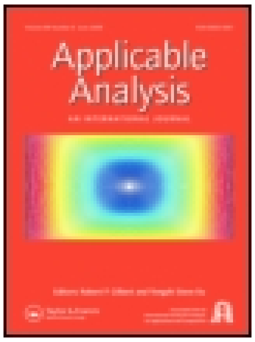

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Iterative scheme for an elliptic non-local free boundary problem

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ABSTRACT
In this paper we treat a non-local free boundary problem arising in financial bubbles, where the model is set in the framework of viscosity solutions. We suggest an iterative scheme which consists of a sequence of obstacle problems at each step to be solved, that in turn gives the next obstacle function in the iteration scheme. The suggested approach gives a solution to the theoretical problem and paves the way for the numerical implementation, as done in the text below.

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1. Introduction

The problem of financial bubbles, which arises in the modeling of speculative bubble, is subject of study in this paper. In [1] the authors suggest a model for studying speculative trading, where the ownership of a share of stock allows traders to profit from other's overvaluation. The model assumes that trading agents "agree to disagree" and there should be restrictions on short selling. As a result asset prices may become higher than their fundamental values. Other important part of the model is overconfidence, i.e. the agent believe that his information is more correct for a disagreement as that of others. Our departing point shall be the stationary case of such a model, formulated in terms of viscosity solutions. In a forthcoming paper we shall consider the parabolic model, and the behavior of the problem as time evolves.

The stationary version of this model was introduced and solved by Scheinkman and Xiong [1]. The article [1] considers one-dimensional case and in that specific case it is possible to construct an explicit stationary solution, which was done in [1] based on Kummer functions. Other stationary models were studied in [2,3].

In this paper we shall analyze the free boundary problem for a stationary setting, but shall keep a general framework, from a PDE point of view. Hence the model equation, studied in this paper, is the following free boundary problem formulated as a Hamilton–Jacobi equation:

$$\begin{cases} \min\left(-Lu(x)+f(x), u(x)-u(-x)-\psi(x)\right)=0, & x\in\Omega,\\ u(x)=g(x), & x\in\partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is a bounded symmetric domain such that if $x\in\Omega$ then $-x\in\Omega$, $f\in C(\Omega)$, $g\in C(\partial\Omega)$, and $\psi\in C^2(\Omega)$.

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As mentioned above we consider stationary case, i.e. the operator L is an elliptic operator, and of the form

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u, \qquad a^{ij} = a^{ji},$$

where the coefficients a^{ij}, b^i, c are assumed to be continuous and the matrix $[a^{ij}(x)]$ is positive definite for all $x \in \Omega$. Additionally, we assume that the coefficients are "symmetric" in the domain Ω:

$$\begin{aligned}(L\widetilde{u})(x) &= a^{ij}(x)D_{ij}u(-x) - b^i(x)D_iu(-x) + c(x)u(-x)\\ &\equiv a^{ij}(-x)D_{ij}u(-x) + b^i(-x)D_iu(-x) + c(-x)u(-x)\\ &= (Lu)(-x) = (\widetilde{Lu})(x) \end{aligned} \tag{2}$$

where $\widetilde{u}(x) = u(-x)$. The Equation (2) is the same if we require that a^{ij} and c are even functions and b^i is odd.

The astute reader may have already noticed that if u is a solution to Equation (1), then $u(-x)$ is a solution to the reflected problem, i.e. all ingredients are reflected accordingly. Hence one has

$$u(x) \geq u(-x) + \psi(x) \geq u(x) + \psi(-x),$$

and in particular $\psi(x) + \psi(-x) \leq 0$ is forced as a condition for an existence theory. From here it is obvious that the obstacle function ψ,[1] and the boundary data $g(x)$, should satisfy the following inequality

$$\psi(x) + \psi(-x) \leq 0, \qquad \psi(x) \leq g(x),$$

which will be a standing assumption in this paper. For more details on this problem in parabolic setting and one-space dimension see [4].

In this model, as well as in any Hamilton–Jacobi type problem, the following two sets distinguish themselves as important domains in the problem:

(i) The coincidence set where $\{x \in \Omega : u(x) = u(-x) + \psi(x)\}$.
(ii) The non-coincidence $\{x \in \Omega : u(x) > u(-x) + \psi(x)\}$.

For the regularity and other theoretical aspects of obstacle-type problems you can see in [5,6]. For review of the PDE models in the socio-economic sciences see [7].

