# Degenerate first order differential-operator equations

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#### Abstract

We consider boundary value problem for degenerate first order differentialoperator equation

$$Lu \equiv t^{\alpha}u' - Pu = f, \quad u(0) - \mu u(b) = 0,$$

where  $t \in (0, b)$ ,  $\alpha \geq 0$ ,  $P : H \to H$  is linear operator in separable Hilbert space H,  $f \in L_{2,\beta}((0,b), H)$ ,  $\mu \in \mathbb{C}$ . We prove that under some conditions on the operator P and number  $\mu$  boundary value problem has unique generalized solution  $u \in L_{2,\beta}((0,b), H)$  when  $2\alpha + \beta < 1$ ,  $\beta \geq 0$ and for any  $f \in L_{2,\beta}((0,b), H)$ .

# 1 Introduction

In the present paper we consider boundary value problem for degenerate differentialoperator equations of the first order

$$Lu \equiv t^{\alpha}u'(t) - Pu = f(t), \quad u(0) - \mu u(b) = 0, \tag{1.1}$$

where  $t \in (0, b), \alpha \ge 0, \mu \in \mathbb{C}, P : H \to H$  is a linear operator in separable Hilbert space  $H f \in L_{2,\beta}((0, b), H), \beta \ge 0$ , i.e.,

$$\|f\|_{\beta}^{2} = \int_{0}^{b} t^{\beta} \|f(t)\|_{H}^{2} dt < \infty.$$

We assume that the operator  $P: H \to H$  has complete system of eigenfunctions  $\{\varphi_k\}_{k=1}^{\infty}$ , which form Riesz base in H, i.e.,  $P\varphi_k = p_k\varphi_k$ ,  $k \in \mathbb{N}$ , all  $x \in H$  have representation

$$x = \sum_{k=1}^{\infty} x_k \varphi_k \tag{1.2}$$

and for some positive constants  $c_1$  and  $c_2$  is valid the inequality

$$c_1 \sum_{k=1}^{\infty} |x_k|^2 \le ||x||_H^2 \le c_2 \sum_{k=1}^{\infty} |x_k|^2.$$
(1.3)

Basics of the theory of differential-operator equations (i.e. ordinary differential equations with operator coefficients) of the first and second order can be found in the monograph of S.G. Krein (see [6]). Differential-operator equations of the first order have been considered in the articles of A.A. Dezin (see [2]), V.P. Glushko (see [1]) and other autors. In the paper of N. Yataev (see [11]) have been regarded operator equations of third order in weighted Sobolev space. In the papers [8] and [9] of the author were considered degenerate operator equations of the fourth order in finite interval (0, b) and operator equations of order 2m on infinite interval  $(1, +\infty)$ . In the article [10] were regarded degenerate operator equations with arbitrary weights. In this papers we explore Dirichlet problem in corresponding weighted Sobolev spaces.

First we consider one-dimensional case of operator equation (1.1), i.e. when  $Pu = pu, p \in \mathbb{C}$ , and then we pass to the general case using general method of A.A. Dezin (see [2]).

# 2 One-dimensional case

In this section we consider one dimensional case of boundary value problem (1.1)

$$Su \equiv t^{\alpha}u' - pu = f, \quad u(0) - \mu u(b) = 0,$$
 (2.1)

were p and  $\mu$  are constant complex numbers,  $\alpha \ge 0$  and  $f \in L_{2,\beta}(0,b)$ .

We investigate the regular case (see [7]), when  $\int_0^b \frac{1}{t^{\alpha}} dt < \infty$ , i.e.  $\alpha < 1$ . Wanting to expand the space  $L_2(0, b)$ , we will assume in the future, that  $\beta \ge 0$ . Observe that for the weighted  $L_{2,\beta}(0,b)$  spaces for  $\beta_1 \le \beta_2$  we have continuous embedding  $L_{2,\beta_1}(0,b) \subset L_{2,\beta_2}(0,b)$ , which for  $\beta_1 < \beta_2$  is not compact. We explore the degeneration at the point t = 0, therefore we do not consider the case  $\mu = \infty$ , i.e. the case when we consider the condition u(b) = 0.

We define the operator  $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$  as closure of the corresponding differential operation S, first defined on smooth functions, satisfying boundary condition  $u(0) - \mu u(b) = 0$  (see. [7]).

