ON A POINTWISE CONVERGENCE OF THE QUASI-PERIODIC TRIGONOMETRIC INTERPOLATION

LUSINE POGHOSYAN AND ARNAK POGHOSYAN

Abstract. The paper investigates the pointwise convergence of the quasi-periodic trigonometric interpolation and derives exact constant of the main term of asymptotic error for smooth functions.

1. Introduction

We continue investigation of the quasi-periodic (QP-) interpolation $I_{N,m}(f,x)$, $m \geq 0$ (m is integer), $x \in [-1,1]$, which interpolates f on equidistant grid

$$x_k = \frac{k}{N}, \ |k| \leq N \tag{1}$$

and is exact for the following set of quasi-periodic functions

$$e^{i\pi n\sigma x}, \ |n| \leq N, \ \sigma = \frac{2N}{2N+m+1} \tag{2}$$

with period $2/\sigma$ which tends to 2 as $N \to \infty$.

The idea of the QP-interpolation was suggested in [2]. Papers [7] and [8] considered the $L_2(-1,1)$-convergence of the QP-interpolation and its behavior at the endpoints $x = \pm 1$ in terms of the limit function. Some results concerning the convergence properties were presented also in [5] and [6].

Here, we explore the pointwise convergence of the QP-interpolation on $(-1,1)$ and obtain exact constant of the main term of asymptotic error. Some results of this research are reported also in [9].

2. The quasi-periodic interpolation

Let us clarify the basic requirements for derivation of the QP-interpolation (see (1) and (2)). Consider a new function $f^*(t)$ defined on $[-\sigma,\sigma]$ by the following change of variable

$$f^*(t) = f\left(\frac{t}{\sigma}\right) = f(x), \ x \in [-1,1], \ t \in [-\sigma,\sigma], \ t = \sigma x.$$

This implies interpolation of $f^*(t)$ on grid

$$t_k = \sigma x_k = \frac{2k}{2N+m+1}, \ |k| \leq N,$$

2010 *Mathematics Subject Classification.* 42A15.

Key words and phrases. Trigonometric interpolation, quasi-periodic interpolation, pointwise convergence.

while interpolating $f(x)$ on grid (1). Thus, the QP-interpolation actually interpolates $f^*(t)$ on grid t_k and is exact for $e^{i\pi nt}$, $|n| \le N$. It is important to note that for $m > 0$, the 2-periodic extension of grid t_k to the real line is non-uniform as

$$t_k - t_{k-1} = \frac{2}{2N+m+1} = h, \ k = -N+1, \dots, \ N,$$

while

$$1 - t_N + t_{-N} - (-1) = (m+1)\frac{2}{2N+m+1} \ne h.$$

This non-uniformity is the main reason of faster ($O(N^{-q-m-1})$) pointwise convergence (see Theorem 1) of the QP-interpolation compared to the convergence ($O(N^{-q-1})$ for even q or $O(N^{-q-2})$ for odd q) of the classical trigonometric interpolation realized by uniform grids (see [3]). Wee see that as bigger is the value of m as more dense are the nodes on $(-1, 1)$ and, as a result, is higher the accuracy. It is also worth noting that f^* depends on N and in convergence theorems it must be taken into account and although $f^* \to f$ as $N \to \infty$ but this dependence essentially changes interpolation properties.

First, let $m = 0$. In this case, grid t_k is uniform. Hence, the QP-interpolation $I_{N,0}(f, x)$ of f is the classical trigonometric interpolation $I_N^*(f^*, t)$ of f^* on uniform grid $2k/(2N+1)$, $|k| \le N$ and thus

$$\begin{aligned} I_N^*(f^*, t) &= \sum_{n=-N}^{N} \left(\frac{1}{2N+1} \sum_{k=-N}^{N} f^* \left(\frac{2k}{2N+1} \right) e^{-i\pi n \frac{2k}{2N+1}} \right) e^{i\pi nt} \\ &= \sum_{n=-N}^{N} \left(\frac{1}{2N+1} \sum_{k=-N}^{N} f \left(\frac{k}{N} \right) e^{-i\pi n\sigma \frac{k}{N}} \right) e^{i\pi n\sigma x} = I_{N,0}(f, x), \end{aligned}$$

where $t = \sigma x$. Theorem 2 explores the pointwise convergence of $I_{N,0}$ on $(-1, 1)$ and derives the exact constant of the main term of asymptotic error.