Our aim in this paper is to construct, through an increasing iterative scheme, a solution of the above problem. The scheme consists of a sequence of obstacle problems at each step that eventually converge to the solution. In doing so several difficulties arises, and these are dealt with in the paper.

2. Definition and notion of the solution

To deal with the problem we first recall the definition of the so-called viscosity solution, following [4].

The notion of viscosity solution was first introduced for first-order Hamilton–Jacobi equations in 1981 by Crandall and Lions (see [8] and [9]). For general framework about the theory of viscosity solutions we refer to [10,11], and references therein.

Definition 1: (Viscosity sub/super solution of the Equation (1)) A function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (1) on Ω, (that is, of the first equation in (1)), if u is upper semi-continuous (resp. lower semi-continuous), and if for any function $\varphi \in C^{2,1}(\Omega)$ and any point $x_0 \in \Omega$ such that $u(x_0) = \varphi(x_0)$ and

$$u \leq \varphi \text{ on } \Omega \,(\text{resp. } u \geq \varphi \text{ on } \Omega)$$

then

$$\min\left(-(L\varphi)(x) + f(x), \varphi(x) - \varphi(-x) - \psi(x)\right) \leq 0,$$

(resp. $\min(-(L\varphi)(x)+f(x), \varphi(x)-\varphi(-x)-\psi(x)) \geq 0$).

Definition 2: (Viscosity solution of the Equation (1)) A function $u : \Omega \to R$ is a viscosity solution of (1) on Ω, if and only if u^* is a viscosity subsolution and u_* is a viscosity supersolution on Ω, where

$$u^*(x) = \limsup_{y \to x} u(y),$$

and

$$u_*(x) = \liminf_{y \to x} u(y).$$

3. The iterative scheme

To construct the scheme, at first let us define a function u_0 (the initial guess) as the solution of the following problem:

$$\begin{cases} -Lu_0(x) + f(x) = 0, & x \in \Omega, \\ u_0(x) = g(x), & x \in \partial\Omega. \end{cases}$$

Inductively, we define the sequence $\{u_k\}$ by

$$\begin{cases} \min(-Lu_{k+1}(x) + f(x), u_{k+1}(x) - u_k(-x) - \psi(x)) = 0, & x \in \Omega, \\ u_{k+1}(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{3}$$

For each k we have an obstacle problem with the obstacle $u_k(-x) + \psi(x)$.

The main aim of the paper is to show that (3) produces a non-decreasing, bounded sequence $\{u_k\}$ which tends to the viscosity solution of (1).

Proposition 3.1: *The sequence $\{u_k\}$ is increasing.*

Proof: It is easy to see that $u_1 \geq u_0$, since the function u_1 is the solution of $-Lu_1 + f = 0$ with obstacle $u_0 + \psi$, and u_0 is the solution to the same problem without an obstacle.

To prove that $u_2(x) \geq u_1(x)$ let us examine Equation (3). When $k = 1$ the obstacle is $\Psi_2 := u_1(-x) + \psi(x)$ and for the case $k = 0$ the obstacle is $\Psi_1 := u_0(-x) + \psi(x)$. Since $u_1(x) \geq u_0(x)$ we have $u_1(-x) \geq u_0(-x)$ and hence $\Psi_2 \geq \Psi_1$. Furthermore, $u_2(x)$ and $u_1(x)$ are solutions of the same obstacle problem with obstacles Ψ_2 and Ψ_1, respectively. Since $\Psi_2 \geq \Psi_1$ we have by comparison principle (see [12], page 30, problem 3) $u_2(x) \geq u_1(x)$.

By inductive steps we have u_{k+1} and u_k solve the obstacle problem with obstacle $\Psi_{k+1} := u_k(-x) + \psi$ and $\Psi_k := u_{k-1}(-x) + \psi$, respectively, with $u_k \geq u_{k-1}$ i.e. $\Psi_{k+1} \geq \Psi_k$, and hence by comparison principle $u_{k+1} \geq u_k$. □