Define maximal operator  $\tilde{S}: L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$  as closure of differential operation S in  $L_{2,\beta}(0,b)$ .

Define minimal operator  $S_0: L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$  as closure of differential operation S in  $L_{2,\beta}(0,b)$ , initially defined on smooth functions which satisfy to the conditions u(0) = u(b) = 0.

**Definition 2.1** Operator  $S: L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$  is called proper operator if

$$S_0 \subset S \subset S \tag{2.2}$$

and exists inverse operator  $S^{-1}: L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ , defined on the whole space  $L_{2,\beta}(0,b)$ .

It follows from Definition 2.1, that inverse operator  $S^{-1}$ :  $L_{2,\beta}(0,b) \rightarrow L_{2,\beta}(0,b)$  is bounded, since it is closed operator, defined on the whole space  $L_{2,\beta}(0,b)$  (see [7]).

Our goal is to find the values of the numbers  $\alpha \geq 0, \beta \geq 0, \mu \in \mathbb{C}$  such that boundary value problem (2.1) has unique solution for any  $f \in L_{2,\beta}(0,b)$ , i.e. to prove that the operator  $S: L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$  is proper operator.

It is easy to find, that the general solution of the differential equation in (2.1) has the following form

$$u(t) = C e^{-\gamma t^{1-\alpha}} + e^{-\gamma t^{1-\alpha}} \int_0^t \tau^{-\alpha} e^{\gamma \tau^{1-\alpha}} f(\tau) \, d\tau, \qquad (2.3)$$

where  $\gamma = \frac{p}{1-\alpha}$ . Now, using boundary condition in (2.1) we obtain

$$C(1 - \mu e^{-\gamma b^{\alpha - 1}}) = \mu e^{-\gamma b^{1 - \alpha}} \int_0^b \tau^{-\alpha} e^{\gamma \tau^{1 - \alpha}} f(\tau) \, d\tau.$$
(2.4)

For  $\mu = 0$  we conclude from the formula (2.3) that C = 0. Thus the solution of the boundary value problem (2.1) has the the following form

$$u(t) = e^{-\gamma t^{1-\alpha}} \int_0^t \tau^{-\alpha} e^{\gamma \tau^{1-\alpha}} f(\tau) \, d\tau.$$
 (2.5)

Now we consider the case  $\mu \neq 0$ . Then the equality  $1 - \mu e^{-\gamma b^{1-\alpha}} = 0$  is equivalent to the equality  $e^{\gamma b^{1-\alpha}} = \mu$ , i.e.

$$\gamma b^{1-\alpha} = \ln |\mu| + i \arg \mu + 2\pi m i, \quad m \in \mathbb{Z}.$$

Since  $\gamma = \frac{p}{1-\alpha}$ , from the last equality we obtain

$$p(m,\alpha) := b^{\alpha - 1} (1 - \alpha) (\ln |\mu| + i \arg \mu + 2\pi m i), \quad m \in \mathbb{Z}.$$
 (2.6)

By the formula (2.6) are defined the values of p, for which the equation (2.4) is unsolvable with respect to C. In other words for this values of the number p boundary value problem (2.1) is unsolvable for every  $f \in L_{2,\beta}(0,b)$ .

If  $p \neq p(m, \alpha), m \in \mathbb{Z}$ , then from the equality (2.4) we find uniquely the number C. Thus the solution of boundary value problem (2.1) has the following form

$$u(t) = \frac{\mu e^{-\gamma t^{1-\alpha}}}{e^{\gamma b^{1-\alpha}} - \mu} \int_0^b \tau^{-\alpha} e^{\gamma \tau^{1-\alpha}} f(\tau) \, d\tau + e^{-\gamma t^{1-\alpha}} \int_0^t \tau^{-\alpha} e^{\gamma \tau^{1-\alpha}} f(\tau) \, d\tau.$$
(2.7)