Now, let $m > 0$. Taking into account the above remarks, we write

$$I_{N,m}(f, x) = \sum_{k=-N}^{N} f \left(\frac{k}{N} \right) c_k(x), \tag{3}$$

where c_k are some unknowns to be determined. As (3) is exact for $e^{i\pi n\sigma x}$, we get the following system of linear equations for determination of the unknowns

$$e^{i\pi n\sigma x} = \sum_{k=-N}^{N} e^{i\pi n\sigma \frac{k}{N}} c_k(x), \ |n| \le N.$$

From here, we get (see details in [7])

$$\begin{aligned} c_k(x) = \frac{1}{2N+m+1} &\left(\sum_{\ell=-N}^{N} e^{\frac{2i\pi\ell(Nx-k)}{2N+m+1}} \right. \\ &\left. - \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(N+m-k)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} \sum_{j=-N}^{N} e^{\frac{2i\pi j(Nx+s-N-m-1)}{2N+m+1}} \right), \end{aligned}$$

where $v_{\ell,s}^{-1}$ are the elements of the inverse of the following Vandermonde matrix

$$v_{s,\ell} = \alpha_\ell^{s-1}, \ \alpha_\ell = e^{\frac{2i\pi(\ell+N)}{2N+m+1}}, \ s, \ell = 1, \dots, m, \tag{4}$$

and have the following explicit form (see [1])

$$v_{\ell,s}^{-1} = -\frac{1}{\alpha_\ell^s \prod\limits_{\substack{k=1\\k\neq\ell}}^{m} (\alpha_\ell - \alpha_k)} \sum_{j=0}^{s-1} \gamma_j \alpha_\ell^j, \ \ \ell, s = 1, ..., m. \tag{5}$$

Here, γ_j are the coefficients of the following polynomial

$$\prod_{j=1}^{m} (x - \alpha_j) = \sum_{j=0}^{m} \gamma_j x^j.$$

Explicit expression of c_k leads to the following explicit representation of the QP-interpolation

$$I_{N,m}(f, x) = \sum_{n=-N}^{N} F_{n,m} e^{i\pi n \sigma x},$$

where

$$F_{n,m} = \check{f}_{n,m} - \sum_{\ell=1}^{m} \theta_{n,\ell} \check{f}_{\ell+N,m}, \tag{6}$$

$$\check{f}_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-\frac{2i\pi nk}{2N+m+1}}$$

and

$$\theta_{n,\ell} = e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi n(s-N-m-1)}{2N+m+1}}. \tag{7}$$

From (4), (6) and (7), it follows that

$$F_{N+k,m} = 0, \ F_{-N-k,m} = 0, \ k = 1, ..., m. \tag{8}$$

Then, taking into account that $\alpha_s - \alpha_i = O(1/N)$, we get from (5)

$$v_{\ell,s}^{-1} = O\left(N^{m-1}\right), \ N \to \infty, \tag{9}$$

and

$$\theta_{n,\ell} = O\left(N^{m-1}\right), \ N \to \infty. \tag{10}$$

We denote by $R_{N,m}$ the error of the QP-interpolation

$$R_{N,m}(f, x) = f(x) - I_{N,m}(f, x).$$

3. Convergence analysis

Let $f \in C^q[-1,1]$ and

$$A_{sk}(f) = f^{(k)}(1) - (-1)^{k+s} f^{(k)}(-1), \ k = 0, \dots, q.$$

We denote by f_n the n-th Fourier coefficient of f

$$f_n = \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi n x} dx.$$

Let

$$\delta_n^p \left(\{f_s\}_{s=-\infty}^{\infty} \right) = \delta_n^p \left(\{f_s\} \right) = \sum_{k=0}^{2p} \binom{2p}{k} f_{n+p-k}.$$

First, we prove some lemmas.

Lemma 1. *The following estimate holds for $|n| \le N$ as $N \to \infty$*

$$\delta_n^p \left(\left\{ (-1)^s e^{\frac{i\pi\beta s}{2N+m+1}} \right\}_{s=-\infty}^{\infty} \right) = \frac{(-1)^n (\pi\beta)^{2p}}{(2N+m+1)^{2p}} e^{\frac{i\pi\beta n}{2N+m+1}} + O\left(N^{-2p-1}\right), \tag{11}$$

where β is a constant and $p > 0$.

