Asymptotic Errors of Accelerated Two-dimensional Trigonometric Approximations

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Abstract

Here the Bernoulli polynomials are exploited for acceleration of convergence of two-dimensional Fourier series and periodic trigonometric interpolations. The principal term of asymptotic expansion of L_2- error is revealed.

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1 Introduction

Approximations of smooth but non periodic functions on the finite interval $[a, b]$ $(f(a) \neq f(b))$ by the partial sums of Fourier expansion or by periodic trigonometric interpolation are non efficient due to slow L_2-convergence and essential influence of the Gibbs phenomenon at the end-points of the interval. For corresponding trigonometric approximation the acceleration of convergence can be reached by the method, based on application of Bernoulli polynomials ([1]-[3]). The efficiency of the numerical realization of this method is based on the calculation of the jumps $f^{(k)}(b) - f^{(k)}(a)$ $(k = 0, 1, \cdots)$ by the Fourier (discrete Fourier) coefficients.

In [4] this approach was generalized for the functions of two variables. The main difficulty was the calculation of jumps since we were not only computing the jumps at the tops but also at the edges of the square.

In [5] L_2-convergence of similar polynomial-periodic approximations of sufficiently smooth on [-1,1] functions by translates of a fixed periodic functions was considered and exact formulae for the principal term of the corresponding asymptotic expansions of errors were obtained.

Here we obtain exact asymptotic formulae for the principal term of L_2-error for approximations derived in [4].

2 Notation and Definitions

We put

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}dx,\ f_{n,m} = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} f(x,y)e^{-i\pi(nx+my)}dxdy,$$

$$\check{f}_{n,m} = \frac{1}{(2N+1)^2}\sum_{k,s=-N}^{N} f(x_k,x_s)e^{-i\pi(nx_k+mx_s)},\ x_k = \frac{2k}{2N+1}.$$

The Bernoulli polynomials on the interval $[-1,1]$ are determined from the following recurrence relations

$$B_0(x) = x/2, \quad B_k(x) = \int B_{k-1}(x)\,dx, \quad \int_{-1}^{1} B_k(x)\,dx = 0, \quad k = 1,2,\cdots.$$

The Fourier coefficients of B_k have the form

$$(B_k)_n = \begin{cases} 0, & n = 0 \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n = \pm 1, \pm 2, \cdots \end{cases} \tag{1}$$

Now denote

$$f^{(k,s)}(x,y) = \frac{\partial^{k+s}}{\partial x^k \partial y^s} f(x,y),$$

$$u_k(y) = f^{(k+1,k+1)}(1,y) - f^{(k+1,k+1)}(-1,y),$$

$$v_k(x) = f^{(k+1,k+1)}(x,1) - f^{(k+1,k+1)}(x,-1)$$

and

$$\widetilde{u}_k(y) = \int_{-1}^{1} B_k(t)u_k(y-t)dt,\ \widetilde{v}_k(x) = \frac{1}{2}\int_{-1}^{1} B_k(t)v_k(x-t)dt. \tag{2}$$

According to well-known Parseval's equality (see [6], Chapter 2, §1) we have

$$\int_{-1}^{1} |\widetilde{u}_k(y)|^2 dy = \frac{2}{\pi^{2k+2}} \sum_{n\in\mathbf{Z}}{}' \frac{|(u_k)_n|^2}{n^{2k+2}} \tag{3}$$

and

$$\int_{-1}^{1} |\widetilde{v}_k(x)|^2 dx = \frac{2}{\pi^{2k+2}} \sum_{n\in\mathbf{Z}}{}' \frac{|(v_k)_n|^2}{n^{2k+2}}. \tag{4}$$

Let $\mathbf{Z}$ be the set of integers and consider the following functions defined on the interval $[-1/2, 1/2]$[1]

$$\Phi_k(x) = \sum_{s\in\mathbf{Z}}{}' \frac{(-1)^s}{(x+s)^{k+1}}, \ k = 0, 1, \cdots. \tag{5}$$

It is easy to check that

$$\Phi_k(x) = (-1)^k \frac{\Phi_0^{(k)}(x)}{k!}, \ \Phi_0(x) = \frac{\pi x - \sin \pi x}{x \sin \pi x}. \tag{6}$$

Finally by $||\cdot||$ we denote the standard norm of the space $L_2(-1,1)$.

