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# The behavior of solutions of the system of two first order linear ordinary differential equations. Part I

Gevorg Avagovich **Grigorian**
*0019 Armenia c. Yerevan, str. M. Bagramian 24/5*
*Institute of Mathematics NAS of Armenia*

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**Abstract:** The Riccati equation method is used for study the behavior of solutions of the system of two linear first order ordinary differential equations. All types of oscillation and regularity of this system are revealed. A generalization of Leighton's theorem is obtained. Three new principles for the second order linear differential equation are derived. Stability and non conjugation criteria are proved for the mentioned system, as well as estimates are obtained for the solutions of the last one.

## 1. Introduction

Let $a_{jk}(t)$ $(j,k = 1,2)$ be real valued continuous functions on $[t_0;+\infty)$. Consider the system of equations

$$\begin{cases} \phi' = a_{11}(t)\phi + a_{12}(t)\psi; \\ \\ \psi' = a_{21}(t)\phi + a_{22}(t)\psi, \quad t \geq t_0. \end{cases} \tag{1.1}$$

Study of the questions of the asymptotic behavior (oscillation, non oscillation, non conjugation, rate of growth) of the solutions and stability of the linear system of ordinary differential equations, in

*E-mail address:* mathohys2@instmath.sci.am

particular of the system (1.1) is an important problem of the qualitative theory of differential equations and many works are devoted to them (see [1] and cited works therein, [2], [3], [4], [5], [6], [7]). Let $p(t)$, $q(t)$ and $r(t)$ be real valued continuous functions on $[t_0; +\infty)$, and let $p(t) > 0$, $t \geq t_0$. Along with the system (1.1) consider the equation

$$(p(t)\phi')' + q(t)\phi' + r(t)\phi = 0, \qquad t \geq t_0. \tag{1.2}$$

The substitution

$$p(t)\phi' = \psi \tag{1.3}$$

in this equation reduces it to the system

$$\begin{cases} \phi' &= \frac{1}{p(t)}\psi; \\ \\ \psi' &= -r(t)\phi - \frac{q(t)}{p(t)}(t)\psi, \;\; t \geq t_0, \end{cases} \tag{1.4}$$

which is a particular case of the system (1.1). For $p(t) \equiv 1$, $q(t) \equiv 0$ Eq. (1.2) takes the forme

$$\phi'' + r(t)\phi = 0, \qquad t \geq t_0. \tag{1.5}$$

It is well known (see for example [8]), that by using different transformations Eq. (1.2) can be reduced to the Eq. (1.5). One can show that the system (1.1) can be reduced to Eq. (1.5), if (for example) $a_{12}(t) \neq 0$, $t \geq t_0$. There exist also other conditions for which the system (1.1) can be reduced to Eq. (1.5). Of course the reduction of the system (1.1) to Eq. (1.5), if it is possible to carry it out (until now, it is not known whether this can always be done), can be very useful for study of different qualitative characteristics of the system (1.1). However this method not always can help to solve the assigned problem. One of effective methods of qualitative investigation of Eq. (1.5), as well as of the system (1.1) is the Riccati equation method. In this work we use this method for the study of the behavior of solutions of the system of two linear first order ordinary differential equations. We reveal all types of oscillation and regularity of this system. We obtain a generalization of Leighton's oscillation theorem. We derive three new principles for the second order linear ordinary differential equation. We prove some stability and non conjugation criteria for the mentioned system. We also obtain estimates for the solutions of the last one. Due to large number of sheets of this article we represent here the first part of obtained results of this work. The second part of it we will represent for publication later.

## 2. Auxiliary propositions

Let $a(t)$, $b(t)$ and $c(t)$ be real valued continuous function on $[t_0; +\infty)$. Consider the Riccati equation

$$x' + a(t)x^2 + b(t)x + c(t) = 0, \qquad t \ge t_0. \tag{2.1}$$

In this paragraph we study some important properties of global solutions (existing on $[t_1; +\infty)$ for some $t_1 \ge t_0$) of this equation which will be used further for the study of asymptotic properties of solutions of the system (1.1). Along with Eq. (2.1) consider the system of equations

$$\begin{cases} \phi' &= a(t)\psi; \\ \psi' &= -c(t)\phi - b(t)\psi, \qquad t \ge t_0. \end{cases} \tag{2.2}$$

The solutions $x(t)$ of Eq. (2.1), existiong on some interval $[t_1; t_2)$ $(t_0 \le t_1 < t_2 \le +\infty)$, are connected with the solutions $(\phi(t), \psi(t))$ of the system (2.2) by the equalities (see [9], pp. 153 – 154)

$$\phi(t) = \phi(t_1)\exp\left\{\int_{t_1}^{t} a(\tau)x(\tau)d\tau\right\}, \ \ \phi(t_1) \ne 0, \ \ \psi(t) = x(t)\phi(t). \tag{2.3}$$

In this paragraph we will take that all solutions of equations and systems of equations are real valued. For brevity we introduce the denotations:

$$J_u(t_1; t) \equiv \exp\left\{\int_{t_1}^{t} a(\tau)u(\tau)d\tau\right\}, \ \ J_u(t) \equiv J_u(t_0; t),$$

$$I^+_{u,v}(t_1; t) \equiv \int_{t_1}^{t} u(\tau)J_{-v}(t_1; \tau)d\tau, \ \ I^+_{u,v}(t_1) \equiv \int_{t_1}^{+\infty} u(\tau)J_{-v}(t_1; \tau)d\tau,$$

$$I^-_{u,v}(t_1; t) \equiv \int_{t_1}^{t} J_{-u}(\tau; t)v(\tau)d\tau, \ \ t_1, t \ge t_0,$$

where $u(t)$ and $v(t)$ be arbitrary continuous functions on $[t_0; +\infty)$. Rewrite Eq. (2.1) in the form:

$$x' + h_x(t)x + b(t) = 0, \qquad t \ge t_0,$$

where $h_x(t) \equiv a(t)x + b(t), \;\; t \ge t_0$. The by virtue of the Cauchy formula Eq. (2.1) is equivalent to the following integral equation

$$x = J_{-h_x}(t_1;t)\left[x(t_1) - \int\limits_{t_1}^{t} J_b(t;\tau)c(\tau)\phi_x(t_1;\tau)d\tau\right], \qquad t \ge t_0, \qquad (2.4)$$

where $\phi_x(t_1;t) \equiv \exp\left\{\int\limits_{t_1}^{t} a(\tau)x(\tau)d\tau\right\}, \;\; t_1, t \ge t_0$. Let $a_1(t), \;\; b_1(t)$ and $c_1(t)$ be real valued continuous function on $[t_0;+\infty)$. Along with Eq. (2.1) consider the equation

$$x' + a_1(t)x^2 + b_1(t)x + c_1(t) = 0, \qquad t \ge t_0, \qquad (2.5)$$

and the differential inequality

$$\eta' + a(t)\eta^2 + b(t)\eta + c(t) \ge 0, \qquad t \ge t_0. \qquad (2.6)$$

Note that for $a(t) \ge 0, \;\; t \ge t_0$ each solution of the linear equation

$$\eta' + b(t)\eta + c(t) = 0, \qquad t \ge t_0,$$

is a solution of (2.6). Therefore for each initial condition $\eta_{(0)}$ inequality (2.6) has a solution $\eta_0(t)$ on $[t_0;+\infty)$ with $\eta_0(t_0) = \eta_{(0)}$.
**Theorem 2.1**. *Let Eq. (2.5) has a solution $x_1(t)$ on $[t_0;\tau_0) \;\; (\tau_0 \le +\infty)$ and let the following condition be satisfied:*

$$a(t) \ge 0, \quad \int\limits_{t_0}^{t} \exp\left\{\int\limits_{t_0} \big[a(\xi)\big(\eta_0(\xi) + x_1(\xi)\big) + b(\xi)\big]d\xi\right\}\times$$
$$\times\big[\big(a_1(\tau) - a(\tau)\big)x_1^2(\tau) + \big(b_1(\tau) - b(\tau)\big)x_1(\tau) + c_1(\tau) - c(\tau)\big]d\tau \ge 0,$$
$$t \in [t_0;\tau_0),$$

*where $\eta_0(t)$ is a solution of (2.6) on $[t_0;\tau_0)$ with $\eta_0(t_0) \ge x_1(t_0)$. Then for each $x_{(0)} \ge x_1(t_0)$ Eq. (2.1) has a solution $x_0(t)$ on $[t_0;\tau_0)$, and $x_0(t) \ge x_1(t), \;\; t \in [t_0;\tau_0)$.*

See proof in [10]. Let $t_1 \ge t_0$.
**Definition 2.1.** *A solution of Eq. (2.1) is called $t_1$-regular, if it exists on $[t_1;+\infty)$. Eq. (2.1) is called regular if it has a $t_1$-regular solution for some $t_1 \ge t_0$.*

**Definition 2.2.** *A $t_1$-regular solution $x(t)$ of Eq. (2.1) is called $t_1$-normal, if there exists a neighborhood $U_x(t_1)$ of the point $x(t_1)$ such that each solution $\widetilde{x}(t)$ of Eq. (2.1) with $\widetilde{x}(t_1) \in U_x(t_1)$ is $t_1$-regular. Otherwise $x(t)$ is called $t_1$-extremal.*

**Remark 2.1.** *From the results of work [11] it follows that for some $t_1 \geq t_0$ the regular equation (2.1) can have: the unique $t_1$-regular solution (then it is $t_1$-extremal); no $t_1$-extremal solution (then its all $t_1$-regular solutions are $t_1$-normal); the unique $t_1$-extremal solution (and all other $t_1$-regular solutions are $t_1$-normal); two $t_1$- extremal solutions (all other $t_1$-regular solutions are $t_1$-normal).*

In what follow we will assume that the functions $a(t)$ and $c(t)$ have unbounded supports (the case when one of these functions has a bounded support is trivial). For arbitrary continuous function $u(t)$ on $[t_0; +\infty)$ denote:

$$\mu_u(t_1; t) \equiv \int\limits_{t_1}^{t} a(\tau) \exp\biggl\{-\int\limits_{t_1}^{\tau}[2a(\xi)u(\xi)+b(\xi)]d\xi\biggr\}d\tau,$$

$$\nu_u(t) \equiv \int\limits_{t}^{+\infty} a(\tau) \exp\biggl\{-\int\limits_{t}^{\tau}[2a(\xi)u(\xi) + b(\xi)]d\xi\biggr\}d\tau, \qquad t_1, t \geq t_0.$$

**Theorem 2.2.** *Let for some $t_1$-regular solution $x_0(t)$ of Eq. (2.1) the integral $\nu_{x_0}(t_1)$ be convergent. Then the following assertions are valid.*
*A) For each $t \geq t_1$ and for all $t_1$-normal solutions $x(t)$ of Eq. (2.1) and only for them the integrals $\nu_x(t)$ converge.*
*B) In order that Eq. (2.1) have $t_1$-extremal solution it is necessary and sufficient that*

$$\nu_{x_0}(t) \neq 0, \qquad t \geq t_1.$$

*Under this condition the unique $t_1$-extremal solution $x_*(t)$ is defined by formula*

$$x_*(t) = x_0(t) - \frac{1}{\nu_{x_0}(t)}, \qquad t \geq t_1, \tag{2.7}$$

*and*

$$\nu_{x_*}(t) = +\infty, \;\; t \geq t_1, \quad \textit{or} \quad \nu_{x_*}(t) = -\infty, \;\; t \geq t_1, \tag{2.8}$$

$$\int\limits_t^{+\infty} a(\tau)\big[x_1(\tau)-x_2(\tau)\big]d\tau = \ln\left[\frac{x_*(t)-x_1(t)}{x_*(t)-x_2(t)}\right], \qquad t \geq t_1, \tag{2.9}$$

$$\int\limits_{t_1}^{+\infty} a(\tau)\big[x_1(\tau)-x_2(\tau)\big]d\tau = -\infty. \tag{2.10}$$