Proof. According to definition of $\delta_n^p(\cdot)$, we have

$$\begin{aligned}
\delta_n^p \left(\left\{ (-1)^s e^{\frac{i\pi\beta s}{2N+m+1}} \right\} \right) &= (-1)^{n+p} e^{\frac{i\pi\beta(n+p)}{2N+m+1}} \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k e^{-\frac{i\pi\beta k}{2N+m+1}} \\
&= (-1)^{n+p} e^{\frac{i\pi\beta(n+p)}{2N+m+1}} \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \sum_{t=0}^{\infty} \frac{(i\pi\beta)^t (-1)^t k^t}{t!(2N+m+1)^t} \\
&= (-1)^{n+p} e^{\frac{i\pi\beta(n+p)}{2N+m+1}} \sum_{t=0}^{\infty} \frac{(-1)^t (i\pi\beta)^t}{t!(2N+m+1)^t} \omega_{2p,t},
\end{aligned}$$

where

$$\omega_{p,t} = \sum_{k=0}^{p} \binom{p}{k} (-1)^k k^t \sim p^t, \ t \to \infty$$

and (see [10])

$$\omega_{p,t} = 0, \ 0 \le t \le p-1, \ \omega_{p,p} = (-1)^p \, p!.$$

These complete the proof. □

Let $f \in C^{q+2m}[-1,1]$. We denote

$$f^*(x) = \begin{cases} f_{left}(x), & x \in [-1, -\sigma), \\ f\left(\frac{x}{\sigma}\right), & x \in [-\sigma, \sigma], \\ f_{right}(x), & x \in (\sigma, 1], \end{cases} \tag{12}$$

where

$$f_{left}(x) = \sum_{j=0}^{q+2m} \frac{f^{(j)}(-1)}{j!} \left(\frac{x}{\sigma} + 1\right)^j, \ f_{right}(x) = \sum_{j=0}^{q+2m} \frac{f^{(j)}(1)}{j!} \left(\frac{x}{\sigma} - 1\right)^j.$$

Let

$$B_n(k) = \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}.$$

Lemma 2. [8] *Let* $f^{(q+2m)} \in AC[-1,1]$ *for some* $m \geq 1$, $q \geq 0$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \ldots, q-1.$$

Then, the following estimate holds for $n, N \to \infty$

$$f_n^* = \sum_{j=q}^{q+2m} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}(2N+m+1)^k}{(j-k)!} B_n(k) + o(n^{-q-2m-1}). \tag{13}$$

Let

$$\Phi_{k,m}(e^{i\pi x}) = e^{\frac{i\pi}{2}(m-1)x} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}}.$$

Lemma 3. *Let* $f^{(q+2m)} \in AC[-1,1]$ *for some* $m \geq 1$, $q \geq 0$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \ldots, q-1.$$

Then, the following estimate holds as $N \to \infty$ *and* $|n| \leq N + 2m$

$$\begin{aligned} F_{n,m} - f_n^* &= \frac{(-1)^{n+1}}{2N+m+1} \sum_{j=q}^{q+m+1} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \left(\sum_{r\neq 0} \frac{(-1)^{r(m+1)}}{\left(2r + \frac{2n}{2N+m+1}\right)^{k+1}} - e^{-\frac{i\pi(m-1)n}{2N+m+1}} \sum_{\tau=0}^{m-1} \frac{\Phi_{k,m}^{(\tau)}(-1)}{\tau!} \left(e^{\frac{2i\pi n}{2N+m+1}} + 1\right)^{\tau} \right. \\ &\left. - e^{-\frac{i\pi(m-1)n}{2N+m+1}} \sum_{\tau=m}^{q-j+2m} \frac{\Phi_{k,m}^{(\tau)}(-1)}{\tau!} \sum_{\ell=1}^{m} \left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}} + 1\right)^{\tau} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2\imath\pi n(s-1)}{2N+m+1}} \right) \\ &+ o\left(N^{-q-m-2}\right). \end{aligned} \tag{14}$$

Proof. We have (details see in [8])

$$F_{n,m} = \sum_{r=-\infty}^{\infty} f_{n+r(2N+m+1)}^* - \sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f_{N+\ell+r(2N+m+1)}^*, \ \ n \in \mathbb{Z} \tag{15}$$

which shows that

$$F_{n,m} - f_n^* = \sum_{r\neq 0} f_{n+r(2N+m+1)}^* - \sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f_{N+\ell+r(2N+m+1)}^*. \tag{16}$$

