without loss of generality we can suppose that ϕ_k are defined on $[0,1)$. Applying Lemma 2.2 for $\mathfrak{M}=\mathfrak{M}_l$, we can find extensions of ϕ_k up to $[0,1+\mu)$, satisfying (2.3) and (2.4) (with $\mathfrak{M}=\mathfrak{M}_l$). By (2.4) ϕ_k turns to be $\mathfrak{M}_l$-multiplicative on $[0,1+\mu)$. Applying Lemma 2.3, we find an $\mathfrak{M}_l$-multiplicative system of step functions f_k on $[0,1+\mu)$ satisfying

$$|\{x\in[0,1+\mu):\ |f_k(x)-\phi_k(x)|>\delta\}|<\delta. \tag{3.1}$$

Finally, we apply Lemma 2.4 and get A_k,B_k-valued step functions g_k defined on $[0,1+\mu)$ and satisfying (2.15) and (2.16) ($a=0,b=1+\mu$). Since $\{f_k\}$ is $\mathfrak{M}_l$-multiplicative, in view of (2.15) so we will have for $\{g_k\}$. By Lemma 2.5 functions $\xi_k(x)=g_k((1+\mu)x)$ turn to be an A_k,B_k-valued $\mathfrak{M}_l$-independent random variables. Observe that

$$\int_0^1\Phi\left(\sum_{k=1}^n a_k\phi_k\right)\le\varepsilon(\delta,\{a_k\})+\int_0^1\Phi\left(\sum_{k=1}^n a_kf_k\right),$$

where by (3.1) we have

$$\varepsilon=\varepsilon(\delta,\{a_k\})\to 0 \text{ as } \delta\to 0, \tag{3.2}$$

and it holds for any choice of coefficients $\{a_k:\ k=1,2,\ldots\}$. Thus, from (2.16) we obtain

$$\begin{aligned}\int_0^1\Phi\left(\sum_{k=1}^n a_k\phi_k\right)&\le\varepsilon+\int_0^1\Phi\left(\sum_{k=1}^n a_kf_k\right)\le\varepsilon+\int_0^{1+\mu}\Phi\left(\sum_{k=1}^n a_kf_k\right)\\&\le\varepsilon+\int_0^{1+\mu}\Phi\left(\sum_{k=1}^n a_kg_k\right)\\&=\varepsilon+(1+\mu)\int_0^1\Phi\left(\sum_{k=1}^n a_kg_k((1+\mu)x)\right)dx\\&=\varepsilon+(1+\mu)\mathbf{E}\left[\Phi\left(\sum_{k=1}^n a_k\xi_k\right)\right].\end{aligned}$$

Therefore, taking into account (3.2), one can easily get (1.4). Indeed, the system $\{\xi_k\}$ depends on δ and is independent of $\{a_k\}$. So let $\{\xi_k^{(m)}\}$ be the system corresponding to $\delta=1/m$. We can suppose that there is a partition $0=x_0^{(m)}\le x_1^{(m)}\le,\ldots,x_{2^n}^{(m)}=1$ such that each function $\xi_k^{(m)}$ is equal to a constant (either A_k or B_k) on every interval $[x_{j-1}^{(m)},x_j^{(m)})$, $1\le j\le 2^n$ and this constant is independent of m. Then we find a sequence m_k such that each sequence $x_j^{(m_k)}$, $k=1,2,\ldots$ is convergence. One can check that this generates a limit sequence of $\mathfrak{M}_l$-multiplicative $\{A_k,B_k\}$-valued random variables ξ_k, $k=1,2,\ldots,n$, satisfying (1.4). □

Proof of Theorem 1.2. Applying Theorem 1.1 for $\Phi(t) = \exp(\gamma t)$, $\gamma > 0$, and $l = n$, we find $\{A_k, B_k\}$-valued independent system $\{\xi_k\}$ satisfying (1.4). Thus, we get

$$\begin{aligned}\left|\left\{\sum_{k=1}^{n} \phi_k > \lambda\right\}\right| &\le e^{-\gamma\lambda}\mathbf{E}\left[\exp\left(\gamma\sum_{k=1}^{n}\phi_k\right)\right]\\ &\le (1+\mu)e^{-\gamma\lambda}\mathbf{E}\left[\exp\left(\gamma\sum_{k=1}^{n}\xi_k\right)\right]\\ &= (1+\mu)e^{-\gamma\lambda}\prod_{k=1}^{n}\mathbf{E}\left[\exp\left(\gamma\xi_k\right)\right].\end{aligned}$$