3 Acceleration of Trigonometric Approximations

Our investigations are based on the following

Lemma [4]. *For any $f \in C^{2q+2}([-1,1]\times[-1,1])$, $q \geq -1$ holds $(n, m \neq 0)$*

$$f_{n,m} = \frac{(-1)^{n+1}}{4} \sum_{k=0}^{q} \frac{1}{(i\pi n)^{k+1}} \int_{-1}^{1} \left(f^{(k,0)}(1,t) - f^{(k,0)}(-1,t)\right) e^{-i\pi m t} dt +$$

$$+ \frac{(-1)^{m+1}}{4} \sum_{s=0}^{q} \frac{1}{(i\pi m)^{s+1}} \int_{-1}^{1} \left(f^{(0,s)}(t,1) - f^{(0,s)}(t,-1)\right) e^{-i\pi n t} dt -$$

$$- \frac{(-1)^{n+m}}{4} \sum_{s=0}^{q} \sum_{k=0}^{q} \frac{\Delta_{k,s}}{(i\pi n)^{k+1}(i\pi m)^{s+1}} +$$

[1] Here and below the prime on summation indicates that the zero term is omitted.

$$+\frac{1}{4(i\pi n)^{q+1}(i\pi m)^{q+1}}\int_{-1}^{1}\int_{-1}^{1} f^{(q+1,q+1)}(t,z)e^{-i\pi nt}e^{-i\pi mz}dzdt \tag{7}$$

where

$$\Delta_{k,s} = f^{(k,s)}(1,1) - f^{(k,s)}(-1,1) - f^{(k,s)}(1,-1) + f^{(k,s)}(-1,-1).$$

Proof can be done easily using integration by parts for variables x and y consequently in $f_{n,m}$.•

Application of (7) with (1) leads to the following expansion

$$f(x,y) = G(x,y) + F(x,y) \tag{8}$$

where $F \in C^{2q+2}([-1,1]\times[-1,1])$ is some 2-periodic function and

$$G(x,y) = \sum_{k=0}^{q} B_k(x)\left(f^{(k,0)}(1,y) - f^{(k,0)}(-1,y)\right) +$$

$$+\sum_{s=0}^{q} B_s(y)\left(f^{(0,s)}(x,1) - f^{(0,s)}(x,-1)\right) - \sum_{s=0}^{q}\sum_{k=0}^{q}\Delta_{ks}B_k(x)B_s(y) -$$

$$-\frac{1}{2}\sum_{k=0}^{q} B_k(x)\int_{-1}^{1}\left(f^{(k,0)}(1,t) - f^{(k,0)}(-1,t)\right)dt -$$

$$-\frac{1}{2}\sum_{s=0}^{q} B_s(y)\int_{-1}^{1}\left(f^{(0,s)}(t,1) - f^{(0,s)}(t,-1)\right)dt +$$

$$+\frac{1}{2}\int_{-1}^{1} f(t,y)dt + \frac{1}{2}\int_{-1}^{1} f(x,t)dt. \tag{9}$$

Now consider the following $q-$*accelerated decomposition*

$$S_N^q(f) = G(x,y) + \sum_{n,m=-N}^{N}(f_{n,m} - G_{n,m})e^{i\pi(nx+my)} \tag{10}$$

and $q-$*accelerated interpolation*

$$I_N^q(f) = G(x,y) + \sum_{n,m=-N}^{N}(\check{f}_{n,m} - \check{G}_{n,m})e^{i\pi(nx+my)}. \tag{11}$$

that correspond to the expansion (8).

4 Asymptotic L_2- errors of $q-$accelerated decomposition and interpolation

First consider the approximation $S_N^q(f)$ for $q=-1$ which is the partial sum of Fourier expansion.

Theorem 1. If $f \in C^1([-1,1]\times[-1,1])$ then for $q=-1$

$$\lim_{N\to\infty}(2N+1)||f-S_N^q(f)||^2 = K_{dec}(q)\left(\int_{-1}^{1}|u_q(x)|^2+|v_q(x)|^2\,dx\right) \tag{12}$$

where

$$K_{dec}(q)=\frac{2}{\pi^2}. \tag{13}$$

Proof. Simple calculations lead to the following equality (here and below in this proof $q=-1$)

$$||f-S_N^q(f)||^2 = I_1+I_2+I_3 \tag{14}$$

where

$$I_1 = 4\sum_{n,m=-N}^{N}\sum_{s\in\mathbf{Z}}{}'|f_{n+s(2N+1),m}|^2,\ I_2 = 4\sum_{n,m=-N}^{N}\sum_{r\in\mathbf{Z}}{}'|f_{n,m+r(2N+1)}|^2,$$

$$I_3 = 4\sum_{n,m=-N}^{N}\sum_{s\in\mathbf{Z}}{}'\sum_{r\in\mathbf{Z}}{}'|f_{n+s(2N+1),m+r(2N+1)}|^2.$$