Proposition 3.2: *The sequence $\{u_k\}$ is bounded.*

Proof: Let h be a symmetric function with the properties

$$-Lh(x) = 1, \quad h(x) \geq 1, \quad \psi(x) \leq Ch(x)$$

for some C large, such that $f(x) - L\psi(x) + C \geq 1$; here we have assumed $\psi \in C^2$. Then from the scheme defined in (3) we have

$$\min(-Lv_{k+1}(x) + 1, v_{k+1}(x) - u_k(-x) - Ch(x)) \leq 0, \tag{4}$$

where $v_k = u_k - \psi + Ch$. Now suppose by induction that

$$\max(\sup v_k(x), \sup(v_k(x) + \psi(x))) \leq M, \tag{5}$$

where $M := \max\{\sup(|g(x)| + |\psi(x)| + |Ch(x)|), \sup v_0(x)\}$. The estimate (5) is obviously true for the starting value u_0. Let the maximum value of v_{k+1} be achieved at a point $z \in \bar{\Omega}$. If it is attained on the boundary then we are done. If it is attained inside the domain, then by ellipticity of the operator (and concavity of the graph for v_{k+1} at z) we have $Lv_{k+1}(z) \leq 0$, which implies $-Lv_{k+1}(z) + 1 > 0$, and hence by the inequality (4) we must have $v_{k+1}(z) \leq u_k(-z) + Ch(z) = v_k(-z) + \psi(-z) \leq M$, where we have used $h(z) = h(-z)$.

It remains to prove $v_{k+1}(x) + \psi(x) \leq M$. We make a similar argument for $w_{k+1}(x) := v_{k+1}(-x) + \psi(-x) = u_{k+1}(-x) + Ch(x)$ which satisfies a similar type of equation, with reflected version of the ingredients

$$\min(-Lw_{k+1}(x) + 1, w_{k+1}(x) - u_k(x) - Ch(x) - \psi(-x)) \leq 0.$$

Again as before let the maximum value of w_{k+1} be achieved at a point $z \in \bar{\Omega}$. Obviously if the maximum is on the boundary then we have the desired estimate. Hence, we assume the maximum is attained inside the domain and by using same arguments as above we will have the following

$$w_{k+1}(x) \leq u_k(x) + \psi(-x) + Ch(x) \leq u_k(x) - \psi(x) + Ch(x) = v_k(x) \leq M,$$

where we have used $\psi(x) + \psi(-x) \leq 0$. Hence we arrive at

$$\max(\sup v_{k+1}(x), \sup(v_{k+1}(x) + \psi(x)) \leq M,$$

in the inductive steps. This completes the proof. □

3.1. Convergence of the scheme

In this section, we are going to prove the convergence of $\{u_k\}$ and show that its limit is a solution of the non-local problem (1). To do this, first we will consider a local double obstacle problem (see [4]).

Set

$$w(x) = u(x) - u(-x),$$
$$\phi(x) = f(x) - f(-x).$$

If the function u is a solution of (1) then w is a solution of the following double obstacle problem

$$\begin{cases} \min(\max(-Lw(x) + \phi(x), w(x) + \psi(-x)), w(x) - \psi(x)) = 0 \\ w(x) = g(x) - g(-x), \quad x \in \partial\Omega. \end{cases} \tag{6}$$

For the double obstacle problem, there is a series of papers,[13–16] where optimal regularity of the solution as well as the other theoretical aspects were studied.

Furthermore, we will assume that $\psi(x) + \psi(-x) < 0$ and the function ϕ is non-negative.

Proposition 3.3: *The pointwise limit*

$$\lim_{k\to\infty} u_k = u$$

exists, and the function $w(x) = u(x) - u(-x)$ is a solution of (6). Furthermore, the limit function w is continuous.

Proof: The existence of a unique limits follows immediately from Propositions 3.1 and 3.2.

To get the continuity we will reformulate (3) at point x and at point $-x$

$$\begin{cases} u_k(x) \geq u_{k-1}(-x) + \psi(x), & x \in \Omega, \\ -Lu_k(x) + f(x) \geq 0, & x \in \Omega, \\ -Lu_k(x) + f(x) = 0, & \text{if } u_k(x) > u_{k-1}(-x) + \psi(x), \end{cases} \tag{7}$$

and

$$\begin{cases} u_k(-x) \geq u_{k-1}(x) + \psi(x), & x \in \Omega, \\ -Lu_k(-x) + f(-x) \geq 0, & x \in \Omega, \\ -Lu_k(-x) + f(-x) = 0, & \text{if } u_k(-x) > u_{k-1}(x) + \psi(-x). \end{cases} \tag{8}$$

We are going to combine two equations above as one double obstacle problem with changing obstacles.