Proof. All assertions of this theorem except (2.8) and (2.10), are proved in [11]. Let as prove (2.8). We will use the equalities (see [11]):

$$\mu_{x_*}(t_2;t) = \frac{\mu_{x_0}(t_2;t)}{1+\lambda_*(t_2)\mu_{x_0}(t_2;t)}, \tag{2.11}$$

$$x_1(t) = x_2(t) + \frac{\lambda_{12}(t_2)\exp\left\{-\int\limits_{t_1}^{t}[2a(\tau)x_2(\tau)+b(\tau)]d\tau\right\}}{1+\lambda_{12}(t_2)\mu_{x_2}(t_2;t)}, \ \ t\geq t_2\geq t_1, \tag{2.12}$$

where $\lambda_*(t_2) \equiv x_*(t_2)-x_0(t_2)$, $\lambda_{12}(t_2) \equiv x_1(t_2)-x_2(t_2)$, $t\geq t_2\geq t_1$, $x_1(t)$ and $x_2(t)$ be arbitrary $t_1$-regular solutions of Eq. (2.1). From (2.12) it follows that $\mu_{x_0}(t_2;t)$ is bounded by $t$ on $[t_2;+\infty)$. Then since obviously

$$\nu_{x_0}(t_2) = \lim_{t\to+\infty}\mu_{x_0}(t_2;t)\neq 0, \qquad t_2\geq t_1, \tag{2.13}$$

necessarily

$$\lim_{t\to+\infty}[1+\lambda_*(t_2)\mu_{x_0}(t_2;t)] = 0, \qquad t_2\geq t_1. \tag{2.14}$$

From here from (2.11) and (2.13) it follows (2.8). Let as prove (2.10). By (2.12) we have:

$$x_*(t)-x_0(t) = \frac{\lambda_*(t_1)\exp\left\{-\int\limits_{t_1}^{t}[2a(\tau)x_0(\tau)+b(\tau)]d\tau\right\}}{1+\lambda_*(t_1)\mu_{x_0}(t_1;t)}, \ \ t\geq t_1.$$

Multiplying both sides of this equality on $a(t)$ and integrating from $t_1$ to $t$ we will get

$$\int\limits_{t_1}^{t} a(\tau)[x_*(\tau)-x_0(\tau)]d\tau = \ln\big[1+\lambda_*(t_1)\mu_{x_0}(t_1;t)\big], \qquad t\geq t_1.$$

Passing to the limit in this equality when $t \to +\infty$ and taking into account (2.14) we will come to (2.10). The theorem is proved.

**Corollary 2.1.** *If for some $t_1$-regular solution $x_*(t)$ the equality $\nu_{x_*}(t_1) = \pm\infty$ is fulfilled, then $x_*(t)$ is the unique $t_1$-extremal solution of Eq. (2.1), and Eq. (2.1) has $t_1$-normal solutions, and for each $t \ge t_1$ and for all $t_1$-normal solutions $x(t)$ of Eq. (2.1) the integrals $\nu_x(t)$ converge; for every normal solutions $x_0(t)$, $x_1(t)$, $x_2(t)$ of Eq. (2.1) and for $x_*(t)$ the correlations (2.7) - (2.10) are satisfied.*

Proof. Let $\nu_{x_*}(t_1) = +\infty$ (the proof in the case $\nu_{x_*}(t_1) = -\infty$ by analogy). Then

$$\mu_{x_*}(t_1; t) > 1, \qquad t \ge T, \tag{2.15}$$

for some $T > t_1$. Let $\overline{\mu}_{x_*} \equiv \max\limits_{t\in[t_0;T]} |\mu_{x_*}(t_1;t)|$, and let $x_0(t)$ be a solution to Eq. (2.1) with $x_*(t_1) < x_0(t_1) < x_*(t_1) + \frac{1}{\max\{1, \overline{\mu}_{x_*}\}}$. Then taking into account (2.15) ve will have:

$$1 + \lambda_*(t_1)\mu_{x_*}(t_1; t) > 0, \qquad t \ge t_1,$$

where $\lambda_*(t_1) \equiv x_)(t_1) - x_*(t_1) > 0$. Hence (see [11]) by (2.11) $x_0(t)$ is a $t_1$-regular solution of Eq. (2.1). Show that the integrals $\nu_{x_*}(t)$ converge for all $t \ge t_1$ and $\nu_{x_*}(t) \ne 0,$ . We use the equality (see [11])

$$\mu_{x_0}(t_2; t) = \frac{\mu_{x_*}(t_2; t)}{1 + \lambda_*(t_2)\mu_{x_*}(t_2; t)}, \qquad t \ge t_2 \ge t_1,$$

where $\lambda_*(t_2) \equiv x_0(t_2) - x_*(t_2) \ne 0$. For enough large values of $t > t_2$ we have $\mu_{x_*}(t_2; t) > 0$. Therefore,

$$\nu_{x_0}(t_2) = \lim_{t\to+\infty} \mu_{x_0}(t_2; t) = \lim_{t\to+\infty} \frac{1}{\lambda_*(t_2) + \frac{1}{\mu_{x_*}(t_2;t)}} = \frac{1}{\lambda_*(t_2)} \ne 0.$$

So for $x_0(t)$ all conditions of Theorem 2.2 are fulfilled. Therefore Eq. (2.1) has $t_1$-normal solutions and for every $t_1$-normal solutions $x(t)$ of Eq. (2.1) and for all $t \ge t_1$ the integrals $\nu_x(t)$ converge; also for every $t_1$-normal solutions $x_0(t)$, $x_1(t)$, $x_2(t)$ of Eq. (2.1) and for $x_*(t)$ the correlations (2.7) - (2.10) are satisfied. The corollary is proved.

Denote by $reg(t_1)$ the set of values $x_{(0)} \in R$, for which the solution $x(t)$ of Eq. (2.1) with $x(t_1) = x_{(0)}$ is $t_1$-regular.

**Lemma 2.1**. *Let $a(t) \geq 0, \;\; t \geq t_0$, and let Eq. (2.1) has $t_1$-regular solution. Then it has the unique $t_1$-extremal solution $x_*(t)$, and $reg(t_1) = [x_*(t_1); +\infty)$.*

See proof in [2].

**Lemma 2.2.** *let $a(t) \geq 0, \;\; t \geq t_0; \;\; t_0 \leq t_1 < t_2$, and let $(t_1; t_2)$ be the maximal existence interval for the solution $x(t)$ of Eq. (2.1). Then $\lim\limits_{t\to t_1+0} x(t) = +\infty$.*

See proof in [10].

**Lemma 2.3**. *Let $a(t) \geq 0, \;\; t \geq t_0, \;\; x_0(t)$ be a $t_0$-normal solution of Eq. (2.1), $x_0(t) \neq \; 0, \;\; t \geq t_0$. Then for its unique $t_0$-extremal solution $x_*(t)$ the equality*

$$\int\limits_{t_0}^{t} a(\tau)x_*(\tau)d\tau = -\ln \nu_{x_0}(t_0) + \ln\Biggl[\exp\Biggl\{\int\limits_{t_0}^{t} a(\xi)x_0(\xi)d\xi\Biggr\}\times$$

$$\times \int\limits_{t}^{+\infty} \frac{a(s)x_0(s)}{x_0(t_0)} \exp\Biggl\{\int\limits_{t_0}^{s}\biggl[\frac{c(\xi)}{x_0(\xi)} - a(\xi)x_0(\xi)\biggr]d\xi\Biggr\}ds\Biggr], \qquad t \geq t_0. \quad (2.16)$$

*holds.*

Proof. By Lemma 2.1 Eq. (2.1) has a $t_0$-normal solution $x_0(t)$. Then since $a(t) \geq 0, \;\; t \geq t_0$ and has unbounded support, the integral $\nu_{x_0}(t)$ converges for all $t \geq t_0$ and $\nu_{x_0}(t) \neq 0, \;\; t \geq t_0$. By virtue of Theorem 2.2 from here it follows that Eq. (2.1) has the unique $t_0$-extremal solution $x_*(t)$, satisfying the equality $x_*(t) = x_0(t) - \frac{1}{\nu_{x_0}(t)}, \;\; t \geq t_0$. From here it follows:

$$\int\limits_{t_0}^{t} a(\tau)x_*(\tau)d\tau = \int\limits_{t_0}^{t} a(\tau)x_0(\tau)d\tau - \int\limits_{t_0}^{t} \frac{a(\tau)}{\nu_{x_0}(\tau)}d\tau =$$

$$= \ln\bigl[\exp\bigl\{\int\limits_{t_0}^{t} a(\tau)x_0(\tau)d\tau\bigr\}\bigr] - \ln \nu_{x_0}(t_0) +$$

$$+ \int\limits_{t_0}^{t} d\Biggl(\ln\Biggl[\int\limits_{\tau}^{+\infty} a(s)\exp\Biggl\{-\int\limits_{t_0}^{s}\bigl[2a(\xi)x_0(\xi)+b(\xi)\bigr]d\xi\Biggr\}ds\Biggr]\Biggr), \; t \geq t_0. \quad (2.17)$$

On the strength of (2.1) from the condition $x_0(t) \neq 0,\ t \geq t_0$, it follows:

$$2a(\xi)x_0(\xi) + b(\xi) = -\frac{x_0'(\xi) - a(\xi)x_0^2(\xi) + c(\xi)}{x_0(\xi)}, \ \ \xi \geq t_0.$$

From here and from (2.17) it follows (2.16). The lemma is proved.

**Lemma 2.4.** *Let $a(t) \geq 0,\ c(t) \geq 0,\ t \geq t_0,\ I^+_{a,b}(t_0) = +\infty$ and let Eq. (2.1) has a solution on $[t_1; +\infty)$ for some $t_1 \geq t_0$. Then Eq. (2.1) has a positive solution on $[t_1; +\infty)$.*

See proof in [5].