Then applying Hoeffding's [9] inequality we get

$$\mathbf{E}\left[\exp\left(\gamma\xi_k\right)\right] \le \exp\left(\frac{\gamma^2(B_k - A_k)^2}{8}\right)$$

and finally,

$$\left|\left\{\sum_{k=1}^{n} \phi_k > \lambda\right\}\right| \le (1+\mu)e^{-\gamma\lambda}\exp\left(\frac{\gamma^2\sum_{k=1}^{n}(B_k - A_k)^2}{8}\right).$$

Choosing $\gamma = \frac{4\lambda}{\sum_{k=1}^{n}(B_k - A_k)^2}$ we get the bound

$$\left|\left\{\sum_{k=1}^{n} \phi_k > \lambda\right\}\right| \le (1+\mu)\exp\left(-\frac{2\lambda^2}{\sum_{k=1}^{n}(B_k - A_k)^2}\right),$$

which completes the proof of theorem. □

4. Lacunary subsystems

In this section we provide applications of the main results in lacunary systems.

Definition 4.1. An infinite sequence of random variables $\phi = \{\phi_k : k = 1, 2, \ldots\}$ satisfying (1.2) with $A_k = -1, B_k = 1$ is said to be quasi-multiplicative if the multiplicative error (1.3) is finite, that is $\mu = \mu(\phi, \mathfrak{M}_\infty) < \infty$, where $\mathfrak{M}_\infty$ denotes the family of all finite nonempty subsets of positive integers $\mathbb{Z}_+ = \{1, 2, \ldots\}$.

Definition 4.2. An infinite sequence of random variables $\phi = \{\phi_k : k = 1, 2, \ldots\}$ is said to be sub-Gaussian if there are constants $c_1, c_2 > 0$ such that the inequality

$$\mathbf{E}\left[\exp\left(c_1\frac{|\sum_{k=1}^{n} a_k\phi_k|^2}{\sum_{k=1}^{n} a_k^2}\right)\right] \le c_2$$

holds for any coefficients a_k, $k = 1, 2, \ldots, n$.

It is well-known that an equivalent condition for a system $\phi = \{\phi_k : k = 1, 2, \ldots\}$ to be sub-Gaussian is the bound $K(\nu) \lesssim \sqrt{\nu}$ in the Khintchin inequality (1.6). The following corollary, in addition to classical results stated in the introduction, provides new examples of sub-Gaussian sequences of random variables. It immediately follows from (1.4) and the Khintchin inequality for the Rademacher independent random variables. Namely,

Corollary 4.1. *Any quasi-multiplicative sequence of random variables is sub-Gaussian.*

We consider lacunary trigonometric system

$$t_k(x) = \sin(2\pi\tau(k)x), \quad x \in [0,1), \quad k = 1,2,\dots, \tag{4.1}$$

where

$$\tau(1) \geq 1, \quad \tau(k+1) > \lambda\tau(k), \quad k = 1,2,\dots, \tag{4.2}$$

for some constant $\lambda > 1$. In the case of integer $\tau(k)$ this sequence is known to be either multiplicative (if $\lambda \geq 3$) or finite union of multiplicative systems. It was proved by Zygmund ([20], chap. 5) that in that case the system $t_k(x)$ is sub-Gaussian. Using Corollary 4.1 and the following result we prove that $t_k(x)$ is sub-Gaussian in the general case.

Proposition 4.1. *If $\lambda > 2$, then the sequence (4.1) is quasi-multiplicative (with parameters $A_k = -1, B_k = 1$ in (1.3)). Moreover, the multiplicative error satisfies*

$$\mu = \mu(\phi, \mathfrak{M}_\infty) \leq \frac{\lambda(\lambda-1)}{\pi(\lambda-2)^2}.$$