Set $w_k(x) = u_k(x) - u_k(-x)$,

$$\psi_k^-(x) = \psi(x) - u_k(-x) + u_{k-1}(-x),$$

$$\psi_k^+(x) = -\psi(-x) + u_k(x) - u_{k-1}(x),$$

combining Equations (7) and (8) and reminding us that $u_k(x)$ is a solution of the obstacle problem, we will get that $w_k \in H^2$ solves the following double obstacle problem (see [16])

$$\begin{cases} w_k(x) \geq \psi_k^-(x), & x \in \Omega, \\ w_k(x) \leq \psi_k^+(x), & x \in \Omega, \\ -Lw_k(x) + \phi(x) \geq 0, & \text{if } \psi_k^-(x) < w_k(x), \\ -Lw_k(x) + \phi(x) \geq 0, & \text{if } w_k(x) < \psi_k^+(x), \\ -Lw_k(x) + \phi(x) = 0, & \text{if } \psi_k^-(x) < w_k(x) < \psi_k^+(x). \end{cases}$$

It is easy to see that $\psi_k^+(x) \geq -\psi(-x) > \psi(x) \geq \psi_k^-(x)$ hence two obstacles do not meet (for any k). Which means locally near each coincidence set (lower and upper ones) we still solve the standard obstacle problem, so our function w_k as defined solves the double obstacle problem.

To finalize the proof, it remains to see that $\psi_k^-(x) \to \psi(x)$, $\psi_k^+(x) \to -\psi(-x)$ as $k \to \infty$, and to invoke the continuous dependence of the solutions on obstacles in variational inequalities, see [15].

From the above analysis it immediately follows that the function $w(x)$ is continuous. □

Theorem 3.4: *The function u is a continuous viscosity solution of* (1).

Proof: Having a bounded increasing sequence of continuous functions, the limit function u is lower semi-continuous, i.e.

$$u(x) = u_*(x) := \liminf_{y \to x} u(y). \tag{9}$$

We also denote by u^* the upper semi-continuous envelop of u, i.e.

$$u^*(x) := \limsup_{y \to x} u(y).$$

Using continuity of the function $u(x) - u(-x)$, it is easy to show that $u^*(x) - u^*(-x) = u(x) - u(-x)$. This follows directly from inequalities below:

$$\begin{aligned} u^*(x) - u^*(-x) &= \limsup_{y \to x} u(y) + \liminf_{y \to x} -u(-y) \\ &\leq \limsup_{y \to x} (u(y) - u(-y)) = u(x) - u(-x). \end{aligned} \tag{10}$$

On the other hand, we have

$$u^*(x) - u^*(-x) = \limsup_{y\to x} u(y) + \liminf_{y\to x} -u(-y)$$
$$\geq \liminf_{y\to x} (u(y) - u(-y)) = u(x) - u(-x). \tag{11}$$

First we show that the function u^* is a subsolution to (1). For that purpose let us suppose u^* is not a subsolution. Then there exists $x_0 \in \Omega$ and a degree two polynomial P satisfying

$$P \geq u^*, \quad P(x_0) = u^*(x_0)$$

such that

$$\min\Big\{-LP(x_0) + f(x_0), P(x_0) - P(-x_0) + \psi(x_0)\Big\} > 0.$$

Assume that the first inequality holds. Then

$$-LP(x_0) + f(x_0) > 0$$

and

$$P(x_0) > P(-x_0) + \psi(x_0).$$

Substituting the values for $P(x_0)$, the last inequality can be rewritten in the following way:

$$u^*(x_0) > u^*(-x_0) + \psi(x_0).$$

Using continuity of f, ψ, and $u^*(x) - u^*(-x)$, we can deduce that there exist number $r > 0$ such that

$$-LP(x) + f(x) > 0, \quad x \in B(x_0, r),$$
$$u^*(x) > u^*(-x) + \psi(x), \quad x \in B(x_0, r).$$

and using continuity of $u^*(x) - u^*(-x)$ there exists a positive number $\mu < u^*(x_0) - u^*(-x_0) - \psi(x_0)$ such that

$$u_j(x) > u_{j-1}(-x) + \psi(x) + \mu, \quad x \in B(x_0, r). \tag{12}$$

Denote

$$P_\varepsilon = P + \varepsilon\eta(x),$$

where $\eta(x)$ is a function which satisfy $L\eta = 1$, $\eta \geq 0$, and $\eta(x_0) = 0$. If $\varepsilon > 0$ is small enough, then

