On a Convergence of a Rational Trigonometric Approximation

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Abstract

We investigate rational trigonometric approximation of smooth functions in a finite interval and show that by appropriate choice of parameters in rational trigonometric approximant we obtain more precise approximation compared with classic Fourier-Pade scheme.

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1 Introduction

It is well known that reconstruction of smooth in the finite interval function $f(x),\ x \in [-1,1]$, by its finite number of Fourier coefficients

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}dx,\ \ n = 0, \pm 1, \cdots, \pm N,\ \ 0 < N < \infty \tag{1}$$

by the partial sum of Fourier expansion

$$S_N(f) = \sum_{n=-N}^{N} f_n e^{i\pi nx} \tag{2}$$

is non efficient if the periodic extension (with the period 2) of f is not smooth enough.

If $f(1) \neq f(-1)$, then uniform convergence of $S_N(f),\ N \to \infty$ in the segment $[-1,1]$ is impossible due to the Gibbs phenomenon and the order of L_2-convergence in a segment inside of the interval $(-1,1)$ is much greater than in the whole interval. In a case when $f \in C^{q+1}[-1,1],\ q \geq 0$, $f^{(k)}(-1) = f^{(k)}(1),\ k = 0,1,...,q$ and $f^{(q+1)}(-1) \neq f^{(q+1)}(1)$ although the uniform convergence takes place in the whole segment $[-1,1]$ its order in a segment inside of $(-1,1)$ is higher.

The same is true for the well-known Fourier-Pade approximation ([1]-[3]). Here the faster convergence is more sharp revealed compared with approximation by Fourier partial sums.

In [1],[2] we minimize L_2 and uniform errors of a rational linear approximation in the whole interval by appropriate choice of parameters and receive more precise approximation compared with Fourier-Pade approximation. Unfortunately, it leads to less efficient approximation inside of the interval.

In this article we show how it is possible to keep the results of [1],[2] and simultaneously increase the precision inside of $(-1,1)$.

2 Preliminaries.

Consider a finite sequence of complex numbers as a vector $\theta := \{\theta_k\}_{k=-p}^{p},\ p \geq 1$ and denote

$$\Delta_n^0(\theta) = f_n,\ \Delta_n^k(\theta) = \Delta_n^{k-1}(\theta) + \theta_{k\,sgn(n)}\Delta_{(|n|-1)sgn(n)}^{k-1}(\theta),\ k \geq 1, \tag{3}$$

where $sgn(n) = 1$ if $n \geq 0$ and $sgn(n) = -1$ if $n < 0$.

Consider also the following rational approximation in $[-1,1]$ ([1],[2])

$$S_{p,N}(\theta,f) := \sum_{n=-N}^{N} f_n e^{i\pi nx} - e^{i\pi(N+1)x}\sum_{k=1}^{p}\frac{\theta_k\Delta_N^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_s e^{i\pi x})} - $$

$$-e^{-i\pi(N+1)x}\sum_{k=1}^{p}\frac{\theta_{-k}\Delta_{-N}^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_{-s}e^{-i\pi x})} \tag{4}$$

with error

$$R_{p,N}(\theta,f) := f(x) - S_{p,N}(\theta,f) = R_{p,N}^{+}(\theta,f) + R_{p,N}^{-}(\theta,f), \tag{5}$$

where

$$R^{\pm}_{p,N}(\theta, f) := \frac{1}{\prod_{k=1}^{p}(1+\theta_{\pm k}e^{\pm i\pi x})} \sum_{n=N+1}^{\infty} \Delta^{p}_{\pm n}(\theta) e^{\pm i\pi n x}. \qquad (6)$$

If θ is a solution of the systems

$$\Delta^{p}_{n}(\theta) = 0,\ n = -N-p, \cdots, -N-1 \ \textit{and}\ n = N+1, \cdots, N+p, \qquad (7)$$

then approximation $S_{p,N}(\theta, f)$ is Fourier-Pade approximation ([3]).

3 Theoretical Results.

Let $f \in C^{q}[-1,1]$. By definition, put

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1),\ k = 0, \cdots, q. \qquad (8)$$

By $\gamma_k(p), k = 0, \cdots, p$, we denote the coefficients of the polynomial

$$\prod_{k=1}^{p}(1+\tau_k x) \equiv \sum_{k=0}^{p} \gamma_k(p) x^k. \qquad (9)$$

Lemma 1 *([1],[2]) Suppose* $f \in C^{q+p}[-1,1]$, $q \geq 0$, $p \geq 1$, $f^{(q+p+1)} \in L_1[-1,1]$ *and* $A_j(f) = 0$ *for* $j = 0, \cdots, q-1$. *If*

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N},\ \ k = 1, \cdots, p, \qquad (10)$$

then asymptotic expansion

$$\Delta^{p}_{n}(\theta) = A_q(f)\frac{(-1)^{n+p+1}}{2(i\pi)^{q+1}q!} \sum_{k=0}^{p} \frac{(q+p-k)!(-1)^k\gamma_k(p)}{N^k(n-k)^{q+1}|n-k|^{p-k}} + o(n^{-q-p-1}) \qquad (11)$$

holds as $N \to \infty$, $|n| \geq N+1$.

If in (3) $\theta_k \equiv 1$, $|k| \leq p$, we put $\Delta^{k}_{n} := \Delta^{k}_{n}(\theta)$. Notice that Δ^{k}_{n} are well-known classic finite differences. From Lemma 1 we derive.

