A REMARK ON THE DIVERGENCE OF STRONG POWER MEANS OF WALSH-FOURIER SERIES

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ABSTRACT. F. Schipp in 1969 proved almost everywhere p-strong summability of Walsh-Fourier series and if $\lambda(n) \to \infty$ there exists a function $f \in L^1[0,1)$ which Walsh partial sums $S_k(x,f)$ satisfy the divergence condition

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |S_k(x,f)|^{\lambda(k)} = \infty$$

almost everywhere on $[0,1)$. In the present paper we show that this condition may hold everywhere.

1. Introduction

In the study of almost everywhere convergence and summability of Fourier series the trigonometric and Walsh systems have many common properties. Kolmogorov [?] in 1926 constructed the first example of everywhere divergent trigonometric Fourier series. The existence of the almost everywhere divergent Walsh-Fourier series first proved Stein [?]. Then Schipp [?] constructed an example of everywhere divergent Walsh-Fourier series. A significant complement to these divergence theorems are the investigations on almost everywhere summability of Fourier series.

Let $\Phi(t) : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, be an increasing continuous function. A numerical series with partial sums $s_1, s_2, \ldots$ is said to be (strong) Φ-summable to a number s, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi\left(|s_k - s|\right) = 0. \tag{1.1}$$

We note that the condition (??) is as strong as rapidly growing is Φ, and in the case of $\Phi(t) = t^p$, $p > 0$, the condition (??) coincides with H^p-summability, well known in the theory of Fourier series. Marcinkiewicz-Zygmund in [?], [?] established the almost everywhere H^p-summability for an arbitrary trigonometric Fourier series (ordinary and conjugate). Oskolkov in [?] proved a.e. Φ-summability for trigonometric Fourier series if $\Phi(t) = O(t/\log\log t)$. Then Rodin [?] established the analogous with Φ satisfying the condition

$$\limsup_{t\to\infty} \frac{\log \Phi(t)}{t} < \infty, \tag{1.2}$$

which is equivalent to the bound $\Phi(t) < \exp(Ct)$ with some $C > 0$. Moreover, Rodin invented an interesting property, that is almost everywhere BMO-boundedness of

2010 *Mathematics Subject Classification.* 42C10, 42A24.
Key words and phrases. Walsh series, strong summability, everywhere divergent Walsh-Fourier series.
The first author is supported by project TÁMOP-4.2.2.A-11/1/KONV-2012-0051.

Fourier series, and the a.e. Φ-summability immediately follows from this results, applying John-Nirenberg theorem. Karagulyan in [?, ?] proved that the condition (??) is sharp for a.e. Φ-summability for Fourier series. That is if

$$\limsup_{t\to\infty}\frac{\log\Phi(t)}{t}=\infty, \tag{1.3}$$

then there exists an integrable function $f\in L^1(0,2\pi)$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi(|S_k(x,f)|)=\infty,\quad \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi(|\tilde{S}_k(x,f)|)=\infty,$$

hold everywhere on $\mathbb{R}$, where $S_k(x,f)$ and $\tilde{S}_k(x,f)$ are the ordinary and conjugate partial sums of Fourier series of $f(x)$.

Analogous problems are considered also for Walsh series. Almost everywhere H^p-summability of Walsh-Fourier series with $p>0$ was proved by F. Schipp [?]. The almost everywhere Φ-summability with the condition (??) was proved by V. Rodin [?] and F. Schipp [?]. Recently Gát, Goginava and Karagulyan [?] established, that the bound (??) is sharp for a.e. Φ-summability of Walsh-Fourier series too. Moreover, as in the trigonometric case [?], it is constructed a function $f\in L^1[0,1)$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi\left(|S_k(x,f)|\right)=\infty$$

holds everywhere on $[0,1)$, where Φ satisfies the condition (??) and $S_k(x,f)$ are the partial sums of Walsh-Fourier series of $f(x)$. Let $\lambda(n)\to\infty$ be an arbitrary sequence. Schipp [?] constructed an example of function $f\in L^1[0,1)$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|S_k(x,f)|^{\lambda(k)}=\infty \tag{1.4}$$

holds almost everywhere on $[0,1)$. Using a version of the basic lemma from [?], in this paper we shall show that (??) may hold everywhere. That is