**Theorem 2.3**. *Let $a(t) \geq 0,\ c(t) \leq 0,\ t \geq t_0$. Then the following assertions are valid.*

*I°). For each $x_{(0)} \geq \frac{-1}{I^+_{a,b}(t_0)}$ (for $I^+_{a,b}(t_0) = +\infty$ we take that $\frac{1}{I^+_{a,b}(t_0)} = 0$) Eq. (2.1) has a $t_0$-regular solution $x_0(t)$ with $x_0(t_0) = x_{(0)}$, and*

$$\frac{x_{(0)}J_{-b}(t)}{1 + x_{(0)}I^+_{a,b}(t_0;t)} \leq x_0(t) \leq x_{(0)}J_{-b}(t) - I^-_{a,b}(t_0;t), \qquad t \geq t_0, \qquad (2.18)$$

*moreover if $x_{(0)} = 0$, then there exists $t_1 \geq t_0$ such that $x_0(t) = 0$, $t \in [t_0; t_1], x_0(t) > 0,\ t > t_1$. If $x_{(0)} > 0$ then $x_0(t) > 0,\ t \geq t_0$.*

*II°). The unique $t_0$-extremal solution $x_*(t)$ of Eq. (2.1) is negative.*

*III°). If $I^+_{a,b}(t_0) = +\infty$ or $I^+_{c,-b}(t_0) = -\infty$, then for each solution $x(t)$ of Eq. (2.1) with $x(t_0) \in (x_*(t_0); 0)$ there exists $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) < 0,\ t \in [t_0; t_1),\ x(t) = 0,\ t \in [t_1; t_2]$ and $x(t) > 0$, $t > t_2$.*

*IV°). If $I^+_{a,b}(t_0) = +\infty$, then foe each $t_0$-normal solution $x_N(t)$ of Eq. (2.1) the equality $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = +\infty$. is fulfilled.*

*V°). If $I^+_{c,-b}(t_0) = -\infty$, then $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = -\infty$, where $x_*(t)$ is the unique $t_0$-extremal solution of Eq. (2.1).*

*VI°). If $I^+_{a,b}(t_0) < +\infty$ and $I^+_{-c,-b}(t_0) < +\infty$, then Eq. (2.1) has a negative $t_0$-normal solution $x_N^-(t)$ such that for each solution $x(t)$ of Eq. (2.1) with $x_0(t_0) \in (x_N^-(t_0); 0)$ there exists $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) < 0,\ t \in [t_0; t_1),\ x(t) = 0,\ t \in [t_1; t_2],\ x(t) > 0,\ t > t_2$.*

*VII°). If*

$$I^+_{a,b}(t_0) = +\infty, \quad \int\limits_{t_0}^{+\infty} |c(\tau)| I^-_{-b,a}(t_0;\tau) d\tau < +\infty, \tag{2.19}$$

*then* $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau > -\infty$, $\int\limits_{t_0}^{+\infty} c(\tau)u_*(\tau)d\tau = +\infty$, *where* $u_*(t)$ *is the unique* $t_0$*-extremal solution of the equation*

$$u' - c(t)u^2 - b(t)u - a(t) = 0, \qquad t \ge t_0.$$

Proof. Set $a_1(t) = a(t)$ $b_1(t) = b(t)$, $t \ge t_0$, $c_1(t) \equiv 0$. Then for each $x_{(0)} \ge \frac{-1}{I^+_{a,b}(t_0)} \stackrel{def}{=} A$ the function $x_1(t) \equiv \frac{x_{(0)} J_{-b}(t)}{1 + x_{(0)} I^+_{a,b}(t_0;t)}$ is a $t_0$-regular solution of Eq. (2.5), and the conditions of Theorem 2.1 are fulfilled. Therefore for each $x_{(0)} \ge A$ Eq. (2.1) has a $t_0$-regular solution $x_0(t)$ with $x_0(t_0) = x_{(0)}$ and the first of conditions of inequalities (2.18) is satisfied. Set $a_1(t) = a(t)$, $b_1(t) = b(t)$, $c_1(t) = c(t)$, $t \ge t_0$. Then by already proven Eq. (2.5) will have $t_0$-regular solutions, coinciding wit the $t_0$-regular solutions of Eq. (2.1). In the Eq. (2.1) set: $a(t) \equiv 0$. Then $x_2(t) \equiv x_{(0)} J_{-b}(t_0;t) - I^-_{b,c}(t_0;t)$ is a $t_0$-regular solution of Eq. (2.1). Obviously in this case the conditions of Theorem 2.1. are satisfied. Therefore the second of the inequalities (2.18) is fulfilled. Let $x_{(0)} = 0$. Then since $c(t)$ has unbounded support by virtue of (2.17) from the inequality $c(t) \le 0$, $t \ge t_0$, it follows existence of $t_1 > t_0$ such that $x(t) = 0$, $t \in [t_0;t_1]$ and $x(t) > 0$, $t > t_1$. The assertion I° is proved. Prove II°. Let $x_0(t)$ be a solution of Eq. (2.1) with $x_0(t_0) > 0$. By virtue of I° $x_0(t)$ is $t_0$-normal and positive. Therefore from Theorem 2.2 it follows that for each $t \ge t_0$ the integral $\nu_{x_0}(t)$ converges. Obviously $\nu_{x_0}(t) > 0$, $t \ge t_0$. Then by virtue of the same Theorem 2.2 $x_*(t) \equiv x_0(t) - \frac{1}{\nu_{x_0}(t)}$ is the unique $t_0$-extremal solution of Eq. (2.1). Show that $x_*(t) \le 0$, $t \ge t_0$. By virtue of the first of inequalities (2.18) we have:

$$\frac{x_0(t) J_{-b}(t;s)}{1 + x_0(t) I_{a,b}(t;s)} \le x_0(s), \qquad t_0 \le t \le s.$$

Multiplying both sides of this inequality on $a(s)$ and integrating by $s$ from $t$ to $\tau$. we will get: $\ln[1 + x_0(t) I_{a,b}(t;\tau)] \le \int\limits_{t_0}^{t} a(s)x_0(s)ds$, $t_0 \le t \le s$.

Then

$$\nu_{x_0}(t) \le \int\limits_t^{+\infty} \frac{a(\tau)J_{-b}(t;\tau)d\tau}{1 + x_0(t)I^+_{a,b}(t;\tau)} = -\frac{1}{x_0(t)} \int\limits_t^{+\infty} d\left(\frac{1}{1 + x_0(t)I^+_{a,b}(t;\tau)}\right) =$$

$$= \frac{1}{x_0(t)}\left[1 - \frac{1}{1 + x_0(t)I^+_{a,b}(t;+\infty)}\right] \le \frac{1}{x_0(t)}, \qquad t \ge t_0.$$

From here it follows that $x_*(t) \le 0,\ \ t \ge t_0$. Show that

$$x_*(t) < 0, \qquad t \ge t_0. \tag{2.20}$$

Suppose that it is not true. Then since $x_*(t) \le 0,\ \ t \ge t_0$, there exists $t_1 \ge t_0$ such that $x_*(t_1) = 0$. By the first of the inequalities (2.18) from here it follows that $x_*(t) \ge 0,\ \ t \ge t_1$. Hence $x_*(t) \equiv 0$ on $[t_1;+\infty)$, which is impossible (since on $[t_1;+\infty)$ $c(t) \not\equiv 0$.) The obtained contradiction proves (2.20), and therefore the assertion II°. Prove III°. Let $x_0(t)$ and $x_1(t)$ be solutions of Eq. (2.1) with the initial conditions $x_0(t_0) > 0,\ \ x_1(t_0) \in (x_*(t_0);0)$. By virtue of Lemma 2.1 $x_0(t)$ and $x_1(t)$ are $t_0$-normal. Therefore by (2.8) we have

$$\int\limits_{t_0}^{+\infty} a(\tau)[x_0(\tau) - x_1(\tau)]d\tau < +\infty. \tag{2.21}$$

Let $I^+_{a,b}(t_0) = +\infty$. Then from the first of inequalities (2.18) it follow:

$$\int\limits_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau \ge \ln[1 + x_0(t_1)I^+_{a,b}(t_0)] = +\infty, \tag{2.22}$$

Show that there exists $\widetilde{t}_1 \ge t_0$ such that $x_1(\widetilde{t}_1) = 0$. Suppose that it is not true. Then $x_1(t) < 0,\ \ t \ge t_0$. Taking into account (2.18) from here we will get:

$$\int\limits_{t_0}^{+\infty} a(\tau)(x_0(\tau) - x_1(\tau))d\tau \ge \int\limits_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau = +\infty, \tag{2.23}$$

which contradicts (2.21). The obtained contradiction shows that $x_1(\widetilde{t}_1) = 0$ for some $\widetilde{t}_1 > t_0$. Since $c(t)$ has unbounded support by virtue of (2.4) from here and from non positivity of $c(t)$ it follows that $x_1(t) < 0,\ t \in [t_0; t_1),\ x_1(t) = 0,\ t \in [t_1; t_2]$ and $x_1(t) > 0,\ t > t_2$, for some $t_2 \ge t_1 > t_0$. Let $I^+_{c,-b}(t_0) = -\infty$. Consider the equation

$$u' - c(t)u^2 - b(t)u - a(t) = 0, \qquad t \ge t_0. \tag{2.24}$$

By II° the unique $t_0$-extremal solution $u_*(t)$ of this equation is negative. Therefore $\widetilde{x}_*(t) \equiv \frac{1}{u_*(t)}$ is a $t_0$-regular solution of Eq. (2.1). By already proven, from here and from the equality $I^+_{c,-b}(t_0) = -\infty$ it follows that each solution $u(t)$ of Eq. (2.24) with $u(t_0) \in (u_*(t_0); 0)$ vanishes on $[t_0; +\infty)$. Therefore each solution $x(t)$ of Eq. (2.1) with $x(t_0) < \widetilde{x}(t_0)$ is not $t_0$-regular. By virtue of Lemma 2.1 from here it follows that

$$u_*(t) = \frac{1}{x_*(t)}, \qquad t \ge t_0. \tag{2.25}$$

Suppose that some solution $\widetilde{x}(t)$ of Eq. (2.1) with $\widetilde{x}(t_0) \in (x_*(t_0); 0)$ is negative. Then $\widetilde{u}(t) \equiv \frac{1}{\widetilde{x}(t)},\ t \ge t_0$, is a $t_0$-regular solution of Eq. (2.24), and $\widetilde{u}(t_0) = \frac{1}{\widetilde{x}(t_0)} < \frac{1}{x_*(t_0)}$. From here and from (2.25) it follows that $\widetilde{u}(t_0) < u_*(t_0)$, which contradicts Lemma 2.1. The obtained contradiction shows that for each solution $x(t)$ of Eq. (2.1) with $x(t_0) \in (x_*(t_0); 0)$ there exists $t_1 = t_1(x) > t_0$ such that $x(t_1) = 0,\ x(t) < 0,\ t \in [t_0; t_1)$. Since $c(t)$ has unbounded support by (2.4) from here and from non negativity of $c(t)$ it follows that there exists $t_2 = t_2(x) \ge t_1$ such that $x(t) = 0,\ t \in [t_1; t_2],\ x(t) > 0, t > t_2$. The assertion III° is proved. Prove IV°. Let $x_+(t)$ be a solution of Eq. (2.1) with $x_+(t_0) = 1$. On the strength of Lemma 2.1 from the assertion I° it follows that $x_+(t)$ is $t_0$-normal. By the first of the inequalities (2.18) we have $x_+(t) \ge J_{-b}(t),\ t \ge t_0$. Let $I^+_{a,b}(t_0) = +\infty$. Then from the last inequality it follows that

$$\int\limits_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau \ge I^+_{a,b}(t_0) = +\infty. \tag{2.26}$$

Let $x_N(t)$ be an arbitrary $t_0$-normal solution of Eq. (2.1). By (2.9) we have: $\int\limits_{t_0}^{+\infty} |x_+(\tau) - x_N(\tau)|d\tau < +\infty$.