Now, for $|n| \leq N + 2m$, according to Lemma 2, equations (7) and (10), we get

$$\begin{aligned} \sum_{r\neq 0} f_{n+r(2N+m+1)}^* &= \frac{(-1)^{n+1}}{2N+m+1} \sum_{j=q}^{q+2m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \sum_{r\neq 0} \frac{(-1)^{r(m+1)}}{\left(2r + \frac{2n}{2N+m+1}\right)^{k+1}} + o\left(N^{-q-2m-1}\right), \end{aligned} \tag{17}$$

and

$$\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty} f^{*}_{N+\ell+r(2N+m+1)} = \frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+2m}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}$$
$$\times e^{-\frac{i\pi(m-1)n}{2N+m+1}}\sum_{\ell=1}^{m}\Phi_{k,m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}\right)\sum_{s=1}^{m} v^{-1}_{\ell,s} e^{\frac{2\imath\pi n(s-1)}{2N+m+1}} + o\left(N^{-q-m-2}\right).$$

Then, by the Taylor expansion

$$\Phi_{k,m}\left(e^{\frac{2\imath\pi(N+\ell)}{2N+m+1}}\right) = \sum_{\tau=0}^{2m}\frac{\Phi^{(\tau)}_{k,m}(-1)}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau} + O(N^{-2m-1}),$$

we derive

$$\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty} f^{*}_{N+\ell+r(2N+m+1)} = \frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+2m}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}e^{-\frac{i\pi(m-1)n}{2N+m+1}}$$
$$\times \sum_{\tau=0}^{2m}\frac{\Phi^{(\tau)}_{k,m}(-1)}{\tau!}\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=1}^{m} v^{-1}_{\ell,s} e^{\frac{2\imath\pi n(s-1)}{2N+m+1}} + o\left(N^{-q-m-2}\right).$$

Finally, taking into account the following relations

$$\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=0}^{m-1} v^{-1}_{\ell,s+1} e^{\frac{2\imath\pi ns}{2N+m+1}} = \left(e^{\frac{2i\pi n}{2N+m+1}}+1\right)^{\tau},\ \tau=0,\ldots,m-1,$$

we get

$$\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty} f^{*}_{N+\ell+r(2N+m+1)} = \frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+m+1}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}e^{-\frac{i\pi(m-1)n}{2N+m+1}}$$
$$\times\left(\sum_{\tau=0}^{m-1}\frac{\Phi^{(\tau)}_{k,m}(-1)}{\tau!}\left(e^{\frac{2i\pi n}{2N+m+1}}+1\right)^{\tau}\right.$$
$$\left.+\sum_{\tau=m}^{q-j+2m}\frac{\Phi^{(\tau)}_{k,m}(-1)}{\tau!}\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=1}^{m} v^{-1}_{\ell,s} e^{\frac{2\imath\pi n(s-1)}{2N+m+1}}\right) + o\left(N^{-q-m-2}\right).$$

Substituting this and (17) into (16), we get the required. □

Lemma 4. *Let* $f^{(q+2m)} \in AC[-1,1]$ *for some* $m \ge 1$, $q \ge 0$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0,\ k = 0,...,q-1.$$

Then, the following estimates hold as $N \to \infty$

$$F_{N-p,m} = C_{q,m}(f)\frac{(-1)^{N+p+1}}{N^{q+m+1}}\binom{m+p}{m} + O\left(N^{-q-m-2}\right),\ p \ge 0, \tag{18}$$

and

$$F_{-N+p,m} = -F_{N-p,m} + O\left(N^{-q-m-2}\right),\ p \ge 0, \tag{19}$$

where

$$C_{q,m}(f) = \sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k+1}i^k\pi^{k-m+1}(q-k)!}\Phi^{(m)}_{k,m}(-1). \tag{20}$$

Proof. We have from (15)

$$F_{N-p,m} = \sum_{r=-\infty}^{\infty} f^*_{N-p+r(2N+m+1)} - \sum_{\ell=1}^{m} \theta_{N-p,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)}.$$

We write in view of Lemma 2 and (10)

$$F_{N-p,m} = \frac{(-1)^{N+1}}{2N+m+1} \sum_{j=q}^{q+2m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}$$
$$\times \left((-1)^p \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{\left(2r + \frac{2(N-p)}{2N+m+1}\right)^{k+1}} - \sum_{\ell=1}^{m} (-1)^{\ell} \theta_{N-p,\ell} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{\left(2r + \frac{2(N+\ell)}{2N+m+1}\right)^{k+1}} \right)$$
$$+ o\left(N^{-q-m-2}\right).$$