For estimation of I_1 note that

$$f_{n,m}=J^1_{n,m}+J^2_{n,m} \tag{15}$$

where $(n\neq 0)$

$$J^1_{n,m}=\frac{(-1)^{n+1}(u_q)_m}{2i\pi n},$$

$$J^2_{n,m}=\frac{\varepsilon_{n,m}}{4i\pi n},\quad \varepsilon_{n,m}=\int_{-1}^{1}\int_{-1}^{1}f^{(1,0)}(x,y)e^{-i\pi(nx+my)}dxdy.$$

Since

$$\sum_{n,m=-N}^{N}\sum_{s\in\mathbf{Z}}{}'\left|J^2_{n+s(2N+1),m}\right|^2 \le \frac{Const}{N^2}\sum_{n,m=-N}^{N}\sum_{s\in\mathbf{Z}}{}'\frac{|\varepsilon_{n+s(2N+1),m}|^2}{|s-1/2|^2}\le$$

$$\frac{Const}{N^2}\sum_{m=-N}^{N}\sum_{|n|>N}|\varepsilon_{n,m}|^2=\frac{o(1)}{N^2},\ N\to\infty$$

then taking into account the well known inequality

$$||a||-||b||\leq||a+b||\leq||a||+||b||$$

we obtain

$$\lim_{N\to\infty}(2N+1)I_1=$$

$$4\lim_{N\to\infty}\sum_{n,m=-N}^{N}\sum_{s\in\mathbf{Z}}{}'\left|J^1_{n+s(2N+1),m}\right|^2=\frac{4}{\pi^2}\sum_{n\in\mathbf{Z}}|(u_q)_n|^2\,.$$

Similarly we have

$$\lim_{N\to\infty}(2N+1)I_2=\frac{4}{\pi^2}\sum_{n\in\mathbf{Z}}|(v_q)_n|^2$$

and

$$\lim_{N\to\infty}(2N+1)I_3=0.$$

Finally

$$\lim_{N\to\infty}||f-S^q_N(f)||^2=\frac{4}{\pi^2}\sum_{n\in\mathbf{Z}}\left(|(u_q)_n|^2+|(v_q)_n|^2\right).$$

Application of Parseval's equality completes the proof.•

Now consider the case $q\geq 0$.

Theorem 2. *If* $f\in C^{2q+3}([-1,1]\times[-1,1])$, $q\geq 0$ *then*

$$\lim_{N\to\infty}(2N+1)^{2q+3}||f-S^q_N(f)||^2=$$

$$=K_{dec}(q)\int_{-1}^{1}\left(|\tilde{u}_q(x)|^2+|\tilde{v}_q(x)|^2\right)dx \qquad (16)$$

where

$$K_{dec}(q)=\frac{2^{2q+3}}{(2q+3)\pi^{2q+4}}. \qquad (17)$$

Proof. According (8)

$$||f-S^q_N(f)||=||F-S^r_N(F)||,\ r=-1.$$

Besides

$$F_{0,m} = 0,\ \ m \neq 0,\ \ F_{n,0} = 0,\ \ n \neq 0.$$

Hence, from (14) with similar conclusions we obtain

$$\lim_{N\to\infty} (2N+1)^{2q+3} ||f - S_N^q(f)||^2 = \frac{2^{2q+4}}{\pi^{4q+6}(2q+3)} \sum_{m\in\mathbf{Z}}{}' \frac{|(u_q)_m|^2 + |(v_q)_m|^2}{m^{2q+2}}.$$

The remaining follows from (2) − (4). •

Consider now the case $q = -1$ for $q-$accelerated interpolation which is the classic trigonometric interpolation.

