Finite Difference Method for Two-Phase Obstacle Problem

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Abstract. We propose an algorithm to solve the *two-phase obstacle problem* by finite difference method. We prove the existence and uniqueness of the solution of the discrete nonlinear system and obtain an error estimate for finite difference approximation.

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This work is based on a joined work with M. Poghosyan and A. Arakelyan.

THE MATHEMATICAL SETTING OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open subset with Lipschitz-regular boundary. Let $g : \partial\Omega \to \mathbb{R}$ be a continuous function taking both positive and negative values over $\partial\Omega$, and $\lambda^+, \lambda^- : \Omega \to \mathbb{R}$ are Lipschitz-continuous functions satisfying

$$\lambda^+(x) \geq 0, \quad \lambda^-(x) \geq 0, \quad \text{and} \quad \lambda^+(x) + \lambda^-(x) > 0, \quad x \in \Omega.$$

The *two-phase obstacle problem*, or the *two-phase membrane problem*, is the problem of minimization of the cost functional

$$\mathscr{J}(v) := \int_\Omega \left[\frac{1}{2}|\nabla v|^2 + \lambda^+ \max(v,0) + \lambda^- \max(-v,0)\right] dx \tag{1}$$

over the set of admissible "deformations" $\mathbb{K} := \{v \in H^1(\Omega) : \ v - g \in H_0^1(\Omega)\}$.

Writing down the Euler-Lagrange equation for the minimization problem for the energy functional (1), we obtain

$$\begin{cases} \Delta u = \lambda^+ \cdot \chi_{\{u>0\}} - \lambda^- \cdot \chi_{\{u<0\}}, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases} \tag{2}$$

where χ_A stands for the characteristic function of the set A. It is easy to see (cf. [1]), that the solution (in the weak sense) of (2) must coincide with the minimizer $u \in \mathbb{K}$ of (1).

In the presented work we propose an algorithm for solving the two phase obstacle problem based on a finite difference method with 5 point stencil.

The *two-phase obstacle problem* (2) has been studied from different viewpoints. As it has been mentioned above, the existence of minimizers is straightforward and is obtained by the direct methods of calculus of variations. The optimal $C^{1,1}_{loc}$ regularity for the solution to (2) has been proved in [2] for constant coefficients $\lambda^\pm$, and the result was extended in [3] for Lipschitz-regular $\lambda^\pm$ and in [4] for Hölder-regular $\lambda^\pm$. The regularity and the geometry of the free boundary has been studied in [5], [6], [7]. As regards to numerical solution of two-phase obstacle problem, in his recent paper [8] Bozorgnia discussed three algorithms for numerical solution of two-phase obstacle problem.

CONSTRUCTION OF FINITE DIFFERENCE SCHEME

Now we consider the following nonlinear problem, which we will refer as the *Min-Max form of the two-phase obstacle problem*:

$$\begin{cases} \min\left(-\Delta u + \lambda^+, \max(-\Delta u - \lambda^-, u)\right) = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

If we introduce a function $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ by

$$F(x,r,p,X) = \min(-trace(X) + \lambda^+, \max(-trace(X) - \lambda^-, r)),$$

then the equation in (3) can be rewritten as

$$\mathscr{F}[u](x) = F(x,u,Du,D^2u) = 0 \quad \text{in } \Omega, \tag{4}$$

and by **solution to (3)** we mean a function $u \in C(\overline{\Omega})$ which is a viscosity solution to (4) and satisfies $u = g$ along the boundary $\partial\Omega$.

Theorem 1. *The equation* (4) *is degenerate elliptic.*

Proposition 1. *If u is the solution (in the weak sense) to* (2)*, then it is a viscosity solution to* (3)*. Moreover, u satisfies* (3) *a.e.*

Now we are going to construct a Finite Difference Scheme (FDS) for one- and two-dimensional two-phase obstacle problems based on its Min-Max form (3). For the sake of simplicity, we will assume that $\Omega = (-1,1)$ in one-dimensional case and $\Omega = (-1,1) \times (-1,1)$ in two-dimensional case in the rest of the paper, keeping in mind that the method works also for more complicated domains.

Let $N \in \mathbb{N}$ be a positive integer, $h = 2/N$ and

$$x_i = -1 + ih, y_i = -1 + ih, \quad i = 0,1,...,N.$$

We are interested in computing approximate values of the two-phase obstacle problem solution at the grid points x_i or (x_i, y_j) in one- and two-dimensional cases, respectively. We will develop the one-dimensional and two-dimensional cases parallelly in this section, hoping that the same notations for this two cases will not make confusion for reader. We use the notation u_i and $u_{i,j}$ (or simply u_α, where α is one- or two-dimensional multi-index) for finite-difference scheme approximation to $u(x_i)$ and $u(x_i, y_j)$, $\lambda_i^\pm = \lambda^\pm(x_i)$ and $\lambda_{i,j}^\pm = \lambda^\pm(x_i, y_j)$, $g_i = g(x_i)$ and $g_{i,j} = g(x_i, y_j)$ in one- and two-dimensional cases, respectively, assuming that the functions g and $\lambda^\pm$ are extended to be zero everywhere outside the boundary $\partial\Omega$ and outside Ω, respectively. In this section we will use also notations $u = (u_\alpha)$, $g = (g_\alpha)$ and $\lambda^\pm = (\lambda_\alpha^\pm)$ (not to be confused with functions u, g and $\lambda^\pm$).