From here and from (2.26) we will get:

$$\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int\limits_{t_0}^{+\infty} a(\tau)(x_N(\tau) - x_+(\tau))d\tau + \int\limits_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau = +\infty.$$

The assertion IV° is proved. Prove V°. Since on the strength of II° $I_*(t) \equiv \int\limits_{t_0}^{t} a(\tau)x_*(\tau)d\tau$ is a monotonically non increasing function on $[t_0; +\infty)$, from Lemma 2.3 it follows (after differentiation (2.10)):

$$a(t)x_0(t)\exp\Big\{\int\limits_{t_0}^{t} a(\xi)x_0(\xi)d\xi\Big\}\cdot$$
$$\cdot\int\limits_{t}^{+\infty} \frac{a(s)x_0(s)}{x_0(t_0)}\exp\Big\{\int\limits_{t_0}^{s}\big[\frac{c(\xi)}{x_0(\xi)} - a(\xi)x_0(\xi)\big]d\xi\Big\}ds \le$$
$$\le \frac{a(t)x_0(t)}{x_0(t_0)}\exp\Big\{\int\limits_{t_0}^{t}\frac{c(\xi)}{x_0(\xi)d\xi}\Big\}, \qquad t \ge t_0. \qquad (2.27)$$

where $x_0(t)(> 0,\ \ t \ge t_0)$ is a $t_0$-normal solution of Eq. (2.1) (by virtue of Lemma 2.1 from I° it follows the existence of $x_0(t)$). Since $u_0(t) \equiv \frac{1}{x_0(t)}$ is a $t_0$-normal solution of Eq. (2.24) and $I^+_{-c,-b}(t_0) = +\infty$, by IV° we have:

$$\int\limits_{t_0}^{+\infty} c(\tau)u_0(\tau)d\tau = \int\limits_{t_0}^{+\infty}\frac{c(\tau)}{x_0(\tau)}d\tau = -\infty. \qquad (2.28)$$

Since $a(t)$ has unbounded support there exists infinitely large sequence $t_0 < t_1 < ... < t_m < ...$ such that $a(t_m) > 0,\ \ m = 1, 2, ...$ . Then from (2.27) it follows

$$\exp\Big\{\int\limits_{t_0}^{t_m} a(\xi)x_0(\xi)d\xi\Big\}\int\limits_{t_m}^{+\infty}\frac{a(s)x_0(s)}{x_0(t_0)}\exp\Big\{\int\limits_{t_0}^{s}\big[\frac{c(\xi)}{x_0(\xi)} - a(\xi)x_0(\xi)\big]d\xi\Big\}ds \le$$
$$\le \exp\Big\{\int\limits_{t_0}^{t_m}\frac{c(\xi)}{x_0(\xi)}d\xi\Big\},$$

$m = 1, 2, \ldots$ . Due to Lemma 2.3 From here and from (2.28) it folloes that $I_*(t_m) \to -\infty$ for $m \to +\infty$. Hence, $\int\limits_{t_0}^{+\infty} a(\tau)x(\tau)d\tau = -\infty$. The assertion V° is proved. Prove VI°. Show that Eq. (2.1) has a $t_0$-normal negative solution. In Eq. (2.1) make the change: $x = J_{-b}(t_0; t)X, \ \ t \ge t_0$. We will come to the equation

$$X' + a(t)J_{-b}(t)X^2 + c(t)J_b(t) = 0, \qquad t \ge t_0. \tag{2.29}$$

Due to conditions of VI° chose $t_1(> t_0)$ so large that

$$\Bigl[\int\limits_{t_1}^{+\infty} a(\tau)J_{-b}(\tau)d\tau\Bigr]^{-1} > -\int\limits_{t_1}^{+\infty} c(\tau)J_b(\tau)d\tau.$$

Then

$$-\bigl[I_{a,b}^+(t_0)\bigr]^{-1} < I_{c,-b}^+(t_0) < 0. \tag{2.30}$$

Let then $X_-(t)$ be a solution to Eq. (2.29) with

$$X_-(t_1) \in \bigl(-\bigl[I_{a,b}^+(t_0)\bigr]^{-1}; I_{c,-b}^+(t_0)\bigr). \tag{2.31}$$

By (2.18) the inequalities

$$\frac{X_-(t_1)}{1 + X_-(t_1)I_{a,b}^+(t_1; t)} \le X_-(t) \le X_-(t_1) - I_{c,-b}^+(t_1; t), \qquad t \ge t_1, \tag{2.32}$$

are fulfilled.

From here and from (2.30) and (2.31) it follows that $X_-(t)$ is defined on $[t_1; +\infty)$ negative $t_1$-normal solution of Eq. (2.29). Then $x_-(t) \equiv X_-(t)J_{-b}(t_1; t)$ is defined on $[t_1; +\infty)$ negative $t_1$-normal solution to Eq. (2.1). Show that $x_-(t)$ is continuable on $[t_0; +\infty)$ as a solution to Eq. (2.1). Suppose $x_-(t)$ can not be continued on $[t_0; +\infty)$ as a solution of Eq. (2.1). Let then $(t_2; +\infty)$ be the maximum existence interval for $x_-(t)$, where $t_2 \ge t_0$. By Lemma 2.2 there exists $t_3 > t_2$ such that $x_-(t_3) > 0$. On the strength of the first of the inequalities (2.18) from here it follows that $x_-(t) > 0, \ \ t \ge t_3$. The obtained contradiction shows that $x_-(t)$ is continuable on $[t_0; +\infty)$. By virtue of the first of the inequalities (2.18) the supposition that $x_-(t_4) \ge 0$ for some $t_4 \ge t_0$ also leads to the contradiction.

So, $x_-(t) < 0$, $t \geq t_0$. Since $x_-(t)$ is $t_1$-normal, by continuable dependence of solutions of Eq. (2.1) from their initial values, the solution $x_-(t)$ also is $t_0$-normal. According to I° the solution $x_0(t)$ of Eq. (2.1) with $x_0(t_0) = 0$ starting with some $t_1 = t_1(x_0) \geq t_6$ becomes positive. Then by continuable dependence of solutions of Eq. (2.1) from their initial values, all initial values $x_{(0)}$, for which the solutions $x(t)$ of Eq. (2.1) with $x(t_0) = x_{(0)}$ eventually become positive, form an open set. From here from Lemma 2.1 and from the fact that $x_-(t)$ is negative it follows that there exists a negative $t_0$-normal solution $X_N^-(t)$ of Eq. (2.1) such that each solution $x(t)$ of Eq. (2.1) with $x(t_0) > X_N^-(t_0)$ eventually becomes positive.

By (2.4) from here it follows that for each solution $x(t)$ of Eq. (2.1) with $x(t_0) \in (x_N^-(t_0); 0)$ there exists $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) < 0$, $t \in [t_0; t_1)$, $x(t) = 0$, $t \in [t_1; t_2]$, $x(t) > 0$, $t > t_2$. By (2.8) $\int\limits_{t_0}^{+\infty} a(\tau)[x_*(\tau) - x_N(\tau)]d\tau = -\infty$.

Then $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau \leq \int\limits_{t_0}^{+\infty} a(\tau)[x_*(\tau) - x_N(\tau)]d\tau = -\infty$. Let $x_+(t)$ is a solution to Eq. (2.1) with $x_+(t_0) = 1$. On the strength of Lemma 2.1 from I° it follows that $x_+(t)$ is $t_0$-normal. Then by (2.9) we have:

$$0 < \int\limits_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau \leq \int\limits_{t_0}^{+\infty} a(\tau)[x_+(\tau) - x_N^-(\tau)]d\tau < +\infty. \tag{2.33}$$

Let $x_N(t)$ be an arbitrary $t_0$-normal solution to Eq. (2.1). Then since $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int\limits_{t_0}^{+\infty} [x_N(\tau) - x_+(\tau)]d\tau + \int\limits_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau$ and by virtue of (2.9) $\int\limits_{t_0}^{+\infty} a(\tau)|x_N(\tau) - x_+(\tau)|d\tau < +\infty$, from (2.33) it follows that the integral $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau$ converges. The assertion VI° is proved. Prove VII°. Due to the second of inequalities (2.18) taking into account the inequality $c(t) \le 0, \ \ t \ge t_0$ we have:

$$\int\limits_{t_0}^{t} \frac{c(\tau)}{x_0(\tau)} d\tau \ge \frac{1}{x_0(t_0)} \int\limits_{t_0}^{t} c(\tau) J_b(t) \Big[1 + x_0(t_0) I^+_{a,b}(t_0;\tau)\Big] d\tau, \qquad t \ge t_0, \tag{2.34}$$

where $x_0(t)$ is a positive $t_0$-normal solution of Eq. (2.1), existence of which follows from Lemma 2.1 and from I°. By virtue of Fubini's theorem from the second relation of (2.19) it follows that $I^+_{-b,-c}(t_0;+\infty) < +\infty$. From here and from the second relation of (2.19) and from (2.34) it follows that

$$\int\limits_{t_0}^{+\infty} \frac{c(\tau)}{x_0(\tau)} d\tau > -\infty. \tag{3.35}$$

From the first relations of (2.18) and (2.19) it follows that

$$\int\limits_{t_0}^{+\infty} a(\tau) x_0(\tau) d\tau = +\infty. \tag{2.36}$$

Let $g(t) \equiv \exp\Big\{-\int\limits_{t_0}^{t} a(\xi) x_0(\xi) d\xi\Big\}$. Obviously the inverse function $g^{-1}(t)$ of $g(t)$ exists on *supp* $a(t)$. Denote:

$$g_1(t) \equiv \begin{cases} g^{-1}(t), \ \ t \in supp \ \ a(t); \\ \\ t_0, \qquad t \notin supp \ \ a(t), \ \ t \ge t_0. \end{cases}$$

Then taking into account (2.36) the equality (2.16) can be rewritten in the form

$$\int\limits_{t_0}^{t} a(\tau) x_*(\tau) d\tau = -\ln \nu_{x_0}(t_0) + \ln\left[\frac{\int\limits_0^{g(t)} \exp\Big\{\int\limits_{t_0}^{g^{-1}(\zeta)} \frac{c(\xi)}{x_0(\xi)} d\xi\Big\} d\zeta}{x_0(t_0) g(t)}\right] \ge 1, \ t \ge t_0.$$

From here and from (2.35) it follows that

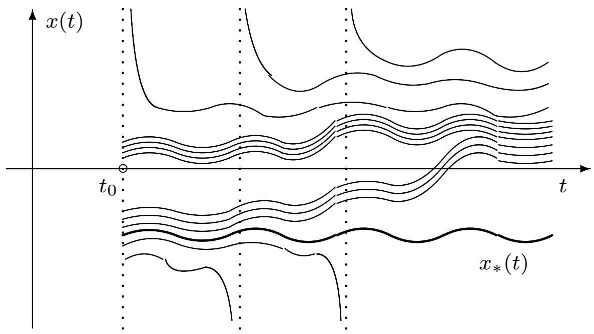
$$\int\limits_{t_0}^{+\infty} a(\tau) x_*(\tau) d\tau \ge -\ln[x_0(t_0)\nu_{x_0}(t_0)] + \int\limits_{t_0}^{+\infty} \frac{c(\xi)}{x_0(\xi)} d\xi > -\infty.$$

By virtue of Lemma 2.1 $u_0(t) \equiv \frac{1}{x_0(t)}$ is a $t_0$-normal solution of Eq. (2.24) (since for $x_1(t_0) > x_0(t_0)$ the function $u_1(t) \equiv \frac{1}{x_1(t)}$ is a $t_0$-regular solution of Eq. (2.24), where $x_1(t)$ is a $t_0$-regular solution to Eq. (2.1)). Then by (2.8) taking into account (2.35) we will have:

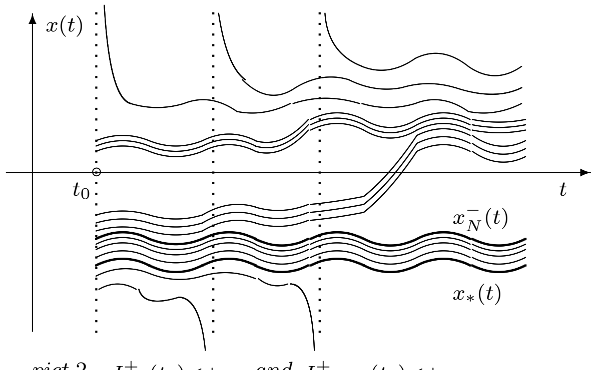
$$\int\limits_{t_0}^{+\infty} c(\tau)u_*(\tau)d\tau = \int\limits_{t_0}^{+\infty} c(\tau)(u_*(\tau) - u_0(\tau))d\tau + \int\limits_{t_0}^{+\infty} \frac{c(\tau)}{x_0(\tau)}d\tau = +\infty.$$

The assertion VII° is proved. The theorem is proved.

On the basis of Theorem 2.3 it can be make phase portrait of solutions of Eq. (2.1) in the case $a(t) \geq 0$, $c(t) \leq 0$, $t \geq t_0$, for the following two possible restrictions:
a) $I^+_{a,b}(t_0) = +\infty$ or $I^+_{c,-b}(t_0) = -\infty$ (see pict. 1) ;
b) $I^+_{a,b}(t_0) < +\infty$ and $I^+_{-c,-b}(t_0) < +\infty$ (see pict. 2).
In pict 1. we see only one negative global solution of Eq (2.1), meanwhile in pict. 2 we see whole slice of negative global solutions of Eq. (2.1).