According to (7), we obtain

$$\begin{aligned} F_{N-p,m} &= \frac{(-1)^{N+p+1}}{2N+m+1} \sum_{j=q}^{q+2m} \frac{e^{-\frac{i\pi(m-1)(N-p)}{2N+m+1}}}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \left(\Phi_{k,m}\left(e^{\frac{2i\pi(N-p)}{2N+m+1}}\right) - \sum_{\ell=1}^{m} \Phi_{k,m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}\right) \sum_{s=0}^{m-1} v^{-1}_{\ell,s+1} e^{\frac{2i\pi(N-p)s}{2N+m+1}} \right) \\ &+ o\left(N^{-q-m-2}\right). \end{aligned} \tag{21}$$

Now, we simplify the expression in the brackets which we denote by S (see also (4))

$$\begin{aligned} S &= \Phi_{k,m}\left(\alpha_{-p}\right) - \sum_{\ell=1}^{m} \Phi_{k,m}\left(\alpha_{\ell}\right) \sum_{s=0}^{m-1} v^{-1}_{\ell,s+1} \alpha^s_{-p} \\ &= \sum_{j=1}^{m} \underset{z=\alpha_j}{res} \frac{\omega\left(\alpha_{-p}\right)\Phi_{k,m}(z)}{\omega(z)\left(z-\alpha_{-p}\right)} + \underset{z=\alpha_{-p}}{res} \frac{\omega\left(\alpha_{-p}\right)\Phi_{k,m}(z)}{\omega(z)\left(z-\alpha_{-p}\right)}, \end{aligned}$$

where $\omega(z) = \prod_{\ell=1}^{m}(z-\alpha_\ell)$. Hence

$$S = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega\left(\alpha_{-p}\right)\Phi_{k,m}(z)}{\omega(z)\left(z-\alpha_{-p}\right)} dz,$$

where Γ contains the points $\{\alpha_\ell\}_{\ell=1}^{m}$ and α_{-p}. Then, we get

$$\begin{aligned} S &= \frac{(i\pi)^m}{N^m 2\pi i} \frac{(m+p)!}{p!} \int_{\Gamma} \frac{\Phi_{k,m}(z)}{(z+1)^{m+1}} dz + O\left(N^{-m-1}\right) \\ &= \frac{(i\pi)^m \Phi^{(m)}_{k,m}(-1)}{N^m} \binom{m+p}{m} + O\left(N^{-m-1}\right). \end{aligned}$$

Substituting this into (21), we get the first statement. The second one can be proved similarly. □

Next theorems present the main results of this paper.

Theorem 1. *Let $f^{(q+2m)} \in AC[-1,1]$ for some $m \geq 1$, $q \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \dots, q-1.$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$\begin{aligned} R_{N,m}(f,x) = iC_{q,m}(f)\frac{(-1)^N}{N^{q+m+1}} \Bigg[\sin\left(\pi(N+1)\sigma x\right) \sum_{k=0}^{\acute{m}} \binom{m-k}{k} \frac{(-1)^k}{2^{2k+1}\cos^{2k+2}\frac{\pi x}{2}} \\ - \sin\left(\pi N \sigma x\right) \sum_{k=0}^{\acute{m}-1} \binom{m-k-2}{k} \frac{(-1)^k}{2^{2k+3}\cos^{2k+4}\frac{\pi x}{2}} \Bigg] + o(N^{-q-m-1}), \end{aligned} \tag{22}$$

where $\acute{m} = \left[\frac{m}{2}\right]$ and $C_{q,m}(f)$ is defined by (20).

Proof. According to definition of f^* (see (12)), we can write for fixed N

$$f^*(x) = \sum_{n=-\infty}^{\infty} f_n^* e^{i\pi n x}, \ \ x \in (-1,1).$$

Hence,

$$f(x) = \sum_{n=-\infty}^{\infty} f_n^* e^{i\pi n \sigma x}, \ \ x \in [-1,1].$$

Therefore,

$$R_{N,m}(f,x) = \sum_{n=-N}^{N} (f_n^* - F_{n,m}) e^{i\pi n\sigma x} + \sum_{|n|>N} f_n^* e^{i\pi n \sigma x}.$$