Theorem. *For any sequence $\lambda(n)\to\infty$, there exists a function $f\in L^1[0,1)$ such that (??) holds everywhere on $[0,1)$.*

We note also, that the problem of uniformly Φ-summability of trigonometric Fourier series, when $f(x)$ is a continuous function was considered by V. Totik [?, ?]. He proved that the condition (??) is necessary and sufficient for the uniformly Φ-summability of Fourier series of continuous functions. For the Walsh series the analogous problem is considered by S. Fridli and F. Schipp [?, ?], V. Rodin [?], U. Goginava and L. Gogoladze [?].

2. Proof of theorem

Recall the definitions of Rademacher and Walsh functions (see [?] or [?]). We consider the function

$$r_0(x)=\begin{cases}1, & \text{if}\quad x\in[0,1/2),\\ -1, & \text{if}\quad x\in[1/2,1),\end{cases}$$

periodically continued over the real line. The Rademacher functions are defined by $r_k(x)=r_0(2^kx)$, $k=0,1,2,\ldots$. Walsh system is obtained by all possible products of Rademacher functions. We shall consider the Paley ordering of Walsh system. We set $w_0(x) \equiv 1$. To define $w_n(x)$ when $n \geq 1$ we write n in the dyadic form

$$n = \sum_{j=0}^{k} \varepsilon_j 2^j, \tag{2.1}$$

where $\varepsilon_k = 1$ and $\varepsilon_j = 0$ or 1 if $j = 0, 1, \dots, k-1$, and set

$$w_n(x) = \prod_{j=0}^{k} (r_j(x))^{\varepsilon_j}.$$

The partial sums of Walsh-Fourier series of a function $f \in L^1[0,1)$ have a formula

$$S_n(x, f) = \int_0^1 f(t) D_n(x \oplus t) dt,$$

where $D_n(x)$ is the Dirichlet kernel and $\oplus$ denotes the dyadic addition. We note that

$$D_{2^k}(x) = \begin{cases} 2^k, & \text{if} \quad x \in [0, 2^{-k}), \\ 0, & \text{if} \quad x \in [2^{-k}, 1). \end{cases}$$

The Dirichlet kernel can be expressed by the modified Dirichlet kernel $D_n^*(x)$ by

$$D_n(x) = w_n(x) D_n^*(x).$$

If $n \in \mathbb{N}$ have the form (**??**), then we have

$$D_n^*(x) = \sum_{j=0}^{k} \varepsilon_j D_{2^j}^*(x) = \sum_{j=0}^{k} \varepsilon_j r_j(x) D_{2^j}(x).$$

We shall write $a \lesssim b$, if $a < c \cdot b$ and $c > 0$ is an absolute constant. The notation $\mathbb{I}_E$ stands for the indicator function of a set E. An interval is said to be a set of the form $[a, b)$. For a dyadic interval δ we denote by δ^+ and δ^- the left and right halves of δ. We denote the spectrum of a Walsh polynomial $P(x) = \sum_{k=0}^{m} a_k w_k(x)$ by

$$\operatorname{sp} P(x) = \{k \in \mathbb{N} \cup 0 : \; a_k \neq 0\}.$$

The following lemma is proved in [**?**]. In its proof we use a well known inequality

$$\left| \left\{ x \in (0,1) : \left| \sum_{k=1}^{n} a_k r_k(x) \right| \leq \lambda \right\} \right| \geq 1 - 2 \exp \left(-\lambda^2 / 4 \sum_{k=1}^{n} a_k^2 \right), \quad \lambda > 0, \tag{2.2}$$

for Rademacher polynomials (see for example [**?**], chap. 2, theorem 5).