*pict.1.* $I^+_{a,b}(t_0)=+\infty$ *or* $I^+_{-c,-b}(t_0)=+\infty$




*pict.2.* $I^+_{a,b}(t_0)<+\infty$ *and* $I^+_{-c,-b}(t_0)<+\infty$

**Theorem 2.4.** *Let $a(t) \geq 0,\ c(t) \geq 0,\ t \geq t_0$, and let Eq. (2.1) has a $t_0$-regular solution. Then the following assertions are valid.*
*I* If $I^+_{a,b}(t_0) = +\infty$, then each $t_0$-regular solution of Eq. (2.1) is positive and for its each $t_0$-normal solution $x_N(t)$ the equality $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = +\infty$ is fulfilled. Moreover if in addition $\int\limits_{t_0}^{+\infty} c(\tau)I^+_{a,-b}(t_0;\tau)d\tau = +\infty$, then also $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = +\infty$, where $x_*(t)$ is the unique $t_0$-extremal solution of Eq. (2.1).*
*II* If $I^+_{c,-b}(t_0) = +\infty$, then for each $t_0$ -regular solution $x(t)$ of Eq. (2.1) with $x(t_0) > 0$ there exist $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) > 0,\ t \in [t_0;t_1),\ x(t) = 0,\ t \in [t_1;t_2],\ x(t) < 0,\ t > t_2$, and if $x(t_0) = 0(< 0)$, there exists $t_1 = t_1(x) \geq t_0$ such that $x(t) = 0$, $t \in [t_0;t_1],\ x(t) < 0,\ t > t_1$ (then $x(t) < 0,\ t \geq t_0$), and $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = -\infty$. Moreover if in addition $\int\limits_{t_0}^{+\infty} a(\tau)I^-_{b,c}(t_0;\tau)d\tau = +\infty$, then for each $t_0$-normal solution $x_N(t)$ of Eq. (2.1) the equality $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = -\infty$ is fulfilled.*
*III* If $I^+_{a,b}(t_0) < +\infty$ and $I^+_{c,-b}(t_0) < +\infty$, then there exist $t_1 \geq t_0$ such that $x_*(t) < 0,\ t \geq t_1$; the solutions $x(t)$ of Eq. (2.1) with $x(t_1) \in (x_*(t_1);0)$ are $t_0$-regular and $x(t) < 0,\ t \geq t_1$; there exists a $t_1$-normal positive on $[t_1;+\infty)$ solution $x^+_N(t)$ of Eq. (2.1) such that fir each solution $x(t)$ of Eq. (2.1) with $x(t_1) \in (0;x^+_N(t_1))$ there exist $t_3 = t_3(x) \geq t_2 = t_2(x) > t_1$ such that $x(t) > 0,\ t \in [t_1;t_2)$, $x(t) = 0,\ t \in [t_2;t_3],\ x(t),0,\ t\ < t_3$, and if $x(t_1) = 0\ (x(t_1) < 0)$, there exists $t_2 = t_2(x) \geq t_1$ such that $x(t) = 0,\ t \in [t_1;t_2],\ x(t) < 0,\ t > t_2$ (then $x(t) < 0,\ t \geq t_1$); for each $t_0$-normal solution $x_N(t)$ of Eq. (2.1) the integral $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau$ converges and $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = -\infty$.*

Proof. Let us prove I$^*$. Let $x_*(t)$ be the $t_0$-extremal solution of Eq. (2.1). Show that $x_*(t) > 0,\ t \geq t_0$. Suppose for some $t_1 > t_0$ the inequality $x_*(t_1) < 0$ is satisfied. Let then $x(t)$ be a solution to Eq. (2.1) with $x(t_1) \in (x_*(t_1);0)$. By virtue of Lemma 2.1 $x(t)$ is $t_1$-normal. Since $x(t_1) < 0$ and $c(t) \geq 0,\ t \geq t_0$, by (2.4) we have $x(t) < 0,\ t \geq t_1$.

From here it follows that

$$\nu_x(t_1) \ge I^+_{a,b}(t_1). \tag{2.37}$$

Let $I^+_{a,b}(t_0) = +\infty$. Then from the easily verifiable equality

$$I^+_{a,b}(t_0) = I^+_{a,b}(t_0;t) + J_b(t_1) I^+_{a,b}(t_1) \tag{2.38}$$

and from (2.37) it follows that $\nu_x(t_1) = +\infty$. But on the other hand since $x(t)$ is $t_1$-normal by virtue of Theorem 2.2 we have $\nu_x(t_1), +\infty$. The obtained contradiction shows that $x_*(t) \ge 0, \ t \ge t_0$. Show that the equality $x_*(t) = 0$ impossible for all $t \ge t_0$. Suppose for some $t_2 \ge t_0$ the equality $x_*(t_2) = 0$ is satisfied. Then by (2.4) from the inequality $c(t) \ge 0, \ t \ge t_0$ it follows that $x_*(t) \le 0, \ t \ge t_2$. Hence, $x_*(t) \equiv 0$ on $[t_2; +\infty)$, which is impossible (since on $[t_2; +\infty)$ we have $c(t) \not\equiv 0$). On the strength of Lemma 2.1 from here it follows that each $t_0$-regular solution of Eq. (2.1) is positive. Let $x_N(t)$ be a $t_0$-normal solution of Eq. (2.1). Then since $x_*(t) > 0, \ t \ge t_0$, by (2.8) we have: $\int\limits_{t_0}^{+\infty} a(\tau) x_N(\tau) d\tau = = \int\limits_{t_0}^{+\infty} a(\tau)[x_N(\tau) - x_*(\tau)]d\tau + \int\limits_{t_0}^{+\infty} a(\tau) x_*(\tau) d\tau \ge \int\limits_{t_0}^{+\infty} a(\tau)[x_N(\tau) - x_*(\tau)]d\tau = +\infty$. Let

$$\int\limits_{t_0}^{+\infty} c(\tau) I_{a,b}(t_0;\tau) d\tau = +\infty. \tag{2.39}$$

Suppose $\int\limits_{t_0}^{+\infty} a(\tau) x_*(\tau) d\tau < +\infty$. Show that then

$$x_*(t) = \int\limits_{t}^{+\infty} J_b(t;\tau) c(\tau) \phi_*(t;\tau) d\tau, \qquad t \ge t_0, \tag{2.40}$$

where $\phi_*(t;\tau) \equiv \exp\{\int\limits_t^\tau a(s) x_*(s) ds\}, \ \tau \ge t \ge t_0$. By (2.4) we have:

$$x_*(t) = J_{-h_*}(t_1;t)\left[x_*(t_1) - \int\limits_{t_1}^{t} J_b(t_1;\tau) c(\tau) \phi_*(t;\tau) d\tau\right], \qquad t \ge t_1 \ge t_0, \tag{2.41}$$

where $h_*(t) \equiv a(t)x_*(t) + b(t), \ \ t \ge t_0$. From here and from the positivity of $x_*(t)$ it follows that

$$x_*(t) \ge \int\limits_{t_1}^{+\infty} j_b(t_1;\tau)c(\tau)\phi_*(t_1;\tau)d\tau, \qquad t_1 \ge t_0.$$

Show that the strict inequality

$$x_*(t) > \int\limits_{t_1}^{+\infty} J_b(t_1;\tau)c(\tau)\phi_*(t_1;\tau)d\tau \tag{2.42}$$

impossible for all $t_1 \ge t_0$. Multiplying both sides of (2.41) on $a(t)\phi_*(t_1;t)$ and integrationg from $t_1$ to $+\infty$ we will get:

$$\exp\Big\{\int\limits_{t_1}^{+\infty} a(\tau)x_*(\tau)d\tau\Big\} = \\ = 1 + \int\limits_{t_1}^{+\infty} J_{-b}(t_1;\tau)\Big[x_*(t_1) - \int\limits_{t_1}^{t} J_b(t_1;\tau)c(\tau)\phi_*(t_1;\tau)\Big]d\tau \ge \\ \ge 1 + I^+_{a,b}(t_1)\Big[x_*(t_1) - \int\limits_{t_1}^{+\infty} J_b(t_1;\tau)c(\tau)\phi_*(t_1;\tau)d\tau\Big]. \tag{2.43}$$

Suppose for some $t_1 \ge t_0$ the inequality (2.42) is satisfied. Then by (2.38) from the equality $I^+_{a,b}(t_0) = +\infty$ and from (2.43) it follows that $\int\limits_{t_1}^{+\infty} a(\tau)x_*(\tau)d\tau = +\infty$. The obtained contradiction proves (2.40). Multiplying both sides of (2.40) on $a(t)\phi_*(t_0;t)$ and integrating from $t_0$ to $+\infty$ we will get:

$$\exp\Big\{\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau\Big\} = \\ = 1 + \int\limits_{t_0}^{+\infty} a(\tau)\phi_*(t_0;\tau)d\tau \int\limits_{t}^{+\infty} J_b(t;\tau)c(\tau)\phi_*(t;\tau)d\tau \ge \\ \ge 1 + \int\limits_{t_0}^{+\infty} a(t)dt \int\limits_{t}^{+\infty} J_b(t;\tau)c(\tau)d\tau.$$

By virtue of Fubini's theorem from here and from (2.39) it follows that $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = +\infty$. We came to the contradiction. The assertion I* is proved. Let us prove II*. Let $x(t)$ be a $t_0$-regular solution of Eq. (2.1). If $x(t_0) < 0$, then by (2.4) from inequality $c(t) \ge 0,\ \ t \ge t_0$ it follows that $x(t) < 0,\ \ t \ge t_0$. Let $x(t_0) \ge 0$. Show that in this case impossible that

$$x(t) \ge 0, \qquad t \ge t_0. \tag{2.44}$$

Suppose that this relation takes place. Then since $a(t) \ge 0$, $c(t) \ge 0,\ \ t \ge t_)$, we have

$$\int\limits_{t_0}^{t} J_b(\tau)c(\tau)\exp\biggl\{\int\limits_{t_0}^{\tau} a(s)x(s)ds\biggr\}d\tau \ge I^+_{c,-b}(t_0;t), \qquad t \ge t_0.$$

By (2.4) from here and from equality $I^+_{c,-b}(t_0) = +\infty$ it follows that $x(t_1) < 0$ for some $t_1 > t_0$. We came to the contradiction. Hence by (2.4) if $x(t_0) > 0$, then there exists $t_2 = t_2(x) \ge t_1 = t_1(x) > t_0$ such that $x(t) > 0,\ \ t \in [t_0;t_1),\ \ x(t) = 0,\ \ t \in [t_1;t_2],\ \ x(t) < 0,\ \ t > t_2$, and if $x(t_0) = 0$, then there exists $t_1 = t_1(x) \ge t_0$ such that $x(t) = 0,\ \ t \in [t_0;t_1]$, and $x(t) < 0,\ \ t \ge t_1$. Let $x_0(t)$ be a $t_0$-normal solution of Eq. (2.1) with $x_0(t_0) \ge 0$, and let $t_1 = t_1(x) \ge t_0$ such that $x_0(t_1) = 0,\ \ x_0(t) < 0$, $t > t_1$. Then since by (2.8) $\int\limits_{t_1}^{+\infty} a(\tau)[x_*(\tau) - x_0(\tau)]d\tau = -\infty$, we have

$$\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau =$$

$$= \int\limits_{t_0}^{t_1} a(\tau)x_*(\tau)d\tau + \int\limits_{t_1}^{+\infty} a(\tau)[x_*(\tau) - x_0(\tau)]d\tau + \int\limits_{t_1}^{+\infty} a(\tau)x_0(\tau)d\tau \le$$

$$\le \int\limits_{t_0}^{t} a(\tau)x_*(\tau)d\tau + \int\limits_{t_1}^{+\infty} a(\tau)[x_*(\tau) - x_0(\tau)]d\tau = -\infty.$$