The following expansion of the error is easy to verify (see also [4] with similar transformations)

$$\begin{aligned} R_{N,m}(f,x) = e^{i\pi N\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{N+1}^k(\{F_{n,m}\})}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} \\ - e^{i\pi(N+1)\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{N}^k(\{F_{n,m}\})}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} \\ + e^{-i\pi N\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{-N-1}^k(\{F_{n,m}\})}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} \\ - e^{-i\pi(N+1)\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{-N}^k(\{F_{n,m}\})}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} + r_{N,m}(f,x), \end{aligned} \tag{23}$$

where

$$\begin{aligned} r_{N,m}(f,x) = \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+1}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+1}} \sum_{n=-N}^{N} \delta_n^{\acute{m}+1}(\{f_s^* - F_{s,m}\}) e^{i\pi\sigma n x} \\ + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+1}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+1}} \sum_{|n|>N} \delta_n^{\acute{m}+1}(\{f_s^*\}) e^{i\pi\sigma n x}. \end{aligned}$$

First, we show that

$$r_{N,m}(f,x) = o(N^{-q-m-1}), \ \ N \to \infty, \ \ |x| < 1. \tag{24}$$

Application of similar transformation leads to the following expansion for $r_{N,m}(f,x)$

$$
\begin{aligned}
r_{N,m}(f,x) = {} & \frac{\delta_{-N-1}^{\acute{m}+1}\left(\{F_{n,m}\}\right)e^{-i\pi N\sigma x} - \delta_{N}^{\acute{m}+1}\left(\{F_{n,m}\}\right)e^{i\pi(N+1)\sigma x}}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \\
& + \frac{\delta_{N+1}^{\acute{m}+1}\left(\{F_{n,m}\}\right)e^{i\pi N\sigma x} - \delta_{-N}^{\acute{m}+1}\left(\{F_{n,m}\}\right)e^{-i\pi(N+1)\sigma x}}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \\
& + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \sum_{n=-N}^{N} \delta_n^{\acute{m}+2}(\{f_s^* - F_{s,m}\})e^{i\pi\sigma n x} \\
& + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \sum_{|n|>N} \delta_n^{\acute{m}+2}(\{f_s^*\})e^{i\pi\sigma n x}.
\end{aligned} \tag{25}
$$

According to estimate (13) of Lemma 2, we get

$$
\begin{aligned}
\delta_n^{\acute{m}+2}(\{f_s^*\}) = {} & \sum_{j=q}^{q+2m} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}(2N+m+1)^k}{(j-k)!} \delta_n^{\acute{m}+2}\left(\{B_s(k)\}_{s=-\infty}^{\infty}\right) \\
& + o(n^{-q-2m-1}).
\end{aligned}
$$

We have (see [3])

$$
\delta_n^{\acute{m}+2}(\{B_s(k)\}_{s=-\infty}^{\infty}) = O(n^{-2\acute{m}-k-5}),
$$

and hence,

$$
\delta_n^{\acute{m}+2}(\{f_s^*\}) = O(n^{-2\acute{m}-5}N^{-q}) + o(n^{-q-2m-1}),\ |n| > N,\ N \to \infty.
$$

We see that the last term in the right-hand side of (25) is $o(N^{-q-m-1})$.

Then, according to estimate (14) of Lemma 3, we write

$$
\begin{aligned}
\delta_n^{\acute{m}+2}(\{F_{s,m} - f_s^*\}) = {} & \frac{1}{2N+m+1} \sum_{j=q}^{q+m+1} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!N^j} \\
& \times \Bigg(\frac{(2N+m+1)^{k+1}(i\pi)^{k+1}}{2^k} \delta_n^{\acute{m}+2}\left(\left\{\sum_{r\neq 0} B_{t+r(2N+m+1)}(k)\right\}_{t=-\infty}^{\infty}\right) \\
& - \sum_{s=0}^{m-1} \frac{\Phi_{k,m}^{(s)}(-1)}{s!} \delta_n^{\acute{m}+2}\left(\left\{(-1)^{t+1}\left(e^{\frac{2i\pi t}{2N+m+1}}+1\right)^s e^{-\frac{i\pi(m-1)t}{2N+m+1}}\right\}_{t=-\infty}^{\infty}\right) - \\
& \sum_{\tau=m}^{q-j+2m} \frac{\Phi_{k,m}^{(\tau)}(-1)}{\tau!} \sum_{\ell=1}^{m} \left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau} \sum_{s=0}^{m-1} v_{\ell,s+1}^{-1} \delta_n^{\acute{m}+2}\left(\left\{(-1)^{t+1} e^{\frac{\imath\pi t(2s-m+1)}{2N+m+1}}\right\}_{t=-\infty}^{\infty}\right)\Bigg) \\
& + o(N^{-q-m-2}).
\end{aligned}
$$

In view of the following estimate (see [3])

$$
\delta_n^{\acute{m}+2}\left(\left\{\sum_{r\neq 0} B_{t+r(2N+m+1)}(k)\right\}_{t=-\infty}^{\infty}\right) = O\left(N^{-2\acute{m}-k-5}\right)
$$

and according to Lemma 1 and (9), we have

$$
\delta_n^{\acute{m}+2}\left(\{F_{s,m} - f_s^*\}\right) = o\left(N^{-q-m-2}\right)
$$

and the third term in the right-hand side of (25) is $o(N^{-q-m-1})$.