Lemma 1 ([**?**])**.** *If $n \in \mathbb{N}$, $n > 50$, then there exists a set $E_n \subset [0,1)$, which is a union of some dyadic intervals of the length 2^{-n}, satisfies the inequality*

$$|E_n| > 1 - 2 \exp(-n/36), \tag{2.3}$$

and for any $x \in E_n$ there exists an integer $m = m(x) < 2^n$ such that

$$\int_0^x D_m^*(x \oplus t) dt \geq \frac{n}{30}. \tag{2.4}$$

Proof. We define

$$E_n = \left\{ x \in [0,1) : \left| \sum_{j=1}^{n} r_j(x) r_{j+1}(x) \right| < \frac{n}{3} \right\}. \tag{2.5}$$

Since $\phi_j(x) = r_j(x)r_{j+1}(x)$, $j = 1, 2, \ldots, n$ are independent functions, taking values ± 1 equally, the inequality (**??**) holds for $\phi_j(x)$ functions too. Applying (**??**) in (**??**) we will get the bound (**??**). Observe that for a fixed $x \in E_n$ we have

$$\#\{j \in \mathbb{N} : 1 \le j \le n : r_j(x)r_{j+1}(x) = -1\} > n/3, \tag{2.6}$$

where $\#A$ denotes the cardinality of a set A. On the other hand the value in (**??**) characterizes the number of sign changes in the sequence $r_1(x), r_2(x), \ldots, r_{n+1}(x)$. Using this fact, we may fix integers $1 \le k_1 < k_2 < \ldots < k_\nu \le n$, such that

$$r_{k_i}(x) = 1, \quad r_{k_i+1}(x) = -1, \quad i = 1, 2, \ldots, \nu, \quad \nu \ge \frac{n}{6} - 1. \tag{2.7}$$

Suppose δ_j is the dyadic interval of the length 2^{-j} containing the point x. Observe that (**??**) is equivalent to the condition

$$x \in \left((\delta_{k_j})^+ \right)^- . \tag{2.8}$$

This implies

$$\left((\delta_{k_j})^+ \right)^+ \subset [0, x), \tag{2.9}$$

$$r_{k_j}(x \oplus t) = 1, \quad t \in \delta_{k_j} \cap [0, x). \tag{2.10}$$

Now consider the integer

$$m = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_\nu}.$$

Using (**??**) and (**??**), we obtain

$$\begin{aligned}
\int_0^x D_m^*(x \oplus t)dt &= \sum_{j=1}^{\nu} \int_0^x r_{k_j}(x \oplus t) D_{2^{k_j}}(x \oplus t)dt \\
&= \sum_{j=1}^{\nu} 2^{k_j} \int_{\delta_{k_j} \cap [0,x)} r_{k_j}(x \oplus t)dt \\
&\ge \sum_{j=1}^{\nu} 2^{k_j} \int_{\left((\delta_{k_j})^+\right)^+} r_{k_j}(x \oplus t)dt \\
&= \sum_{j=1}^{\nu} 2^{k_j - 2} |\delta_{k_j}| = \frac{\nu}{4} > \frac{n}{30}.
\end{aligned}$$

□

Lemma 2. *Let $\lambda(n) \to \infty$ be an arbitrary sequence. Then for any integer $n > n_0$ there exists a Walsh polynomial $f(x) = f_n(x)$ such that*

$$\|f\|_1 \le 4, \quad \operatorname{sp} f(x) \subset [p(n), q(n)], \tag{2.11}$$

$$\sup_{N \in [p(n), 2q(n)]} \frac{\#\{k \in \mathbb{N} : 1 \le k \le N, |S_k(x, f)| > n/40\}}{N} \gtrsim 2^{-2n}, \tag{2.12}$$

where n_0 *is an absolute constant and* $q(n) > p(n)$ *are positive integers such that*

$$\min_{k\ge p(n)} \lambda(k) > n. \tag{2.13}$$

Proof. We define

$$\theta_k = \frac{k-1}{2^n} + \frac{k-1}{4^n} \in \Delta_k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right), \quad k = 1, 2, \ldots, 2^n,$$

and a positive integer $s \in \mathbb{N}$, satisfying

$$\min_{k\ge 2^s} \lambda(k) > n, \quad s > n. \tag{2.14}$$

Let E_n be the set obtained in **??**. We define $f(x)$ by

$$f(x) = 2^\gamma \cdot \mathbb{I}_{(E_n)^c}(x) r_s(x) + \frac{1}{2^n}\sum_{j=1}^{2^n}\Big(D_{u_{2^n}}(x\oplus\theta_j) - D_{u_j}(x\oplus\theta_j)\Big), \tag{2.15}$$