Using Theorem 2.1 by analogy of the second of inequalities (2.18) can be obtained the estimation

$$x_0(t) \le x_0(t_1)J_{-b}(t_1;t) - I^-_{b,c}(t_1;t) = -I^-_{b,c}(t_1;t), \qquad t \ge t_1.$$

Then

$$\int\limits_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau \le \int\limits_{t_0}^{t_1} a(\tau)x_0(\tau)d\tau - \int\limits_{t_1}^{+\infty} a(\tau)I^-_{a,b}(t_1;\tau)d\tau. \qquad (2.45)$$

Since $I^-_{b,c}(t_0;t) = I^-_{b,c}(t_0;t_1)J_{-b}(t_1;t) + I^-_{b,c}(t_1;t)$, we have

$$\int\limits_{t_0}^{+\infty} a(\tau)I^-_{b,c}(t_0;\tau)d\tau =$$

$$= \int\limits_{t_0}^{t_1} I^-_{b,c}(t_1;\tau)d\tau + I^-_{b,c}(t_0;t_1)I^+_{a,b}(t_1;+\infty) + \int\limits_{t_1}^{+\infty} a(\tau)I^-_{b,c}(t_1;\tau)d\tau. \quad (2.46)$$

Since $x_0(t) < 0,\ \ t > t_1$, by virtue of I* we will get

$$I^+_{a,b}(t_1) < +\infty. \qquad (2.47)$$

Let $\int\limits_{t_0}^{+\infty} a(\tau)I^-_{b,c}(t_0;\tau)d\tau = +\infty$. Then from (2.45) - (2.47) it follows that

$$\int\limits_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau = -\infty. \qquad (2.48)$$

Let $x_N(t)$ be an arbitrary $t_0$-normal solution of Eq. (2.1). Then since by (2.9) $\int\limits_{t_0}^{+\infty} a(\tau)|x_N(\tau) - x_0(\tau)|d\tau < +\infty$, taking into account (2.48) we will have: $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int\limits_{t_0}^{+\infty} a(\tau)[x_N(\tau) - x_0(\tau)]d\tau + \int\limits_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau = -\infty$. The assertion II* is proved. Let us prove III*. Show that

$$x_*(t_1) < 0 \qquad (2.49)$$

for some $t_1 \ge t_0$. Suppose that it is not true. Then $x_*(t) \ge 0,\ \ t \ge t_0$ and therefore $\nu_{x_*}(t_0) \le I^+_{a,b}(t_0) < +\infty$. But on the other hand by (2.7) we have $\nu_{x_*}(t_0) = +\infty$. The obtained contradiction proves (2.49). By (2.4) from (2.49) and from non negativity of $c(t)$ it follows that

$$x_*(t) < 0,\ \ t \ge t_1. \qquad (2.50)$$

Hence $v_*(t) \equiv \frac{1}{x_*(t)}$, $t \ge t_1$ is a $t_1$-regular solution of the equation

$$v' + c(t)v^2 - b(t)v + a(t) = 0, \qquad t \ge t_0. \tag{2.51}$$

Let $I^+_{c,-b}(t_0) < +\infty$. By (2.38) from here it follows that $I^+_{c,-b}(t_1) < +\infty$. Then by already proven $v_*(t) < 0$, $t \ge t_2$ for some $t_2 \ge t_1$, vhere $v_*(t)$ is the $t_1$-extremal solution of Eq. (2.51). From here it follows that $x_1(t) \equiv -\frac{1}{v_*(t)}$ is an positive solution of Eq. (2.1) defined on $[t_2; +\infty)$. Since according to (2.50) $x_*(t_2) < 0$, by virtue of Lemma 2.1 $x_1(t)$ is $t_2$-normal. By virtue of continuously dependence of solutions of Eq. (2.1) from their initial values from here from (2.4) and (2.50) it follows that there exists $t_1$-normal positive solution $x^+_N(t)$ of Eq. (2.1) on $[t_2; +\infty)$ having the property: for each solution $x(t)$ of Eq. (2.1) with $x(t_2) \in (0; x^+_N(t_2))$ there exists $t_4 = t_4(x) \ge t_3 = t_3(x) > t_2$ such that $x(t) > 0$, $t \in [t_2; t_3)$, $x(t) = 0$, $t \in [t_3; t_4]$, $x(t) < 0$, $t > t_4$; if $x(t_2) = 0$, then there exists $t_3 = t_3(x) \ge t_2$ such that $x(t) = 0$, $t \in [t_2; t_3]$, $x(t) < 0$, $t > t_3$; if $x(t_2) \in (x_*(t_2); 0)$, then $x(t) < 0$, $t \ge t_2$. Let $x_-(t)$ be a solution of Eq. (2.1) with $x_-(t_2) \in (x_*(t_2); 0)$. Then by virtue of Lemma 2.1 $x_-(t)$ is $t_2$-normal and, as it was already proved, $x_-(t) < 0$, $t \ge t_2$. Therefore taking into account (2.9) we will have:

$$0 < \int\limits_{t_0}^{+\infty} a(\tau)x^+_N(\tau)d\tau \le \int\limits_{t_0}^{t_2} a(\tau)x^+_N(\tau)d\tau + \int\limits_{t_2}^{+\infty} a(\tau)[x^+_N(\tau) - x_-(\tau)]d\tau < +\infty. \tag{2.52}$$

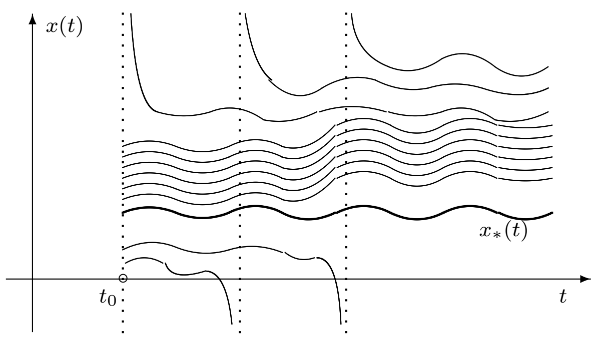
Let $x_N(t)$ be an arbitrary $t_0$-normal solution of Eq. (2.1). Then $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int\limits_{t_0}^{t_2} a(\tau)x_N(\tau)d\tau + \int\limits_{t_2}^{+\infty} a(\tau)[x_N(\tau) - x^+_N(\tau)]d\tau + + \int\limits_{t_2}^{+\infty} a(\tau)x^+_N(\tau)d\tau$. Since by (2.9) $\int\limits_{t_2}^{+\infty} a(\tau)|x_N(\tau) - x^+_N(\tau)|d\tau < +\infty$, from the last equality and from (2.52) it follows convergence of the integral $\int\limits_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau$. Since $x^+_N(t) > 0$, $t \ge t_2$; $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = \int\limits_{t_0}^{t_2} a(\tau)x_*(\tau)d\tau + \int\limits_{t_2}^{+\infty} a(\tau)[x_*(\tau) - x^+_N(\tau)]d\tau + \int\limits_{t_2}^{+\infty} a(\tau)x^+_N(\tau)d\tau$. and by (2.8) $\int\limits_{t_2}^{+\infty} a(\tau)[x_*(\tau) - x^+_N(\tau)]d\tau = -\infty$, taking into account (2.52) we will have: $\int\limits_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = -\infty$. The theorem is proved.

**Remark 2.2**. *Existence criteria of $t_1$-regular solutions of Eq. (2.1) are proved in [5] and [12].*
**Remark 2.3.** *If $a(t) > 0, \ \ t \geq t_)$, then existence of $t_1$-regular solutions of Eq. (2.1) is equivalent to the non oscillation of the equation*

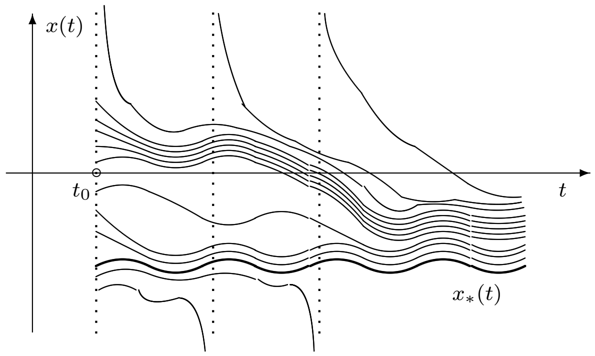
$$\left(\frac{\phi'}{a(t)}\right)' - a(t)b(t)\phi' - c(t)\phi = 0, \qquad t \geq t_0.$$

*Non oscillatory criteria for the last equation is proved in [13].*
**Corollary 2.2.** *Let $a(t) \geq 0, \ \ c(t) \geq 0, \ \ t \geq t_0, \ \ I^+_{a,b}(t_0) = I_{c,-b}(t_0) = +\infty$. Then Eq. (2.1) has no $t_1$-regular solutions for all $t_1 \geq t_0$.*
Proof. Suppose that for some $t_1 \geq t_)$ Eq. (2.1) has $t_1$-regular solution $x(t)$. Then by virtue of Theorem 2.4 from the equality $I^+_{a,b}(t_1) = I^+_{a,b}(t_0)$ (see (2.38)) it follows that $x(t) > 0, \ \ t \geq t_1$. Therefore $v(t) \equiv -\frac{1}{x(t)}$, $t \geq t_1$, is a negative $t_1$-regular solution of Eq. (2.51). But on the other hand by virtue of Theorem 2.4 I* from the equality $I^+_{c,-b}(t_1) = I^+_{a,b}(t_0) = +\infty$ it follows that $v(t) > 0, \ \ t \geq t_1$. We came to the contradiction. The corollary is proved.

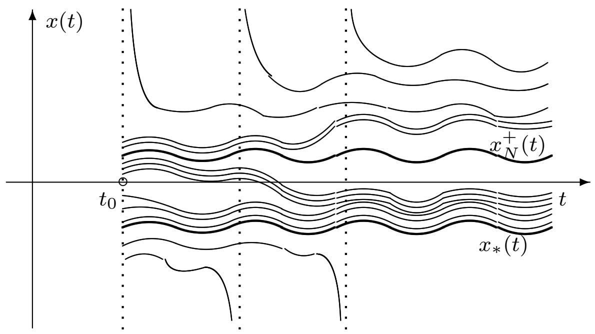
On the basis of Theorem 2.1 and corollary 2.2 we can make the phase portrait of solutions of Eq. (2.1) if $a(t) \geq 0, \ \ c(t) \geq 0, \ \ t \geq t_0$ for the following four cases:
$\alpha$) $I^+_{a,b}(t_0) = +\infty$; and Eq. (2.1) has a $t_1$-regular solution for some $t_1 \geq t_0$ (see pict. 3);



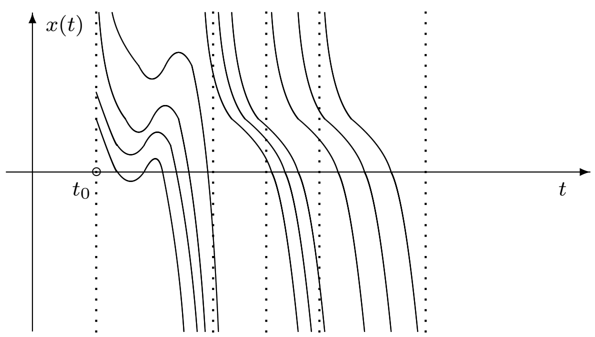

*pict.3.* $I^+_{a,b}(t_0)=+\infty$




*pict.4.* $I^+_{c,-b}(t_0)=+\infty$




*pict.5.* $I^+_{a,b}(t_0)<+\infty$ *and* $I^+_{c,-b}(t_0)<+\infty$




*pict.6.* $I^+_{a,b}(t_0)=I^+_{c,-b}(t_0)=+\infty$

$\beta$) $I^+_{c,-b}(t_0) = +\infty$; and Eq. (2.1) has a $t_1$-regular solution for some $t_1 \ge t_0$ (see pict. 4);
$\gamma$) $I^+_{a,b}(t_0) < +\infty$, $I^+_{c,-b}(t_0) < +\infty$; and Eq. (2.1) has a $t_1$-regular solution for some $t_1 \ge t_0$ (see pict. 5);
$\delta$) $I^+_{a,b}(t_0) = I^+_{c,-b}(t_0) = +\infty$ (see pict. 6).