$$-LP_\varepsilon(x) + f(x) > 0, \quad x \in B(x_0, r). \tag{13}$$

Next observe that

$$P_\varepsilon(x) > P(x) \geq u^*(x), \quad x \neq x_0.$$

As $u_k \uparrow u^*$, we can choose j large enough to satisfy

$$\inf_{B(x_0,r)} \left(P_\varepsilon - u_j\right) < \min_{\partial B(x_0,r)} \left(P_\varepsilon - u^*\right) \tag{14}$$

and

$$\inf_{B(x_0,r)} \left(P_\varepsilon - u_j\right) < \mu. \tag{15}$$

Take $Q_\varepsilon = P_\varepsilon - c$, where c is a constant, chosen in such a way, that Q_ε touches u_j from above at some $x' \in B(x_0, r)$ (the inequality (14) guarantees that the first touch point in $B(x_0, r)$ will be not on the boundary of $B(x_0, r)$).

At this point, we have constructed a functino Q_ε satisfying the following conditions:

$$Q_\varepsilon(x) \geq u_j(x), \quad x \in B(x_0, r) \tag{16}$$
$$Q_\varepsilon(x') = u_j(x'), \quad \text{where} \quad x' \in B(x_0, r). \tag{17}$$

Since u_j is a viscosity subsolution (and, in fact, a solution) of

$$\min\left\{-Lu_j + f, u_j(x) - u_{j-1}(-x) - \psi(x)\right\} = 0, \quad u_j|_{\partial\Omega} = g_1,$$

then, by the definition of viscosity subsolution and (16)–(17), we obtain

$$\min\left\{-LQ_\varepsilon(x') + f(x'), Q_\varepsilon(x') - u_{j-1}(-x') - \psi(x')\right\} \leq 0. \tag{18}$$

Using (13), we have that $-LQ_\varepsilon(x') - f(x') > 0$, so the only possibility to have (18) is

$$Q_\varepsilon(x') \leq u_{j-1}(-x') + \psi(x').$$

This means that

$$P(x') \leq u_{j-1}(-x') + \psi(x') + c - \varepsilon\eta(x).$$

Then

$$u_j(x') \leq u^*(x') \leq P(x') \leq u_{j-1}(-x') + \psi(x') + c - \varepsilon\eta(x),$$

hence

$$u_j(x') \leq u_{j-1}(-x') + \psi(x') + c.$$

But, from (12) we deduce

$$u_j(x') > u_{j-1}(x') + \psi(x') + \mu.$$

This is a contradiction, since by (15), it follows $c < \mu$. Hence u^* is a subsolution of (1).

Let us now discuss the supersolution properties of $u = u_*$, see (9). Suppose u is not a supersolution. Then there exists $x_0 \in \Omega$ and a degree two polynomial P satisfying

$$P \leq u, \quad P(x_0) = u(x_0)$$

such that

$$\min\left\{-LP(x_0) + f(x_0), P(x_0) - P(-x_0) - \psi(x_0)\right\} < 0.$$

Then

$$-LP(x_0) + f(x_0) < 0$$

or

$$P(x_0) < P(-x_0) + \psi(x_0). \tag{19}$$

Let us consider the first inequality. Using continuity of u_j, f we can deduce that there exist number $r > 0$ such that

$$-LP(x) + f(x) < 0, \quad x \in B(x_0, r).$$

Like in the previous case we will construct new polynomial $Q = P - c$ which will touch u_j at some point $x' \in B(x_0, r)$ i.e.

$$Q_\varepsilon(x) \leq u_j(x), \quad x \in B(x_0, r)$$
$$Q(x') = u_j(x'), \quad \text{where} \quad x' \in B(x_0, r).$$

Since u_j is a viscosity supersolution (and, in fact, a solution) of

$$\min\left\{-Lu_j+f, u_j(x)-u_{j-1}(-x)-\psi(x)\right\}=0, \quad u_j|_{\partial\Omega}=g_1, \tag{20}$$

we will get contradiction as $-LQ+f<0$

It remains to show that inequality (19) also cannot be hold, for that purpose let us substitute the values for $P(x_0)$ and rewrite (19):

$$u(x_0)-u(-x_0)<\psi(x_0).$$

The function $u(x)-u(-x)$ is continuous in Ω which means that there exist N such that if $j>N$ then

$$u_j(x)-u_{j-1}(-x)<\psi(x), \quad x\in B(x_0,r).$$

This is a contradiction as $u_j(x)$ should satisfy (20)