Now, we estimate the first two terms in the right-hand side of (25). We have

$$\delta_N^{\acute{m}+1}\left(\{F_{n,m}\}\right) = \sum_{k=0}^{2\acute{m}+2} \binom{2\acute{m}+2}{k} F_{N+\acute{m}+1-k,m}.$$

Taking into account (8), we get

$$\delta_N^{\acute{m}+1}\left(\{F_{n,m}\}\right) = \sum_{k=\acute{m}+1}^{2\acute{m}+2} \binom{2\acute{m}+2}{k} F_{N+\acute{m}+1-k,m}.$$

In view of Lemma 4, we derive

$$\delta_N^{\acute{m}+1}\left(\{F_{n,m}\}\right) = C_{q,m}(f)\frac{(-1)^{N+\acute{m}}}{N^{q+m+1}} \sum_{k=\acute{m}+1}^{2\acute{m}+2} (-1)^k \binom{2\acute{m}+2}{k}\binom{m+k-\acute{m}-1}{m} + O\left(N^{-q-m-2}\right).$$

Taking into account the identity (see [10])

$$\sum_{k=\acute{m}+1}^{2\acute{m}+2} (-1)^k \binom{2\acute{m}+2}{k}\binom{m+k-\acute{m}-1}{m} = 0,$$

we conclude that

$$\delta_N^{\acute{m}+1}\left(\{F_{n,m}\}\right) = O\left(N^{-q-m-2}\right).$$

Similarly, we estimate the other terms and see that (24) is true.

Now, we return to the first four terms in the right hand-side of (23) which we denote by I_1, I_2, I_3 and I_4, respectively.

We have for the first term in the right-hand side of (23)

$$\delta_{N+1}^k(\{F_{n,m}\}) = \sum_{s=0}^{2k}\binom{2k}{s}F_{N+1+k-s,m} = \sum_{s=k+1}^{2k}\binom{2k}{s}F_{N+1+k-s,m} = \sum_{s=0}^{k-1}\binom{2k}{s+k+1}F_{N-s,m}.$$

Then,

$$I_1 = e^{i\pi N\sigma x}\sum_{k=0}^{\acute{m}} \frac{\delta_{N+1}^k(\{F_{n,m}\})}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}}$$
$$= e^{i\pi N\sigma x}\sum_{k=0}^{\acute{m}} \frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}}\sum_{s=0}^{k-1}\binom{2k}{s+k+1}F_{N-s,m}.$$

In view of Lemma 4, we get

$$I_1 = C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{i\pi N\sigma x}\sum_{k=1}^{\acute{m}} \frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}}\sum_{s=0}^{k-1}(-1)^s\binom{2k}{s+k+1}\binom{m+s}{m}$$
$$+ O(N^{-q-m-2}).$$

We apply identity (see [10])

$$\sum_{s=0}^{k-1}(-1)^s\binom{2k}{s+k+1}\binom{m+s}{m} = (-1)^{k+1}\binom{m-k-1}{k-1}$$

and obtain

$$I_1 = C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{i\pi N\sigma x}\sum_{k=0}^{\acute{m}-1}\binom{m-k-2}{k}\frac{(-1)^k}{2^{2k+4}\cos^{2k+4}\frac{\pi\sigma x}{2}} + O(N^{-q-m-2}).$$

Similarly, in view of relations (8), we have for the third term in right-hand side of (23) and Lemma 4 that

$$\delta^k_{-N-1}(\{F_{n,m}\}) = \sum_{s=0}^{2k}\binom{2k}{s}F_{-N-1+k-s,m} = \sum_{s=0}^{k-1}\binom{2k}{s}F_{-N-1+k-s,m}$$

$$= \sum_{s=0}^{k-1}\binom{2k}{k-1-s}F_{-N+s,m} = -\sum_{s=0}^{k-1}\binom{2k}{k+s+1}F_{N-s,m} + O(N^{-q-m-2}).$$