Let $a(t) > 0$, $t \ge t_0$, and let $x_0(t)$ is a solution of Eq. (2.1) with $x_0(t_0) = 0$. Then by virtue of Theorem 2.3. I° $x_0(t)$ is $t_0$-normal and non negative. Obviously

$$x_0(t) + \int\limits_{t_0}^{t} a(\tau)\Big(x_0(\tau) + \frac{b(\tau)}{2a(\tau)}\Big)^2 d\tau = \int\limits_{t_0}^{t} \frac{b^2(\tau) - 4a(\tau)c(\tau)}{4a(\tau)} d\tau, \qquad t \ge t_0.$$

Then since $x_0(t) \ge 0$, $t \ge t_0$, we have

$$\int\limits_{t_0}^{t} a(\tau)\Big(x_0(\tau) + \frac{b(\tau)}{2a(\tau)}\Big)^2 d\tau \le \int\limits_{t_0}^{t} \frac{b^2(\tau) - 4a(\tau)c(\tau)}{4a(\tau)} d\tau, \qquad t \ge t_0. \quad (2.53)$$

According to Cauchy - Schwarz inequality we have:

$$\int\limits_{t_0}^{t} a(\tau)\Big(x_0(\tau) + \frac{b(\tau)}{2a(\tau)}\Big) d\tau \le \sqrt{\int\limits_{t_0}^{t} a(\tau) d\tau} \sqrt{\int\limits_{t_0}^{t} a(\tau)\Big(x_0(\tau) + \frac{b(\tau)}{2a(\tau)}\Big)^2 d\tau}$$

From here and from (2.53) we get:

$$\int\limits_{t_0}^{t} a(\tau)x_0(\tau) d\tau \le -\frac{1}{2}\int\limits_{t_0}^{t} b(\tau) d\tau + \frac{1}{2}\sqrt{\int\limits_{t_0}^{t} a(\tau) d\tau \Big[\int\limits_{t_0}^{t} \frac{b^2(\tau) - 4a(\tau)c(\tau)}{a(\tau)}\Big] d\tau}, \quad (2.54)$$

$t \ge t_0$. Let $x_*(t)$ be a $t_0$-extremal solution to Eq. (2.1). Then (see [11])

$$a(t)x_*(t) = \frac{\nu_{x_0}(t)}{\nu_{x_0}(t)} - a(t)x_0(t) - b(t), \qquad t \ge t_0.$$

From here and from (2.54) we carry out:

$$\int\limits_{t_0}^{t} a(\tau)x_*(\tau) d\tau \ge -\frac{1}{2}\int\limits_{t_0}^{t} b(\tau) d\tau -$$

$$-\frac{1}{2}\sqrt{\int\limits_{t_0}^{t} a(\tau) d\tau \Big[\int\limits_{t_0}^{t} \frac{b^2(\tau) - 4a(\tau)c(\tau)}{a(\tau)}\Big] d\tau} + \ln\frac{\nu_{x_0}(t)}{\nu_{x_0}(t_0)}, \qquad t \ge t_0. \quad (2.55)$$

**Remark 2.2** *The estimates (2.54) and (2.55) are sharp in the sense that for $a(t) = const$, $b(t) = const$, $c(t) = const$ the estimate (2.54) becomes an equality up to constant summand and the inequality (2.55) becomes an equality.*

Let $I^+_{a,b}(t_0) < +\infty$, $I^+_{-c,-b}(t_0) < +\infty$. Then due to Theorem 2.3. VI° Eq. (2.1) has a negative $t_0$-normal solution. Therefore,

$$\nu_{x_0}(t) = \int\limits_t^{+\infty} a(\tau)\exp\Big\{-\int\limits_t^{\tau}\big[2a(s)\big(x_0(\xi)-x_N^-(\xi)\big)+2a(\xi)x_N^-(\xi)+b(\xi)\big]d\xi\Big\}d\tau \geq$$
$$\geq \exp\Big\{-\int\limits_t^{+\infty} 2a(s)\big(x_0(\xi)-x_N^-(\xi)\big)d\xi\Big\} I^+_{a,b}(t,+\infty), \qquad t \geq t_0.$$

From here and from (2.55) we get:

$$\int\limits_{t_0}^{t} a(\tau)x_*(\tau)d\tau \geq -\frac{1}{2}\int\limits_{t_0}^{t} b(\tau)d\tau -$$
$$-\frac{1}{2}\sqrt{\int\limits_{t_0}^{t} a(\tau)d\tau\Big[\int\limits_{t_0}^{t}\frac{b^2(\tau)-4a(\tau)c(\tau)}{a(\tau)}\Big]d\tau} + \ln I^+_{a,b}(t) + c, \qquad t \geq t_0. \tag{2.56}$$

where

$$c \equiv -\int\limits_{t_0}^{+\infty} a(\tau)\big(x_0(\tau)-x_N^-(\tau)\big)d\tau - \ln \nu_{x_0}(t_0). \tag{2.57}$$

Let $a(t) > 0$, $c(t) \geq 0$, $t \geq t_0$, $I^+_{a,b}(t_0) = +\infty$, and let Eq. (2.1) has a $t_0$-regular solution. Obviously

$$x_1(t) + \int\limits_{t_0}^{t} a(\tau)\Big(x_1(\tau)+\frac{b(\tau)}{2a(\tau)}\Big)^2 d\tau = x_1(t_0) + \int\limits_{t_0}^{t}\frac{b^2(\tau)-4a(\tau)c(\tau)}{4a(\tau)}d\tau,$$

$t \geq t_0$. Then since by Theorem 2.4. I* $x_1(t) > 0$, $t \geq t_0$, we have

$$\int\limits_{t_0}^{t} a(\tau)\Big(x_1(\tau)+\frac{b(\tau)}{2a(\tau)}\Big)^2 d\tau \leq x_1(t_0) + \int\limits_{t_0}^{t}\frac{b^2(\tau)-4a(\tau)c(\tau)}{4a(\tau)}d\tau, \qquad t \geq t_0.$$

From here using Cauchy - Schwarz inequality by analogy of (2.54) we get:

$$\int\limits_{t_0}^{t} a(\tau)x_0(\tau)d\tau \le -\frac{1}{2}\int\limits_{t_0}^{t} b(\tau)d\tau +$$
$$+\frac{1}{2}\sqrt{\int\limits_{t_0}^{t} a(\tau)d\tau\left[4x_1(t_0) + \int\limits_{t_0}^{t}\frac{b^2(\tau)-4a(\tau)c(\tau)}{a(\tau)}\right]d\tau}, \qquad t \ge t_0. \tag{2.58}$$

## 3. The behavior of solutions of the system (1.1)

**Definition 3.1.** *The function $u(t)$ is called oscillatory if it has arbitrary large zeroes, otherwise $u(t)$ is called non oscillatory.*

**Definition 3.2.** *The system (1.1) is called oscillatory (non oscillatory), if for its each non trivial solution $(\phi(t), \psi(t))$ the functions $\phi(t)$ and $\psi(t)$ are oscillatory (non oscillatory).*

**Remark 3.1.** *Some oscillatory and non oscillatory criteria are proved in [6] (see also [5]).*

**Definition 3.3.** *The system (1.1) is called weak oscillatory (weak non oscillatory), if for its each non trivial solution $(\phi(t), \psi(t))$ at least one of the functions $\phi(t)$ and $\psi(t)$ is oscillatory (non oscillatory) and there exist two solutions $(\phi_j(t), \psi_j(t)), j = 1, 2,$ such that $\phi_1(t)$ and $\psi_1(t)$ are oscillatory (non oscillatory), and at least one of the functions $\phi_2(t)$ and $\psi_2(t)$ is non oscillatory (oscillatory).*

**Definition 3.4.** *The system (1.1) is called half oscillatory if for its each non trivial solution $(\phi(t), \psi(t))$ one of the functions $\phi(t)$, $\psi(t)$ is oscillatory and other is non oscillatory.*

**Definition 3.5.** *The system (1.1) is called singular, if it has two non trivial solutions $(\phi_j(t), \psi_j(t))$, $j = 1, 2,$ such that $\phi_1(t)$ and $\psi_1(t)$ are oscillatory, and $\phi_2(t)$ and $\psi_2(t)$ are non oscillatory.*

**Remark 3.2.** *It is evident that each system (1.1) is or else oscillatory or else non oscillatory or else weak oscillatory or else weak non oscillatory or else half oscillatory or else singular.*

Example 3.1. Consider the system

$$\begin{cases} \phi' = \cos(\lambda t)\psi; \\ \psi' = -\cos(\lambda t)\phi, \quad t \geq t_0 \end{cases} \tag{3.8}$$

where $\lambda = const > 0$ is a parameter. The general solution $(\phi(t), \psi(t))$ to this system is given by formulas:

$$\phi(t) = c_1 \sin\left(\frac{1}{\lambda}\sin\lambda t + c_2\right), \qquad \psi(t) = c_1 \cos\left(\frac{1}{\lambda}\sin\lambda t + c_2\right),$$

where $c_1$ and $c_2$ are arbitrary constants. It is not difficult to verify that if:
1) $0 < \lambda \leq \frac{2}{\pi}$, then the system (3.8) is oscillatory;
2) $\frac{2}{\pi} < \lambda \leq \frac{4}{\pi}$, then the system (3.8) is weak oscillatory;
3) $\lambda > \frac{4}{\pi}$, then the system (3.8) is weak non oscillatory.
Example 3.2. Consider the system

$$\begin{cases} \phi' = \psi; \\ \psi' = \left(-\cos^2 t - \sin t\right)\phi, \qquad t \geq t_0. \end{cases} \tag{3.9}$$

The general solution $(\phi(t), \psi(t))$ of this system is given by formulas:

$$\phi(t) = e^{\sin t}\left(c_1 + c_2\int\limits_{t_0}^{t} e^{-2\sin\tau} d\tau\right), \quad \psi(t) =$$

$$= e^{\sin t}\left[c_1 \cos t + c_2\left\{\cos t\int\limits_{t_0}^{t} e^{-2\sin\tau} d\tau + e^{-2\sin t}\right\}\right],$$

where $c_1$ and $c_2$ are arbitrary constants. Obviously $\phi(t)$ is non oscillatory and $\psi(t)$ is oscillatory. Hence the system (3.9) is half oscillatory.
Example 3.3. Consider the system

$$\begin{cases} \phi' = 3\cos t\,\phi - 2\cos t\,\psi; \\ \psi' = 4\cos t\,\phi - 3\cos t\,\psi, \quad t \geq t_0. \end{cases} \tag{3.10}$$

It has the solutions

$$\left(e^{\sin t},\ e^{\sin t}\right), \qquad \left(e^{\sin t} - e^{-\sin t},\ e^{\sin t} - 2e^{-\sin t}\right), \qquad t \ge t_0.$$

Obviously the components of the firs solution are non oscillatory; the firs component of the second solution vanishes in the points $\pi k \ge t_0$, $k = 0, \pm 1, \pm 2, ...$, and the nulls of the second component of the second solution are all solutions of the equation $\sin t = \ln \sqrt{2}$ on $[t_0; +\infty)$. Therefore the system (3.10) is singular.