where s is defined in (**??**) and

$$\gamma = \left[\log_2(\exp(n/36))\right], \tag{2.16}$$

$$u_j = 2^{s+10j}, \quad j = 0, 1, \ldots, 2^n. \tag{2.17}$$

We have

$$\operatorname{sp}\left(\mathbb{I}_{(E_n)^c}(x) r_s(x)\right) \subset [2^s, 2^{s+1}) = [u_0, 2u_0),$$

$$\operatorname{sp}\left(D_{u_{2^n}}(x\oplus\theta_j) - D_{u_j}(x\oplus\theta_j)\right) \subset (u_j, u_{2^n}] \subset [u_0, u_{2^n}],$$

and therefore

$$\operatorname{sp} f(x) \subset [p(n), q(n)], \quad p(n) = u_0 = 2^s, \quad q(n) = u_{2^n}.$$

Using (**??**) and (**??**), we obtain

$$\|f\|_1 \le 2^\gamma(1-|E_n|) + 2 \le \exp(n/36)\cdot 2\exp(-n/36) + 2 = 4.$$

From the expression (**??**) it follows that any value taken by $f(x)$ is either 0 or a sum of different numbers of the form $\pm 2^k$ with $k \ge \gamma$. This implies

$$|f(x)| \ge 2^\gamma \ge \frac{\exp(n/36)}{2} > \frac{n}{40}, \quad n > n_0 = 150,$$

whenever

$$x \in \operatorname{supp} f = (E_n)^c \bigcup \left(\bigcup_{j=1}^{2^n-1} (\theta_j \oplus \operatorname{supp} D_{u_j})\right). \tag{2.18}$$

On the other hand if $l \ge q(n)$ and x satisfies (**??**), then we have

$$|S_l(x,f)| = |f(x)| > \frac{n}{40}.$$

Thus we obtain

$$\frac{\#\{k\in\mathbb{N} :\ 1\le k\le 2q(n),\ |S_k(x,f)| > n/40\}}{2q(n)} \ge \frac{1}{2} > 2^{-2n},$$

which implies (**??**). Now consider the case when (**??**) doesn't hold. We may suppose that

$$x \in \Delta_k \setminus \operatorname{supp} f, \quad 1 \le k \le 2^n. \tag{2.19}$$

According to Lemma **??**, there exists an integer $m = m(x) < 2^n$ satisfying the inequality (**??**). Together with m we consider

$$p = p(x) = m(x)(1 + 2^n) < 2^{2n}.$$

Using the definition of θ_j, observe that

$$\begin{aligned} w_m(\theta_k) &= w_m\left(\frac{k-1}{2^n}\right), \\ w_{m\cdot 2^n}(\theta_k) &= w_{m\cdot 2^n}\left(\frac{k-1}{4^n}\right) = w_m\left(\frac{k-1}{2^n}\right), \end{aligned}$$

and therefore we get

$$w_p(\theta_k) = w_m(\theta_k)w_{m\cdot 2^n}(\theta_k) = 1, \quad k = 1, 2, \ldots, 2^n. \tag{2.20}$$

Define

$$L(x) = \{l \in \mathbb{N} : \ l = p + \mu \cdot 2^{2n}, \mu \in \mathbb{N}\}. \tag{2.21}$$

Once again using the definition of θ_k as well as (**??**), we conclude

$$w_l(\theta_k) = w_p(\theta_k)w_{\mu\cdot 2^{2n}}(\theta_k) = 1, \quad k = 1, 2, \ldots, 2^n, \quad l \in L(x). \tag{2.22}$$

Suppose

$$l \in L(x) \cap [u_{k-1}, u_k), \quad k \le 2^n. \tag{2.23}$$

Since x is taken outside of $\operatorname{supp} f$, we have

$$\begin{aligned} S_l(x, f) &= \frac{1}{2^n}\left(\sum_{j=1}^{k-1} D_l(x \oplus \theta_j) - \sum_{j=1}^{k-1} D_{u_j}(x \oplus \theta_j)\right) \\ &= \frac{1}{2^n}\sum_{j=1}^{k-1} D_l(x \oplus \theta_j). \end{aligned} \tag{2.24}$$