Denote

$$\mathscr{N} = \{i : \ 0 \le i \le N\} \quad \text{or} \quad \mathscr{N} = \{(i,j) : \ 0 \le i,j \le N\},$$
$$\mathscr{N}^o = \{i : \ 1 \le i \le N-1\} \quad \text{or} \quad \mathscr{N}^o = \{(i,j) : \ 1 \le i,j \le N-1\},$$

in one- and two- dimensional cases, respectively, and

$$\partial\mathscr{N} = \mathscr{N} \setminus \mathscr{N}^o.$$

In one-dimensional case we consider the following approximation for Laplace operator: for any $i \in \mathscr{N}^o$,

$$L_h u_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},$$

and for two-dimensional case we introduce the following 5-point stencil approximation for Laplacian:

$$L_h u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2}$$

for any $(i,j) \in \mathscr{N}^o$.

Applying the finite difference method to (3), we obtain the following nonlinear system:

$$\begin{cases} \min(-L_h u_\alpha + \lambda_\alpha^+ , \max(-L_h u_\alpha - \lambda_\alpha^- , u_\alpha)) = 0, & \alpha \in \mathscr{N}^o, \\ u_\alpha = g_\alpha, & \alpha \in \partial\mathscr{N}. \end{cases} \tag{5}$$

It is not clear a priori, whether this system has a solution, or, in the case of existence, this solution is unique. To this end, we consider the following functional:

$$J_h(v) = -\frac{1}{2}\Big(L_h v, v\Big) + \Big(\lambda^+, v \vee 0\Big) - \Big(\lambda^-, v \wedge 0\Big) - \Big(L_h g, v\Big),$$

defined on the finite dimensional space

$$\mathcal{K} = \{v \in \mathcal{H} : \ v_\alpha = 0, \ \alpha \in \partial\mathcal{N}\}, \quad \text{where} \quad \mathcal{H} = \{v = (v_\alpha) : v_\alpha \in \mathbb{R}, \ \alpha \in \mathcal{N}\}.$$

Here $v \vee 0 = \max(v,0)$, $v \wedge 0 = \min(v,0)$ and for $w = (w_\alpha)$ and $v = (v_\alpha)$, $\alpha \in \mathcal{N}$, the inner product $(\cdot,\cdot)$ is defined by

$$(w,v) = \sum_{\alpha \in \mathcal{N}} w_\alpha \cdot v_\alpha.$$

Theorem 2. *The element $u \in \mathcal{H}$ solves* (5) *if and only if $\tilde{u} = u - g$ solves the following minimization problem:*

$$\tilde{u} \in \mathcal{K} : \qquad J_h(\tilde{u}) = \min_{v \in \mathcal{K}} J_h(v). \tag{6}$$

Theorem 3. *The nonlinear system* (5) *has a unique solution.*

NUMERICAL EXAMPLES

For the solution of nonlinear system (5) we will use iterative sequence, based on well-known PSOR (Projected Successive Over-Relaxation) method (see [9]). We will call our algorithm *Projected Gauss-Seidel (PGS)* method, since the main ingredient here is the Gauss-Seidel iteration combined with projection step (see [10]).

Example 1. We consider the following one-dimensional two-phase obstacle problem:

$$\begin{cases} \Delta u = 8 \cdot \chi_{\{u>0\}} - 8 \cdot \chi_{\{u<0\}}, & x \in (-1,1) \\ u(-1) = -1, \quad u(1) = 1. \end{cases}$$

In this case the exact solution can be written down as a piecewise polynomial function:

$$u(x) = \begin{cases} 4x^2 - 4x + 1, & 0.5 \le x \le 1, \\ 0, & -0.5 < x < 0.5, \\ -4x^2 - 4x - 1, & -1 \le x \le -0.5. \end{cases}$$

We use the above described discretization with $N = 20$. The PGS algorithm produces the result given in Figure 1, and the error between numerical and exact solution (after 10 and 20 iterations) is represented in Figure 1 on the right side.