Theorem 3. *If* $f \in C^1([-1,1] \times [-1,1])$ *then for* $q = -1$

$$\lim_{N\to\infty} (2N+1) ||f - I_N^q(f)||^2 = K_{int}(q) \int_{-1}^{1} \left(|u_q(x)|^2 + |v_q(x)|^2 \right) dx \tag{18}$$

where

$$K_{int}(q) = K_{dec}(q) + \frac{1}{2\pi^2} \int_{-1/2}^{1/2} \left| \Phi_)(x) \right|^2 dx. \tag{19}$$

Proof. Taking into account the easy derived relation between Fourier and discrete Fourier coefficients

$$\check{f}_{n,m} = \sum_{r,s\in\mathbf{Z}} f_{n+r(2N+1),m+s(2N+1)}$$

we obtain ($q = -1$)

$$||f - I_N^q(f)||^2 = I_1 + I_2 + I_3 + I_4$$

where I_1, I_2, I_3 are the same as in the proof of Theorem 1 with the same estimations and

$$I_4 = 4 \sum_{n,m=-N}^{N} |I_{41} + I_{42} + I_{43}|^2$$

with

$$I_{41} = \sum_{p\in\mathbf{Z}}{}' f_{n+p(2N+1),m}, \qquad I_{42} = \sum_{k\in\mathbf{Z}}{}' f_{n,m+k(2N+1)}$$

$$I_{43} = \sum_{k,p\in\mathbf{Z}}{}' f_{n+p(2N+1),m+k(2N+1)}.$$

For estimation of I_{41} we use (15) and obtain that the second term is of no importance. The same is true for I_{42}. Easy to check that the term I_{43} we can omit. Finally

$$\lim_{N\to\infty}(2N+1)||f-I_N^q(f)||^2=\sum_{n\in\mathbf{Z}}(|(u_q)_n|^2+|(v_q)_n|^2)\times$$

$$\times\left(\frac{4}{\pi^2}+\frac{1}{\pi^2}\int_{-1/2}^{1/2}\left|{\sum_{r\in\mathbf{Z}}}'\frac{(-1)^r}{x+r}\right|^2dx\right).$$

The rest can be obtained from Parseval's equality and from (5), (6).•

Now consider the case $q\geq 0$. As the proof of the following Theorem doesn't contain any new idea we omit it.

Theorem 4. *If $f\in C^{2q+3}([-1,1]\times[-1,1])$, $q\geq 0$ then*

$$\lim_{N\to\infty}(2N+1)^{2q+3}||f-I_N^q(f)||^2=K_{int}(q)\int_{-1}^{1}(|\tilde{v}_q(x)|^2+|\tilde{u}_q(x)|^2)dx \quad (20)$$

where

$$K_{int}(q)=K_{dec}(q)+\frac{1}{2\pi^{2q+4}}\int_{-1/2}^{1/2}\left|\Phi_{q+1}(x)\right|^2dx. \quad (21)$$

5 Numerical values

Note (see Theorems 1-4) that the constant $K_{int}(q)$ differs from $K_{dec}(q)$ only by the component

$$\delta_q=\frac{1}{2\pi^{2q+4}}\int_{-1/2}^{1/2}\left|\Phi_{q+1}(x)\right|^2dx \quad (22)$$

In Table below we represent the values of $K_{dec}(q)$ and $K_{int}(q)$ for $-1\leq q\leq 4$.

q	-1	0	1	2	3	4
$K_{dec}(q)$	0.202	0.027	0.0066	0.00192	0.000607	0.00020143
$K_{int}(q)$	0.219	0.056	0.0078	0.00198	0.000608	0.00020145

Table. *Numerical values of $K_{dec}(q)$ and $K_{int}(q)$ for $-1\leq q\leq 4$.*

Hence δ_q is not essential for great values of q.

Remark 1. Up to now we suppose that the function $G(x, y)$ is known. If it doesn't so then the problem of recovering this function by the known $f_{n,m}$ or $\check{f}_{n,m}$ $|n|, |m| \leq N$ arises. This problem was successfully solved in [4] by expanding all functions from one variable in $G(x, y)$ into polynomial-periodic approximation with the same idea as in [1], [2]. In that case approximaations $S_N^q(f)$ and $I_N^q(f)$ will be given as polynomial-periodic approximations.

Remark 2. Note that without any complexity we can generalize Lemma (and hence the results of this paper) assuming that $f \in C^{q+1}$ by x and $f \in C^{p+1}$ by y. Besides it is possible to generalize (10) and (11) assuming that $|n| \leq N$ and $|m| \leq M$ in the corresponding summations and $N, M \to \infty$.

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