On the other hand by (**??**) we get

$$\begin{aligned} \frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_l(x \oplus \theta_j)\right| &= \frac{1}{2^n}\left|\sum_{j=1}^{k-1} w_l(\theta_j)\, D_l^*(x \oplus \theta_j)\right| \\ &= \frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_l^*(x \oplus \theta_j)\right|. \end{aligned} \tag{2.25}$$

Using the definition of $D_l^*(x)$, observe that

$$D_l^*(x) = D_p^*(x) + D_{\mu\cdot 2^{2n}}^*(x) = D_m^*(x) + D_{m\cdot 2^n}^*(x) + D_{\mu\cdot 2^{2n}}^*(x).$$

Since the supports of the functions $D_{m\cdot 2^n}^*(t)$ and $D_{\mu\cdot 2^{2n}}^*(t)$ are in Δ_1, we conclude

$$D_l^*(x \oplus \theta_j) = D_m^*(x \oplus \theta_j), \quad x \in \Delta_k, \quad j \ne k. \tag{2.26}$$

Thus, applying Lemma **??** and (**??**), we obtain the bound

(2.27)
$$\begin{aligned}\frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_l(x\oplus\theta_j)\right| &= \frac{1}{2^n}\left|\sum_{j=1}^{k-1} D_m^*(x\oplus\theta_j)\right| \\ &\ge \int_0^x D_m^*(x\oplus t)dt - 1 > \frac{n}{30} - 1 > \frac{n}{40}, \quad n > n_0 = 150,\end{aligned}$$

which holds whenever l satisfies (**??**). Taking into account of (**??**) and (**??**), we get

$$\frac{\#\{l\in\mathbb{N}:\, 1\le l\le u_k,\, |S_l(x,f)| > n/40\}}{u_k} \ge \frac{\#\left(L(x)\cap[u_{k-1},u_k)\right)}{u_k} \gtrsim 2^{-2n},$$

which completes the proof of lemma. □

Proof of theorem. We may choose numbers $\{n_k\}_{k=1}^\infty$ such that

$$p(n_{k+1}) > 2q(n_k), \tag{2.28}$$

$$\frac{1}{2^{2n_k}}\cdot\left(\frac{n_k}{50\cdot 2^k}\right)^{n_k} > k, \tag{2.29}$$

$$n_{k+1} > 800k2^k q(n_k), \tag{2.30}$$

where $p(n)$ and $q(n)$ are the sequences determined in Lemma **??**. Applying Lemma **??**, we get polynomials $g_k(x) = f_{n_k}(x)$, which satisfy (**??**) for any $x\in[0,1)$. We have

$$f(x) = \sum_{k=1}^\infty 2^{-k} g_k(x) \in L^1[0,1).$$

The condition (**??**) provides increasing spectrums of these polynomials. Thus, if $p(n_k) < l \le q(n_k)$, then we have

$$\begin{aligned}|S_l(x,f)| &= \left|\sum_{j=1}^\infty 2^{-j} S_l(x,g_j)\right| = \left|\sum_{j=1}^{k-1} 2^{-j} g_j(x) + 2^{-k} S_l(x,g_k)\right| \\ &\ge 2^{-k}|S_l(x,g_k)| - 4(k-1)q(n_{k-1}).\end{aligned} \tag{2.31}$$

Applying Lemma **??**, for any $x\in[0,1)$ we may find a number $N_k\in[p(n_k),2q(n_k)]$ such that

$$\#\{l\in\mathbb{N}:\, p(n_k) < l \le N_k,\, |S_l(x,g_k)| > n_k/40\} \gtrsim \frac{N_k}{2^{2n_k}}.$$