The continuity of u follows from comparison principle (see [4]). Indeed, from comparison principle it follows that supersolution should be greater or equal to the subsolution, but from the definition of w it follow that $u^*\geq u$, so $u=u^*$ is a continuous viscosity solution of (1). □

3.2. Existence and uniqueness

Theorem 3.5: *There exists a unique viscosity solution of the system* (1).

Proof: Assume that u_1 and u_2 both are viscosity solutions of the system (1). Without loss of generality we may assume that there exists $x\in\Omega$ such that $u_1(x)-u_2(x)>0$ i.e. $\max\,(u_1(x)-u_2(x))>0$. From the boundary condition we have $u_1(x)-u_2(x)=0$ if $x\in\partial\Omega$. First, we prove uniqueness for the difference $w_i(x):=u_i(x)-u_i(-x)$. In Proposition (3.3), it was proved that $w_i(x)$ is a solution of double obstacle problem (6) and hence it is unique $w_1=w_2$.

Let us consider

$$u_1(x)=u_1(-x)+\psi(x) \text{ on coincidence set } K_1,$$
$$u_2(x)=u_2(-x)-\psi(-x) \text{ on coincidence set } K_2.$$

If $K_1\cap K_2\neq\emptyset$, then $\psi(x)=-\psi(-x)$ which is not possible as $\psi(x)+\psi(-x)<0$. Thus, the only possibility is $K_1\cap K_2=\emptyset$. In this case, let x^0 be the maximum point for the function $u_1(x)-u_2(x)=u_1(-x)-u_2(-x)$ and denote $M=u_1(x^0)-u_2(x^0)=u_1(-x^0)-u_2(-x^0)$. It follows from the last equality that $-x^0$ also is a maximum point. To conclude our proof, it is enough to prove that x^0 and $-x^0$ both belongs to K^1, and then we will have:

$$u_1(x_0)=u_1(-x_0)+\psi(x_0), \tag{21}$$
$$u_1(-x_0)=u_1(x_0)+\psi(-x_0). \tag{22}$$

Summing Equations (21), (22) yields a contradiction, as $\psi(x_0)+\psi(-x_0)\neq 0$.

It remains to prove that the function $u_1(x)-u_2(x)$ reached its maximum value on the set K_1. It is easy to see that the $u_1(x)-u_2(x)$ is a viscosity subsolution of the equation $-Lu=0$ so it should reach its maximum value on the boundary of the set $\Omega\setminus K_1$. □

4. Numerical results

In this section, we present two numerical experiments done by iterative scheme. For every step of the scheme we should solve an obstacle problem and to do this numerically we are going to use finite difference scheme. The finite difference scheme was extensively used for numerical solutions of variational inequalities, one-phase obstacle problems of elliptic and parabolic type, and in particular, for valuation of American type option (for details, see [17] and references in these papers).

In 2009, the finite difference scheme has been applied for one-dimensional parabolic obstacle problem in connection with valuation of American type options (see [18]). It has been proved that under some natural conditions, the finite difference scheme converges to the exact solution and the rate of convergence is $o(\sqrt{h}+k)$. Here h and k are space- and time-discretization steps. Recently, in the works [19–21] finite difference scheme and the convergence results have been applied for the one-phase and two-phase elliptic obstacle problems.

To construct a finite difference scheme we start by discretizing the domain Ω into a regular uniform mesh. In one-dimensional case we consider the following approximation for Laplace operator: for any point interior point (i,j),

$$L_h u_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},$$

and for two-dimensional case we introduce the following 5-point stencil approximation for Laplacian:

$$L_h u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2}.$$

The corresponding discretization should be carried out for other operators.

Applying the finite difference method to (3), we obtain the following nonlinear system:

$$\begin{cases} \min\left(-L_h u^\alpha_{k+1} + f^\alpha, u^\alpha_{k+1} - u^\alpha_k - \psi^\alpha \right) = 0, \\ u^\alpha_{k+1} = g^\alpha, \end{cases} \tag{23}$$

where f^α, ψ^α, g^α are the corresponding discretization of the $f(x)$, $\psi(x)$, $g(x)$, and u^α_k is finite difference approximation of the system (3).