Proof. Let

$$\{n_1, n_2, \dots, n_\nu\} \in \mathfrak{M}_\infty \tag{4.3}$$

be arbitrary collection of indexes with the head n_ν, that is $n_1 < n_2 < \dots < n_\nu$. Using the product to sum formulas for trigonometric functions, we can write the integral

$$\int_0^1 \prod_{j=1}^{\nu} \sin(2\pi\tau(n_j)x)dx, \tag{4.4}$$

as an arithmetic mean of $2^{\nu-1}$ integrals of the forms

$$\int_0^1 \sin\left[(2\pi(\tau(n_\nu) \pm \tau(n_{\nu-1}) \pm \dots \pm \tau(n_1))x\right] dx, \tag{4.5}$$

$$\int_0^1 \cos\left[(2\pi(\tau(n_\nu) \pm \tau(n_{\nu-1}) \pm \dots \pm \tau(n_1))x\right] dx. \tag{4.6}$$

A simple calculation shows that

$$\tau(n_\nu) \pm \tau(n_{\nu-1}) \pm \dots \pm \tau(n_1) \in \left(\frac{(\lambda-2)\tau(n_\nu)}{\lambda-1}, \frac{\lambda\tau(n_\nu)}{\lambda-1}\right)$$

for all choices of $\pm$, so the absolute value of each integral in (4.5) and (4.6) can be estimated by $\frac{\lambda-1}{\pi(\lambda-2)\tau(n_\nu)}$. Thus the same bound we will have for the integral (4.4). Namely,

$$\left|\int_0^1 \prod_{j=1}^{\nu} \sin(2\pi\tau(n_j)x)\right| \leq \frac{\lambda-1}{\pi(\lambda-2)\tau(n_\nu)}.$$

On the other hand the number of collections (4.3) with a fixed head $n = n_\nu$ is equal to 2^{n-1} and we have $\tau(n) \ge \lambda^{n-1}$. Thus for the multiplicative error we obtain

$$\mu \le \frac{\lambda - 1}{\pi(\lambda - 2)} \sum_{n=1}^{\infty} \frac{2^{n-1}}{\tau(n)} \le \frac{\lambda(\lambda-1)}{\pi(\lambda-2)^2}.$$

□

The following corollary is an extension of the above mentioned theorem of Zygmund, where an integer sequence $\tau(k)$ is considered (see [20], chap. 5, Theorem 8.20). Note that if $\tau(k)$ are integers, then the system (4.1) becomes orthogonal and that is essential in the proof of the Zygmund theorem. Our proof of Corollary 4.2 essentially uses the main result via Corollary 1.2 and Corollary 4.1. Hence,

Corollary 4.2. *If real numbers $\tau(k)$ satisfy (4.2) with a constant $\lambda > 1$, then lacunary system (4.1) is sub-Gaussian.*

Proof. If the lacunarity order $\lambda \ge 3$, then by Proposition 4.1 our system $\{t_k(x)\}$ is a quasi-multiplicative and so sub-Gaussian by Corollary 4.1. If $1 < \lambda < 3$, then (4.1) can be split into $\lceil \log_\lambda 3 \rceil$ number of systems of lacunarity order greater that 3. It remains just notice that a finite union of sub-Gaussian sequences is sub-Gaussian. □

Definition 4.3. An infinite sequence of random variables $\phi = \{\phi_k : k = 1, 2, \ldots\}$ is said to be unconditional convergence system if under the condition $\sum_{n=1}^\infty a_n^2 < \infty$ the series

$$\sum_{k=1}^{\infty} a_k \phi_k(x)$$

converges a.e. after any rearrangements of the terms.

Corollary 4.3. *If an infinite sequence of real numbers $\tau(k)$ satisfies (4.2) with $\lambda > 1$, then the lacunary system (4.1) is an unconditional convergence system.*

Proof. According to Corollary 4.2, $\{t_k(x)\}$ is sub-Gaussian so satisfies Khintchin's inequality for any $p > 2$. Then, by Stechkin's result of [13] (chap 9.4), we conclude that $\{t_k(x)\}$ is an unconditional convergence system. □

Theorem 4.1. *If ϕ_k is an orthogonal system of random variables and $\|\phi_k\|_\infty \le M$, then for any $\lambda > 1$ one can find a subsequence of integers n_k such that $n_k \le \lambda^k$ for $k \ge k(\lambda)$ and $\{\phi_{n_k}\}$ is sub-Gaussian.*

The following statement is a version of a lemma from [12].