Thus, using also (**??**) and (**??**), we conclude

$$\#\{l\in\mathbb{N}:\, p(n_k) < l \le N_k,\, |S_l(x,f)| > n_k/50\cdot 2^k\} \gtrsim \frac{N_k}{2^{2n_k}}$$

and finally, using (**??**) we obtain

$$\frac{1}{N_k}\sum_{j=1}^{N_k} |S_j(x,f)|^{\lambda(j)} \gtrsim \frac{1}{N_k}\cdot\frac{N_k}{2^{2n_k}}\cdot\left(\frac{n_k}{50\cdot 2^k}\right)^{n_k} \ge k, \quad k = 1,2,\ldots.$$

This implies the divergence of λ-power means at a point $x\in[0,1)$ taken arbitrarily, which completes the proof of the theorem. □

References

[1] G. Gát, U. Goginava, and G. Karagulyan, "On everywhere divergence of the strong Φ-means of Walsh-Fourier series," J. Math. Anal. App., accepted.

[2] U. Goginava and L. Gogoladze, "Strong approximation by Marcinkiewicz means of two-dimensional Walsh-Fourier series," Constr. Approx., **35** (1), 1–19 (2012)

[3] B. I. Golubov, A. V. Efimov and V. A. Skvortsov. *Series and transformations of Walsh* (Nauka, Moscow 1987 (Russian); English translation, Kluwer Academic, Dordrecht, 1991).

[4] S. Fridli and F. Schipp, "Strong summability and Sidon type inequality," Acta Sci. Math. (Szeged), **60** , 277–289 (1985).

[5] S. Fridli and F. Schipp, "Strong approximation via Sidon type inequalities," J. Approx. Theory, **94** , 263–284 (1998).

[6] G. A. Karagulyan, "On the divergence of strong Φ-means of Fourier series," Izv. Acad. Sci. of Armenia, **26** (2), 159–162 (1991).

[7] G. A. Karagulyan," Everywhere divergence Φ-means of Fourier series," Math. Notes, **80** (1–2), 47–56 (2006).

[8] B. S. Kashin and A. A. Sahakian, *Orthogonal series* (Translated from the Russian by Ralph P. Boas. Translation edited by Ben Silver. Translations of Mathematical Monographs, 75. American Mathematical Society, Providence, RI, 1989).

[9] A. N. Kolmogoroff, "Une série de Fourier-Lebesque divergente presgue partout," Comp. Rend., **183** (4), 1327–1328 (1926).

[10] J. Marcinkiewicz, "Sur la sommabilité forte des séries de Fourier," J. Lond. Math. Soc., **14**, 162–168 (1939).

[11] K. I. Oskolkov, "On strong summability of Fourier series," Trudy Mat. Inst. Steklov. **172**, 280–290 (1985).

[12] V. A. Rodin, "BMO -strong means of Fourier series," Functional Analysis and Its Applications **23** (2), 145–147 (1989).

[13] V. A. Rodin, "The space BMO and strong means of Walsh-Fourier series," Mathematics of the USSR-Sbornik, **74** (1), 203–218 (1993).

[14] E. M. Stein, "On the limits of sequences of operators," Annals Math., **74** (2), 140–170 (1961).

[15] F. Schipp, "Über die Divergenz der Walsh-Fourierreihen," Ann Univ. Sci. Budapest, Sec. Math., **12**, 49–62 (1969).

[16] F. Schipp, "Über die Summation von Walsh-Fourierreihen," Acta Sci. Math.(Szeged), **30**, 77–87 (1969).

[17] F. Schipp, "On the strong summability of Walsh series," Publ. Math. Debrecen, **52** (3-4), 611–633 (1998).

[18] F. Schipp, W. R. Wade, P. Simon and J. Pál, *Walsh Series, an Introduction to Dyadic Harmonic Analysis* (Adam Hilger, Bristol, New York, 1990).

[19] V. Totik, "Notes on Fourier series strong approximations," J. Approx. Theory, **43**, 105–111 (1985).

[20] V. Totik, "On the strong approximation of Fourier series," Acta Math. Acad. Sci. Hung., **35** (1-2), 151–172 (1980).

[21] A. Zygmund, "On the convergence and summability of power series on the circle of convergence," Proc. Lond. Math. Soc., **47**, 326–350 (1941).

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