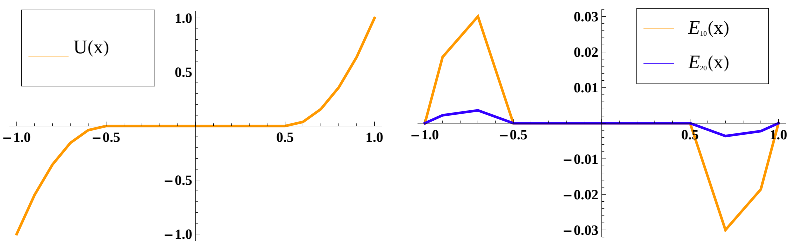

FIGURE 1. Left:Numerical Solution, Right:Error between the exact and numerical solutions

Next, the table 1 we shows maximal errors between the exact and numerical solutions for this example for different numbers of discretization points and iterations ($R_{N,M}$ is the maximal error while using N discretization points and M iterations). It is clearly visible that the error decrases along with the increase of N and M.

Example 2. The second example is the following 2D two-phase problem:

$$\begin{cases} \Delta u = 2 \cdot \chi_{\{u>0\}} - 2 \cdot \chi_{\{u<0\}}, & (x,y) \in (-1,1)^2 \\ u(-1,y) = \left(\frac{1-y}{2}\right)^2, \quad u(1,y) = \left(\frac{1-y}{2}\right)^2, & y \in [-1,1] \\ u(x,-1) = -x|x|, \quad u(x,1) = 0, & x \in [-1,1]. \end{cases}$$

TABLE 1. Error between the exact and numerical solutions

	$N = 20$	$N = 65$	$N = 120$	$N = 175$	$N = 230$
$R_{N,2\times N}$	0.0668629	0.00236045	0.005283	0.0179411	0.0347648
$R_{N,4\times N}$	0.0668629	0.00229779	0.000577501	0.00137638	0.00445856
$R_{N,6\times N}$	0.0668629	0.0022977	0.000556582	0.000227299	0.000658859
$R_{N,8\times N}$	0.0668629	0.0022977	0.000556051	0.000203333	0.000134995
$R_{N,10\times N}$	0.0668629	0.0022977	0.000556037	0.00020249	0.0000872308

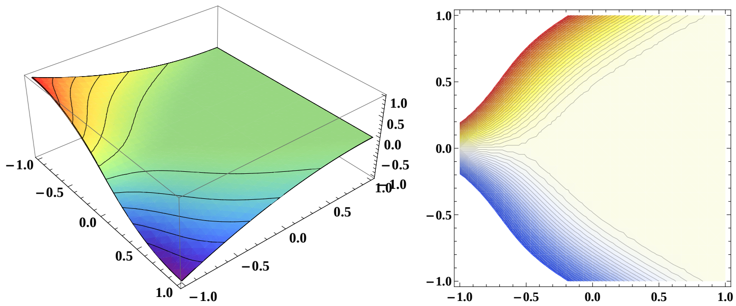

FIGURE 2. Left:Numerical Solution, Right:Level sets

The numerical algorithm produces the result given in Figure 2: the surface is the solution for our problem. Figure 2 was constructed with 100 discretization points and 400 iterations. The free boundary is clearly visible in the right side of Figure 2 (the bell-shaped boundary of the white region, the zero-level set). It is important to mention that in right side of Figure 2 the tangential touch of two branches of the free boundary is clearly visible.

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REFERENCES

1. G. S. Weiss, *Interfaces Free Bound.* **3**, 121–128 (2001), ISSN 1463-9963.
2. N. N. Uraltseva, *J. Math. Sci. (New York)* **106**, 3073–3077 (2001), ISSN 1072-3374, function theory and phase transitions.
3. H. Shahgholian, *Comm. Pure Appl. Math.* **56**, 278–281 (2003), ISSN 0010-3640.
4. E. Lindgren, H. Shahgholian, and A. Edquist, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**, 2359–2372 (2009), ISSN 0294-1449, URL `http://dx.doi.org/10.1016/j.anihpc.2009.03.006`.
5. H. Shahgholian, N. Uraltseva, and G. S. Weiss, *Int. Math. Res. Not. IMRN* pp. Art. ID rnm026, 16 (2007), ISSN 1073-7928.
6. H. Shahgholian, and G. S. Weiss, *Adv. Math.* **205**, 487–503 (2006), ISSN 0001-8708.
7. J. Andersson, N. Matevosyan, and H. Mikayelyan, *Ark. Mat.* **44**, 1–15 (2006), ISSN 0004-2080, URL `http://dx.doi.org/10.1007/s11512-005-0005-2`.
8. F. Bozorgnia, *Applied Numerical Mathematics* **61**, 92–107 (2011).
9. C. W. Cryer, *SIAM J. Control* **9**, 385–392 (1971), ISSN 0363-0129.
10. A. Arakelyan, R. Barkhudaryan, and M. Poghosyan, *Journal of Contemp. Math. Anal.* **46**, 131–141 (2011).