Example 4.1: We consider the following one-dimensional problem.

$$\begin{cases} \min\left(-Lu, u(x) - u(-x) - \psi(x) \right), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega. \end{cases}$$

Here

$$Lu = \frac{1}{2}\sigma^2 x^2 \partial_{xx} u + (r-q)x\partial_x u + ru$$

is an elliptic counter part of the Black–Scholes equation, and the obstacle function is

$$\psi(x) = c(x-1).$$

To illustrate the iterative scheme here we choose $\Omega = [-2,2]$, $\sigma = 2.5$, $r = 10$, $c = 0.01$ $g(-2) = 1.01, g(2) = 1.02961$.

In Figure 1(a), the numerical solution of Example 4.1 is shown in orange and the obstacle function $\psi(x)$ is shown in blue color. The calculations are shown after four steps of iteration and the finite difference scheme is applied with $N = 100$ points. In Figure 1(b), the differences between second and third steps of iteration are shown. It's important to mention that even after three steps of iteration the difference between solutions is less than 10^{-9}.

It easily follows from (1) that the function $w(x) = u(x) - u(-x)$ (here u-is the viscosity solution of the problem) should be greater than $\psi(x)$ and at same time it should be less than $-\psi(-x)$ which is presented in Figure 2.

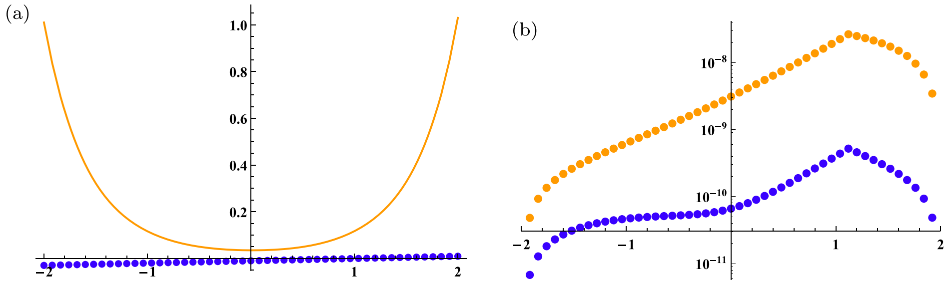

Figure 1. (a) Numerical solution of Example 4.1. (b) The differences between second and third steps of iteration.

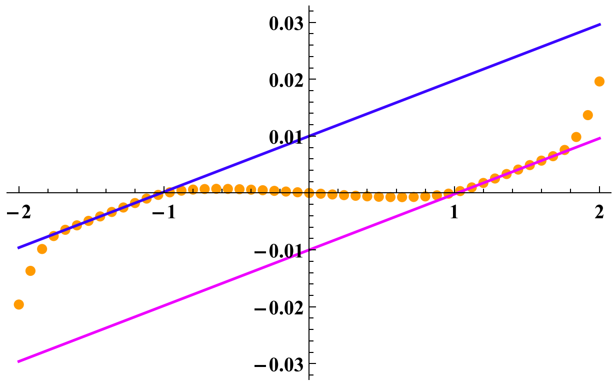

Figure 2. The function $w(x) = u(x) - u(-x)$, and the obstacles $\psi(x), -\psi(-x)$ are presented.

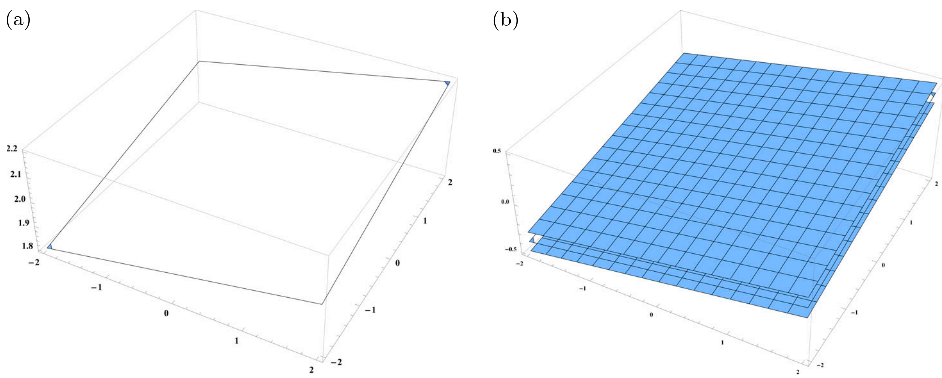

Figure 3. (a) Boundary data of Example 4.2. (b) Obstacle $\psi(x), \psi(-x)$, and the function $g(x) - g(-x)$.