**Definition 3.6** *The system (1.1) is called stable by Lyapunov (asymptotically), if its all solutions are bounded on $[t_0; +\infty)$ (vanish on $+\infty$).*

**Theorem 3.1.** *Let for each solution $(\phi(t), \psi(t))$ of the system (1.1) the function $J_{-S/2}(t)\phi(t)$ is bounded. Then there exists a solution $(\phi_0(t), \psi_0(t))$ of the system (1.1) such that $J_{-S/2}(t)\psi_0(t) \not\to 0$ for $t \to +\infty$. Moreover if in addition $a_{12}(t)$ does not change sign and $\int\limits_{t_0}^{+\infty} |a_{12}(\tau)|d\tau = +\infty$, then the system (1.1) is oscillatory and for each nontrivial solution $(\phi(t), \psi(t))$ of the system (1.1) $J_{-S/2}(t)\psi(t) \not\to 0$ for $t \to +\infty$.*

Proof. By (3.3) from the conditions of the theorem it follows that

$$y_0(t) \ge \varepsilon, \qquad t \ge t_0, \tag{3.11}$$

for some $\varepsilon > 0$. Suppose for each solution $(\phi(t), \psi(t))$ of the system (1.1) $J_{-S/2}(t)\psi(t) \to 0$ for $t \to +\infty$. Then according to (3.4) we have $y_0(t) \to 0$ for $t \to +\infty$, which contradicts (3.11). The obtained contradiction shows the existence of a solution $(\phi_0(t), \psi_0(t))$ of the system (2.1) with $J_{-S/2}(t)\psi_0(t) \not\to 0$ for $t \to +\infty$. If in addition $a_{12}(t)$ does not change sign and $\int\limits_{t_0}^{+\infty} |a_{12}(\tau)|d\tau = +\infty$, then from (3.11) it follows that $|\int\limits_{t_0}^{+\infty} a_{12}(\tau)y_0(\tau)d\tau| = +\infty$. From here and from (3.6) it follows oscillation of the system (1.1). From the last equality from (3.6) and (3.11) it follows that for each solution $(\phi(t), \psi(t))$ for the system (1.1) the relation $J_{-S/2}(t)\psi(t) \not\to 0$ for $t \to +\infty$ is fulfilled. The theorem is proved.

**Theorem 3.2**. *Let for each solution $(\phi(t), \psi(t))$ of the system (1.1) the relation $J_{-S/2}(t)\phi(t) \to 0$ for $t \to +\infty$ be satisfied. Then there exists a solution $(\phi_0(t), \psi_0(t))$ of the system (1.1) such that $J_{-S/2}(t)\psi_0(t)$ is unbounded. Moreover if in addition $a_{12}(t)$ does not change sign and*

$\int\limits_{t_0}^{+\infty} |a_{12}(\tau)|d\tau = +\infty$, *then the system (1.1) is oscillatory, and for each nontrivial solution* $(\phi(t), \psi(t))$ *of the system (1.1) the function* $J_{-S/2}(t)\psi(t)$ *is unbounded.*

Proof. By (3.3) from the condition of the theorem it follows that

$$y_0(t) \to +\infty \ \text{ for } \ t \to +\infty. \tag{3.12}$$

Suppose for each solution $(\phi(t), \psi(t))$ of the system (1.1) the function $J_{-S/2}(t)\psi(t)$ is bounded. Then from (3.4) it follows that $y_0(t)$ is bounded, which contradicts (3.12). Hence for at last one solution $(\phi_0(t), \psi_0(t))$ of the system (1.1) the function $J_{-S/2}(t)\psi_0(t)$ is unbounded. Let $a_{12}(t)$ does not change sign and let $\int\limits_{t_0}^{+\infty} |a_{12}(\tau)|d\tau = +\infty$. Then from (3.6) and (3.12) it follows that the system (1.1) is oscillatory and by virtue of the second of equalities (3.6) from (3.12) it follows that for each nontrivial solution $(\phi(t), \psi(t))$ of the system (1.1) the function $J_{-S/2}(t)\psi(t)$ is unbounded. The theorem is proved.

**Theorem 3.3 (about rings).** *Suppose for each solution* $\Phi(t) \equiv (\phi(t), \psi(t))$ *of the system (1.1) there exists* $R_\Phi > 0$ *such that* $||\Phi(t)|| \le R_\Phi J_{S/2}(t)$, $t \ge t_0$. *Then for each nontrivial solution* $\Phi(t)$ *of the system (1.1) there exists* $r_\Phi$ *such that*

$$||\Phi(t)|| \ge r_\Phi J_{S/2}(t), \qquad t \ge t_0. \tag{3.13}$$

Proof. By (3.3) - (3.5) from the conditions of the theorem it follows that

$$\sqrt{y_0(t)} \le M, \qquad \frac{1}{\sqrt{y_0(t)}} \le M, \qquad \frac{x_0(t)}{y_0(t)} \le M, \qquad t \ge t_0. \tag{3.14}$$

for some $M = const > 0$. Suppose for some solution $\Phi_0(t) \equiv (\phi_0(t), \psi_0(t))$ of the system (1.1) the relation (3.13) does not fulfill. Then there exists infinitely large sequence $\{t_n\}_{n-1}^{+\infty}$ such that

$$J_{-S/2}(t_n)\phi_0(t_n) \to 0, \ \ J_{-S/2}(t_n)\psi_0(t_n) \to 0 \ \text{ for } \ n \to +\infty. \tag{3.15}$$

By (3.6) we have:

$$J_{-S/2}(t_n)\phi_0(t_n) = \frac{\mu_0}{\sqrt{y_0(t_n)}} \sin(\gamma_n);$$

$$J_{-S/2}(t_n)\psi_0(t_n) = \mu_0\sqrt{1+\frac{x_0(t_0)}{y_0(t_0)}}\sqrt{y_0(t_n)}\cos(\gamma_n - \alpha_0(t_n)),$$

where $\gamma_n \equiv \int\limits_{t_0}^{t_n} y_0(\tau)d\tau + \nu_0,\ \ n = 1, 2, ...;\ \ \nu_0$ and $\mu_0$ are some constants. From here from (3.14) and (3.15) it follows:

$$\sin(\gamma_n) \to 0, \ \text{ for } \ n \to +\infty; \tag{3.16}$$

$$\cos(\gamma_n - \alpha_0(t_n)) \to 0, \ \text{ for } \ n \to +\infty; \tag{3.17}$$

From (3.14) and from (3.7) it follows that there exists $\delta > 0$ such that $|\cos(\alpha_0(t_n))| > \ \delta, n = 1, 2, \ldots.$ . From here and from (3.16) it follows that $|\cos(\gamma_n - \alpha_0(t_n))| = 1 \pm \sqrt{1-\sin^2\gamma_n}\cos(\alpha_0(t_n)) + \sin\gamma_n \sin\alpha_0(t_n) \geq$ $\geq \delta/2$ for all enough large values of $n$, which contradicts (3.17). The obtained contradiction proves (3.13). The theorem is proved.

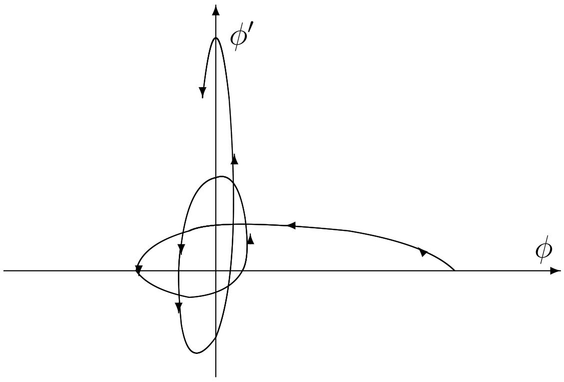
**Remark 3.3.** *The geometrical meaning of Theorem 3.4 is that if for all solutions* $\Phi(t) \equiv (\phi(t), \psi(t))$ *of the system (1.1) the vector functions* $J_{-S/2}(t)\Phi(t)$ *are bounded then each of the last ones lies in some ring of radiuses* $0 < r_\Phi < \ R_\Phi$.

By correlation (1.3) between Eq. (1.2), Eq. (1.5) and the system (1.4) From Theorems 3.1 – 3.4 we deduce the following **three principles for Eq. (1.5)**:

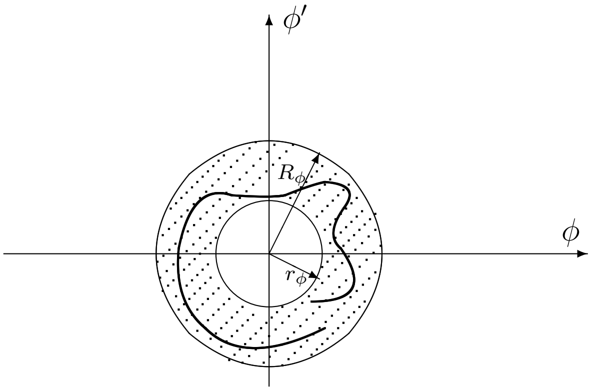
**A)** *If all solutions of eq. (1.5) are bounded then it is oscillatory and for its each nontrivial solution* $\phi(t)$ *the relation* $\phi'(t) \not\to 0$ *for* $t \to +\infty$ *is fulfilled.*

**B)** *If all solutions of Eq. (1.5) vanish on* $+\infty$*, then the derivative of its each nontrivial solution is unbounded.*

**C)** *If Eq. (1.5) is stable by Lyapunov then for its each nontrivial solution* $\phi(t)$ *there exist positive numbers* $r_\phi < R_\phi$ *such that* $r_\phi \leq \sqrt{\phi^2(t) + \phi'(t)^2} \leq R_\phi, \ \ t \geq t_0$.




*Pict.7. An illustration to the principle B)*




*Pict.8. An illustration to the principle C)*

Let us compare these principles wit the following assertion proved in [1] (see [1], p. 222, Corollary 6.2.4).

**Proposition 3.1**. *Let $|r(t)| \le M$, $t \ge t_0$. Then if all solutions of Eq. (1.5) vanish on $+\infty$, then Eq. (1.5) is asymptotically stable.*

Obviously from any of principles A) - C) it follows that the equations (1.5) satisfying the conditions of Proposition 3.1, form an empty set.

Example 3.4. Consider the Mathieu equation (see [14])

$$\phi'' + (\delta + \varepsilon \cos t)\phi = 0, \qquad t \ge t_0, \qquad \delta, \varepsilon \in R.$$

From the principle A) it follows that this equation for all pairs $(\delta, \varepsilon)$ of zone of stability is oscillatory, and from the principle C) it follows that (for this restriction) for its each nontrivial solution $\phi(t)$ there exist $R_\phi > r_\phi > 0$ such that $r_\phi \le \phi^2(t) + \phi'(t)^2 \le R_\phi$, $t \ge t_0$ (see Pict. 8), which agrees quite well with the Floquet's theory. Note that some part of mentioned above zone of stability relates to the extremal case of Eq. (1.5), when $\int\limits_{t_0}^{+\infty} r(t)dt = -\infty$.

Example 3.5. Consider the Airy's equation

$$\phi'' + t\phi = 0, \qquad t \geq t_0.$$

By virtue of L. A. Gusarov's theorem (see [15], Theorem 1) all solutions of this equation vanish on $+\infty$. From the principles A) and B) it follows that this equation is oscillatory and for its each nontrivial solution $\phi(t)$ the function $\phi'(t)$ is unbounded (see Pict. 7).

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