Then,

$$I_3 = e^{-i\pi N\sigma x}\sum_{k=0}^{\acute{m}}\frac{\delta^k_{-N-1}(\{F_{n,m}\})}{(1+e^{-i\pi\sigma x})^{k+1}(1+e^{i\pi\sigma x})^{k+1}}$$

$$= -e^{-i\pi N\sigma x}\sum_{k=0}^{\acute{m}}\frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}}\sum_{s=0}^{k-1}\binom{2k}{k+s+1}F_{N-s} + O(N^{-q-m-2})$$

$$= -C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{-i\pi N\sigma x}\sum_{k=0}^{\acute{m}-1}\binom{m-k-2}{k}\frac{(-1)^k}{2^{2k+4}\cos^{2k+4}\frac{\pi\sigma x}{2}} + O(N^{-q-m-2}).$$

Now, we can write

$$I_1 + I_3 = iC_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}\sin(\pi N\sigma x)\sum_{k=0}^{\acute{m}-1}\binom{m-k-2}{k}\frac{(-1)^k}{2^{2k+3}\cos^{2k+4}\frac{\pi\sigma x}{2}} + O(N^{-q-m-2}).$$

Similarly,

$$I_2 + I_4 = -iC_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}\sin(\pi(N+1)\sigma x)\sum_{k=0}^{\acute{m}}\binom{m-k}{k}\frac{(-1)^k}{2^{2k+1}\cos^{2k+2}\frac{\pi\sigma x}{2}} + O(N^{-q-m-2})$$

which completes the proof. □

Similarly, the case $m = 0$ can be considered.

Theorem 2. *Let $f^{(q+1)} \in AC[-1,1]$ for some $q \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \quad k = 0, \ldots, q-1.$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$R_{N,0}(f,x) = A_{0q}(f)\frac{(-1)^N}{2^{q+2}N^{q+1}}\frac{\sin \pi N x}{\cos\frac{\pi x}{2}}\sum_{k=0}^{[q/2]}\frac{(-1)^k}{(q-2k)!\pi^{2k+1}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{\left(s+\frac{1}{2}\right)^{2k+1}} + o(N^{-q-1}). \quad (26)$$

References

[1] I. Gohberg, and V. Olshevsky, *The fast generalized Parker-Traub algorithm for inversion of Vandermonde and related matrices*, J. of Complexity **13**(2)(1997), 208–234.

[2] A. Nersessian, and N. Hovhannesyan, *Quasiperiodic interpolation*, Reports of the National Academy of Sciences of Armenia **101**(2)(2001), 115–121.

[3] A. Poghosyan, *Asymptotic behavior of the Krylov-Lanczos interpolation*, Analysis and Applications **7**(2) (2009), 199–211.

[4] A. Poghosyan, *On a fast convergence of the rational-trigonometric-polynomial interpolation*, Advances in Numerical Analysis, vol. 2013, article ID 315748, 13 pages, DOI:10.1155/2013/315748.

[5] L. Poghosyan, *On a convergence of the quasi-periodic interpolation*, The International Workshop on Functional Analysis, October 12-14, 2012, Timisoara, Romania.

[6] L. Poghosyan, *On a convergence of the quasi-periodic interpolation*, The III International Conference of the Georgian Mathematical Union, Batumi, Georgia, September 2-9, 2012.

[7] L. Poghosyan, *On L_2-convergence of the quasi-periodic interpolation*, Reports of the National Academy of Sciences of Armenia **113**(3)(2013), 240–247.

[8] L. Poghosyan, and A. Poghosyan, *Asymptotic estimates for the quasi-periodic interpolations*, Armenian Journal of Mathematics, **5**(1)(2013), 34–57.

[9] L. Poghosyan, and A. Poghosyan, *Convergence acceleration of the quasi-periodic interpolation by rational and polynomial corrections (abstract)*, Second International Conference Mathematics in Armenia: Advances and Perspectives, 24-31 August, 2013, Tsaghkadzor, Armenia.

[10] J. Riordan, Combinatorial Identities, Wiley, New York, 1979.

Institute of mathematics, of National Academy of Sciences of Armenia, Bagramian ave. 24/5, 0019 Yerevan, Armenia

E-mail address: lusine@instmath.sci.am

Institute of mathematics, of National Academy of Sciences of Armenia, Bagramian ave. 24/5, 0019 Yerevan, Armenia

E-mail address: arnak@instmath.sci.am