**Theorem 2.2** Generalized solution of boundary value problem (2.1) under condition  $p \neq p(m, \alpha), m \in \mathbb{Z}$  exists and is unique for every  $f \in L_{2,\beta}(0, b)$ , when

$$\alpha \ge 0, \quad \beta \ge 0, \quad 2\alpha + \beta < 1. \tag{2.8}$$

*Proof.* Now we explore the behaviour of the solution (2.7) depending on  $\alpha \geq 0, \beta \geq 0$  for every function  $f \in L_{2,\beta}(0,b)$ . First note, that  $e^{\gamma t^{1-\alpha}}$  is bounded function, since  $|e^{\gamma t^{1-\alpha}}| = e^{\gamma_1 t^{1-\alpha}}$ , where  $\gamma = \gamma_1 + i\gamma_2, t \in (0,b)$  and  $0 \leq \alpha < 1$ . Consequently, to estimate the expression (2.7) it is enough to estimate the function  $F(t) := \int_0^t \tau^{-\alpha} f(\tau) d\tau$   $f \in L_{2,\beta}(0,b)$ . Using inequality of Cauchy we obtain

$$|F(t)|^{2} = \left| \int_{0}^{t} \tau^{-\alpha} \tau^{-\frac{\beta}{2}} \tau^{\frac{\beta}{2}} f(\tau) d\tau \right|^{2} \leq \int_{0}^{t} \tau^{-2\alpha-\beta} d\tau \int_{0}^{t} \tau^{\beta} |f(\tau)|^{2} d\tau + \\ \leq c_{1} t^{1-2\alpha-\beta} ||f||^{2}_{L_{2,\beta}(0,b)}.$$

Thus is valid the following inequality

$$|F(t)| \le ct^{\frac{1-2\alpha-\beta}{2}} ||f||_{L_{2,\beta}(0,b)},$$
(2.9)

so we conclude that for

$$\alpha \ge 0, \quad \beta \ge 0, \quad 2\alpha + \beta < 1 \tag{2.10}$$

the value of the function u(t), given by formula (2.7), is finite for t = 0 for any  $f \in L_{2,\beta}(0,b)$ .

Now we prove that the inequality (2.9) is exact, i.e. for  $\alpha \geq 0, \beta \geq 0$ ,  $2\alpha + \beta \geq 1$  exists function  $f \in L_{2,\beta}(0,b)$ , for which the function F(t) (thus also the solution u(t)) for  $t \to 0$  is unbounded (tends to infinity). Let  $2\alpha + \beta > 1$ . Then as a counterexample we can take, for example, the function  $f(t) = t^{\gamma}$  and choose the number  $\gamma$  such that  $t^{\gamma}$  belongs to  $L_{2,\beta}(0,b)$  but the value of F(t) at the point t = 0 is not finite. Then we obtain the conditions  $\beta + 2\gamma + 1 > 0$  and  $\gamma < \alpha - 1$ , i.e.  $\gamma \in (-\frac{\beta+1}{2}, \alpha - 1)$ , since from the condition  $2\alpha + \beta > 1$  it follows that  $-\frac{\beta+1}{2} < \alpha - 1$ . Now consider the case  $2\alpha + \beta = 1$ . Then as counterexample we can take the function  $f(t) = t^{\gamma} |\ln t|^{\delta}$ . Then for  $2\gamma + \beta = -1$ , i.e.  $\gamma = \alpha - 1$  and for  $-1 < \delta < -\frac{1}{2}$  it is easy to clear, that  $f \in L_{2,\beta}(0,b)$ , but the value of F(t) at the point t = 0 is not finite.

The proof is complete.

Now we estimate the function f(t), given by the formula (2.7), for  $f \in L_{2,\beta}(0,b)$ . Using inequality (2.9) we obtain

$$|u(t)| \le \left(c_1 + c_2 t^{\frac{1-2\alpha-\beta}{2}}\right) \|f\|_{L_{2,\beta}(0,b)}.$$
(2.11)

Before considering operator equation (1.1) we explore the spectrum of the closed operator  $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ . To do this we replace in boundary value problem (2.1) the number p by the number  $p - \lambda$  and try to find the values of  $\lambda \in \mathbb{C}$ , for which boundary value problem (2.1) is uniquely solvable for any  $f \in L_{2,\beta}(0,b)$ . It follows from the considerations for the case  $\mu = 0$  that each number  $\lambda \in \mathbb{C}$  belongs to the resolvent  $\rho(S)$  of the operator S. For the case  $\mu \neq 0$  we require that

$$p - \lambda \neq p(m, \alpha), m \in \mathbb{Z}$$

(see. formula (2.6)), i.e. for any  $m \in \mathbb{Z}$ 

$$\lambda \neq p - p(m, \alpha), \tag{2.12}$$

in both cases under condition (2.8). Thus the spectrum of the operator  $S : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$  is discrete and coinsides with the set of points

$$\sigma(S) = \sigma_p(S) = \{\lambda \in \mathbb{C} : \lambda = p - p(m, \alpha), m \in \mathbb{Z}\}.$$
(2.13)

### **3** Differential-operator equation

In this section we consider boundary value problem for differential-operator equation

$$Lu \equiv t^{\alpha} u'(t) - Pu = f(t), \quad u(0) - \mu u(b) = 0, \tag{3.1}$$

where  $t \in (0, b), \alpha \ge 0, \mu \in \mathbb{C}, P : H \to H$  is linear operator in the separable Hilbert space H and  $f \in L_{2,\beta}((0, b), H)$ .