Example 4.2: We consider the two-dimensional problem.

$$\begin{cases} \min\left(-\Delta u(x) + (x+y)/8 + 1/2, u(x) - u(-x) - \psi(x)\right), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega. \end{cases}$$

Here $\Omega = [-2, 2] \times [-2, 2]$,

$$\psi(x) = \frac{x+y}{10} - .1,$$

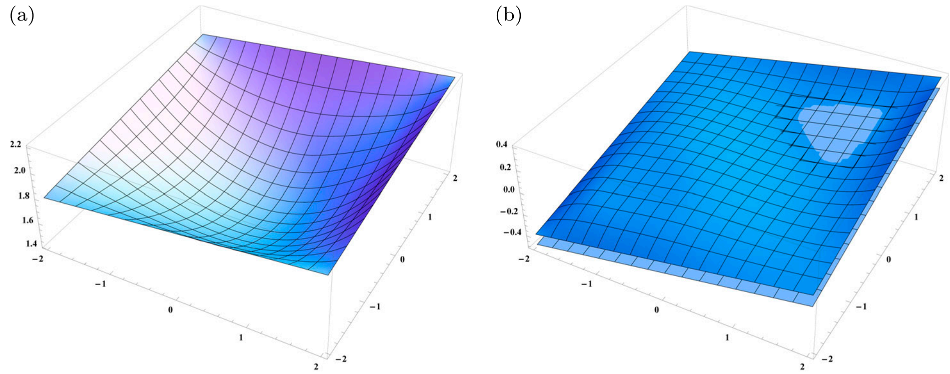

Figure 4. (a) Numerical solution. (b) The function $u(x) - u(-x)$ which touches obstacle ψ.

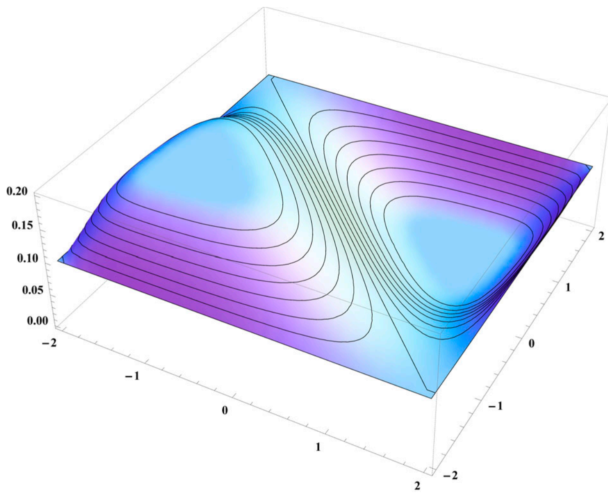

Figure 5. Level sets of the function $u(x) - u(-x) - \psi(x)$.

the boundary data is given by the function

$$g(x) = \begin{cases} 0.05(38 + x_2), & x_1 = -2 \\ 0.05(42 + x_2), & x_1 = 2 \\ 0.05(38 + x_1), & x_2 = -2 \\ 0.05(42 + x_1), & x_2 = 2 \end{cases}$$

In the Figure 3(a), (b) the boundary data and the obstacles are presented. The numerical algorithm produces the result given in Figure 4(a): the surface is the solution for our problem.

The numerical algorithm produces the result given in Figure 4(a): the surface is the solution for our problem. Figure 4(a) was constructed with 50 discretization points and 5 iterations. The free boundary is clearly visible in Figure 4(b) (the triangle-shaped boundary of the blue region).

Figure 5 shows the level sets of the function $u(x) - u(-x) - \psi(x)$. And the two coincidence sets are clearly visible, the first $u(x) - u(-x) = \psi(x)$ is a triangle-shaped zero-level set, the second one is a 0.2-level set $u(-x) - u(x) = -\psi(-x)$.

Note

1. It should be remarked that even though we call this an obstacle, it is indeed not the only obstacle function. The obstacle is actually given by $u(-x) + \psi(x)$.

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