Lemma 4.1. *Let ϕ_k, $k = 1, 2, \ldots, n$ be an orthogonal system of random variables with $\|\phi_k\|_2 \le 1$, and $f_j \in L^2$, $j = 1, 2, \ldots, m$. Then there is an l, $1 \le l \le n$, such that*

$$\sum_{j=1}^{m} |\boldsymbol{E}(f_j \cdot \phi_l)| \le \sqrt{\frac{m \cdot \sum_{j=1}^m \|f_j\|_2^2}{n}}.$$

Proof. Parseval's inequality implies

$$\sum_{j=1}^{m}\sum_{k=1}^{n}|\mathbf{E}(f_j\cdot\phi_k)|^2\le\sum_{j=1}^{m}\|f_j\|_2^2.$$

Thus there exists an l such that

$$\sum_{j=1}^{m}|\mathbf{E}(f_j\cdot\phi_l)|^2\le\frac{\sum_{j=1}^{m}\|f_j\|_2^2}{n}$$

and so by Hölder's inequality we get

$$\sum_{j=1}^{m}|\mathbf{E}(f_j\cdot\phi_l)|\le\sqrt{m}\left(\sum_{j=1}^{m}|\mathbf{E}(f_j\cdot\phi_l)|^2\right)^{1/2}\le\sqrt{\frac{m\cdot\sum_{j=1}^{m}\|f_j\|_2^2}{n}}.$$

□

Proof of Theorem 4.1. Without loss of generality we can suppose that $\|\phi_k\|_\infty\le 1$. First let us prove that there exists a sub-Gaussian subsequence ϕ_{n_k} such that

(4.7) $\quad 8^{k-1}\le n_k<8^k,\quad k=1,2,\ldots.$

We will chose n_k recursively. Set $n_1=1$ and suppose that we have already chosen n_k, $k=1,2,\ldots,m$. Apply Lemma 4.1 as follows. As the collection f_k we consider all possible products of functions ϕ_{n_k}, $k=1,2,\ldots,m$. The number of such products is 2^m-1. So applying Lemma 4.1, we find $\phi_{n_{m+1}}$, $16^m\le n_{m+1}<16^{m+1}$, such that

$$\sum_{1\le k_1<\ldots<k_l\le m}\left|\mathbf{E}\left(\prod_{j=1}^{l}\phi_{n_{k_j}}\cdot\phi_{n_{n_{m+1}}}\right)\right|\le\sqrt{\frac{2^m-1}{8^{m+1}-8^m}}<\frac{1}{2^m}.$$

Clearly, with this we determine a quasi-multiplicative and so sub-Gaussian system ϕ_{n_k} satisfying (4.7). Then, observe that if the statement of theorem is satisfied for a $\lambda_1>1$ then it will hold also for $\lambda=\lambda_1^{2/3}$. Indeed, we can apply the case of $\lambda=\lambda_1$ to the systems ϕ_{2k} and ϕ_{2k-1}. As a result we find sub-Gaussian subsequences ϕ_{2n_k} and ϕ_{2m_k-1} such that $n_k<\lambda_1^k$, $m_k<\lambda_1^k$ for $k>k_0$. Letting $\{r_k\}$ to be the union of sequences $\{n_k\}$ and $\{m_k\}$ arranged in the increasing order of the terms, we consider a new sequence of random variables ϕ_{r_k}. Clearly, it will be sub-Gaussian and one can easily check that $r_k<(\lambda_1^{2/3})^k$ for $k>2k_0$. Thus, starting with $\lambda=8$ we can prove the theorem for parameters $\lambda=8^{(2/3)^k}$, $k=1,2,\ldots,$ and so for arbitrary $\lambda>1$. □

A wide class of multiplicative systems was recently introduced by Rubinshtein [17], who has shown that the system $\phi(2^kx)$ on $[0,1)$ is multiplicative whenever ϕ is 1-periodic function on the real line and on $[0,1)$ it can be written in the form

$$\phi(x) = \begin{cases} f(x) \text{ if } & x \in [0,1/4), \\ f(1/2-x) \text{ if } & x \in [1/4,1/2), \\ f(x-1/2) \text{ if } & [1/2,3/4), \\ f(1-x) \text{ if } & [3/4,1), \end{cases} \tag{4.8}$$

for some $f \in L^\infty[0,1/4)$. Thus, from Corollary 1.1 and Corollary 1.4 it follows that

Corollary 4.4. *For any random variable ϕ of the form (4.8) the sequence $\phi_k(x) = \phi(2^k x)$ satisfies inequalities (1.5) and (1.8) (with $A_k = -1, B_k = 1$).*

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