Note that wide class of linear operators  $P: H \to H$ , having complete system of eigenfunctions, which form Riesz base in H are so called  $\Pi$ -operators (see [4]). We briefly describe these operators. Let  $V := [0, 2\pi]^n \subset \mathbb{R}^n$  and differential expression with constant coefficients

$$P(-iD)u = \sum_{|\gamma| \le m} p_{\gamma} D^{\alpha} u,$$

is first defined on the functions  $C^{\infty}(V)$ , which are periodical (with period  $2\pi$ ) with respect to each variable  $x_k, k = 1, 2, ..., n$ . Define operator  $P : L_2(V) \rightarrow L_2(V)$  as closure of differential expression P(-iD), which are called  $\Pi$ -operators. To each differential operator P(-iD) we can associate polynomial  $P(s), s \in \mathbb{Z}^n$ , such that  $P(-iD)e^{is \cdot x} = P(s)e^{is \cdot x}, s \cdot x = s_1x_1 + \cdots + s_nx_n$ .

Since the system of eigenfunctions  $\{\varphi_k\}_{k\in\mathbb{N}}$  of the operator P form Riesz base in Hilbert space  $H, P\varphi_k = p_k\varphi_k, k\in\mathbb{N}$  and

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\varphi_k, \qquad (3.2)$$

from the boundary value problem (3.1) for operator equations we obtain infinite chain of ordinary differential equations with the boundary conditions

$$L_k u_k \equiv t^{\alpha} u'_k(t) - p_k u_k = f_k(t), \quad u_k(0) - \mu u_k(b) = 0, \quad k \in \mathbb{N}.$$
(3.3)

**Definition 3.1** The function  $u \in L_{2,\beta}((0,b), H)$  is called generalized solution of the boundary value problem (3.1), if it can be represented by the formula (3.2), where the functions  $u_k(t), k \in \mathbb{N}$  are generalized solutions of the boundary value problem (3.1). Actually we defined the operator  $L: L_{2,\beta}((0,b), H) \to L_{2,\beta}((0,b), H)$  as closure of corresponding differential expression 3.1, initially defined on the finite linear combinations of  $u_k(t)\varphi_k$ , where  $u_k \in D(L_k), k \in \mathbb{N}$ .

It follows from the general results of A.A. Dezin ([4]) that is valid the following theorem.

**Theorem 3.2** Operator equation (3.1) is uniquely solvable for any  $f \in L_{2,\beta}((0,b), H)$ if and only if the boundary value problems (3.3) for any  $f_k \in L_{2,\beta}(0,b)$ ,  $k \in \mathbb{N}$ are uniquely solvable and uniformly with respect to  $k \in \mathbb{N}$  are fulfilled the inequalities

$$\|u_k\|_{L_{2,\beta}(0,b)} \le c \|f_k\|_{L_{2,\beta}(0,b)}, \quad c > 0.$$
(3.4)

Now we give sufficient condition to fulfill conditions (3.4).

**Theorem 3.3** For uniformly fulfillment with respect to  $k \in \mathbb{N}$  of the inequalities (3.4) for  $\mu = 0$  are sufficient the conditions

$$\operatorname{Re} p_k \ge M, \quad k \in \mathbb{N},$$

$$(3.5)$$

for some  $M \in \mathbb{R}$ . For the case  $\mu \neq 0$  are sufficient the conditions

$$e^{\gamma_k b^{1-\alpha}} - \mu| \ge \varepsilon, \quad |Re p_k| \le K,$$
(3.6)

for every  $k \in \mathbb{N}$  and some numbers  $\varepsilon > 0, K > 0$ , where  $\gamma_k = \frac{p_k}{1-\alpha}$ .

*Proof.* For the case  $\mu = 0$  the solutions  $u_k(t), k \in \mathbb{N}$  of the boundary value problems (3.3) have the form (2.5) (with replacement  $\gamma$  by  $\gamma_k$  and f by  $f_k$ ,  $k \in \mathbb{N}$ ). Let  $\gamma_k^1 = \operatorname{Re} \gamma_k = \frac{\operatorname{Re} p_k}{1-\alpha}$ . For the  $|u_k(t)|^2$  similar to the reasoning of the proof of Theorem 2.2 we get

$$|u_k(t)|^2 \le \int_0^t \tau^{-2\alpha-\beta} e^{-2\gamma_k^1(t^{1-\alpha}-\tau^{1-\alpha})} d\tau \cdot \|f_k\|_{L_{2,\beta}(0,b)}^2$$

Since the expression  $t^{1-\alpha} - \tau^{1-\alpha} \ge 0$  for  $0 \le \tau \le t$ , under fulfillment of the conditions (3.5) uniformly with respect to  $k \in \mathbb{N}$  are valid the inequalities (3.4) due to conditions  $0 < 2\alpha + \beta < 1, \beta \ge 0$  (see Theorem 2.2).

Let now  $\mu \neq 0$ . The solutions  $u_k(t), k \in \mathbb{N}$  of the boundary value problems (3.3) have the form (2.5) (with substitution f by  $f_k, k \in \mathbb{N}$ ). Estimating this solutions, using inequalities (3.6), similar to the first case we get

$$|u_{k}(t)|^{2} \leq \left(\frac{|\mu|}{\varepsilon} \int_{0}^{b} \tau^{-2\alpha-\beta} e^{-2\gamma_{k}^{1}(t^{1-\alpha}-\tau^{1-\alpha})} d\tau + \int_{0}^{t} \tau^{-2\alpha-\beta} e^{-2\gamma_{k}^{1}(t^{1-\alpha}-\tau^{1-\alpha})} d\tau\right) \cdot \|f_{k}\|_{L_{2,\beta}(0,b)}^{2}.$$

In contrast to the previous case  $(\mu = 0)$  here the expression  $t^{1-\alpha} - \tau^{1-\alpha}$  does not save sign for  $0 \le t \le b$ , therefore we have to require with the first condition in (3.6) more strong condition  $|Rep_k| \le K, k \in \mathbb{N}$ , which ensure fulfilment of the inequalities (3.4).

The proof is complete.

We also give the corresponding counterexample.

**Example 3.4** Consider Cauchy problem (3.1) for  $\alpha = 0, \beta = 0, \mu = 0$ , where as operator P we take closed operator

$$P \equiv -D_x^2, D_x = \frac{d}{dx}, D(P) = \{ u \in L_2(0,\pi), u'' \in L_2(0,\pi), u(0) = u(\pi) = 0 \}.$$
(3.7)

It is easy to calculate that the numbers  $p_k = k^2, k \in \mathbb{N}$  are eigenvalues for the operator P and the role of the eigenfunctions  $\varphi_k, k \in \mathbb{N}$  play the functions  $sin(kx), k \in \mathbb{N}$ . Observe that this system forms orthogonal base in  $L_2(0, b)$ . It is easy to verify that unique solutions of the boundary value problems

$$u'_k(t) - k^2 u_k(t) = e^{k^2 t}, \quad u_k(0) = 0, \quad k \in \mathbb{N}.$$

are the functions  $u_k(t) = te^{k^2 t}$ ,  $k \in \mathbb{N}$  (see formula (2.5)) and are true the following exact inequalities

$$||u_k||_{L_2(0,b)} \le \frac{ce^{k^2b}}{k} ||f_k||_{L_2(0,b)}, \quad c > 0.$$

It follows from the last inequality and Theorem 3.2 that unique solvability of boundary value problem (3.7) is violated, since the number sequence  $c_k = \frac{ce^{k^2b}}{k}, k \in \mathbb{N}$  tends to infinity for  $k \to \infty$ .

<sup>*n*</sup> Observe also that if we take in Example 3.4 the operator  $P \equiv D_x^2$  (with the same domain of definition as above), then it is easy to verify that uniformly with respect to  $k \in \mathbb{N}$  are true the inequalities (3.4). Therefore this boundary value problem will be correct. Here we have "inverse" and "direct" Cauchy problems for the heat equation, and we once again proved incorrectness of the "inverse" Cauchy problem for the heat equation.

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