Lemma 2 *Suppose* $f \in C^{q+p}[-1,1]$, $q \geq 0$, $p \geq 1$, $f^{(q+p+1)} \in L_1[-1,1]$ *and* $A_j(f) = 0$ *for* $j = 0, \cdots, q-1$*; then the following asymptotic expansion holds* $(m \to \infty)$

$$\Delta^{s}_{m} = A_q(f)\frac{(-1)^{m+s+1}(q+s)!}{2(i\pi m)^{q+1}q!|m|^{s}} + o(m^{-s-q-1}),\ \ s = 0, \cdots, p. \qquad (12)$$

Now we investigate pointwise and L_2 convergence of approximations $S_{p,N}(\theta, f)$ inside of $(-1, 1)$.

Theorem 1 *Let* $f \in C^{q+p+2}[-1,1]$, $q \geq 0$, $p \geq 1$, $f^{(q+p+3)} \in L_1[-1,1]$ *and* $A_j(f) = 0$ *for* $j = 0, \cdots, q-1$. *If*

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \ k = 1, \cdots, p, \tag{13}$$

then for $|x| < 1$ *the following asymptotic estimate holds* $(N \to \infty)$

$$(-1)^{N+p} N^{q+p+1} R_{p,N}(\theta, f) =$$

$$= Re \left(\frac{e^{i\pi(N+1)x}}{i^{q+1}(1+e^{i\pi x})^{p+1}} \right) \frac{A_q(f)}{q! \, \pi^{q+1}} \sum_{k=0}^{p} (-1)^k \gamma_k(p)(q+p-k)! + o(1). \tag{14}$$

Proof. It is not hard to prove by induction that

$$\Delta_n^p(\theta) = \sum_{k=0}^{p} \frac{(-1)^k \gamma_k(p)}{N^k} \Delta_{n-k}^{p-k}.$$

Substituting this in (6), we obtain

$$R_{p,N}^{+}(\theta, f) = \frac{1}{\prod_{k=1}^{p}(1+\theta_k e^{i\pi x})} \sum_{k=0}^{p} \frac{(-1)^k \gamma_k(p)}{N^k} \sum_{n=N+1}^{\infty} \Delta_{n-k}^{p-k} e^{i\pi n x}. \tag{15}$$

Applying twice Abel transformation to the last sum, we derive

$$\sum_{n=N+1}^{\infty} \Delta_{n-k}^{p-k} e^{i\pi n x} = -\frac{\Delta_{N-k}^{p-k} e^{i\pi(N+1)x}}{1+e^{i\pi x}} - \frac{\Delta_{N-k}^{p-k+1} e^{i\pi(N+1)x}}{(1+e^{i\pi x})^2} +$$

$$+ \frac{1}{(1+e^{i\pi x})^2} \sum_{n=N+1}^{\infty} \Delta_{n-k}^{p-k+2} e^{i\pi n x}. \tag{16}$$

Using Lemma 2, it is easy to show that the last two terms in (16) are of the order $O(N^{-q-p-2+k})$, $N \to \infty$, $k = 0, \cdots, p$. Substituting the first term in (16) in (15) and tending N to infinity, we obtain

$$N^{q+p+1} R_{p,N}^{+}(\theta, f) = \frac{A_q(f)(-1)^{N+p} e^{i\pi(N+1)x}}{2(i\pi)^{q+1} q! (1+e^{i\pi x})^{p+1}} \sum_{k=0}^{p} (-1)^k \gamma_k(p)(q+p-k)! + o(1).$$

This concludes the proof as the same arguments are valid for $R^-_{p,N}(\theta, f)$. •

Let $f \in L_2(-1, 1)$. By $||\cdot||_\varepsilon$, $0 < \varepsilon \leq 1$ denote the L_2-norm

$$||f||_\varepsilon = \left(\int_{-\varepsilon}^{\varepsilon} |f(x)|^2 dx \right)^{1/2}.$$

From Theorem 1 we immediately derive the following.

Theorem 2 *Under the conditions of Theorem 1 the following asymptotic estimate holds for any* $0 < \varepsilon < 1$

$$\lim_{N\to\infty} N^{q+p+1} ||R_{p,N}(\theta, f)||_\varepsilon =$$

$$= \frac{|A_q(f)|}{2^{p+1}\pi^{q+1} q!} \left| \sum_{k=0}^{p} (-1)^k \gamma_k(p)(q+p-k)! \right| \left(\int_{-\varepsilon}^{\varepsilon} \frac{dx}{\cos^{2p+2}\frac{\pi x}{2}} \right)^{1/2}. \quad (17)$$

For comparison notice that in [1],[2] we show that under the conditions of Theorem 1 (with additional condition $\tau_k > 0$ and $\tau_i \neq \tau_j$, $i \neq j$) the following holds

$$||R_{p,N}(\theta, f)||_1 = \frac{const}{N^{q+1/2}}.$$

Hence approximation $S_{p,N}(\theta, f)$ with $\{\theta_k\}$ as in (13) is $N^{p+1/2}$ times ($N >> 1$) more precise inside of $(-1, 1)$ than in the whole interval (see introduction).

From Theorems 1,2 we see that it is natural to take parameters τ_k, $|k| \leq p$ such that

$$\sum_{k=0}^{p} (-1)^k \gamma_k(p)(q+p-k)! = 0.$$

For example, in the case $p = 1$, we get $\tau_1 = q + 1$ and

$$R_{1,N}(\theta, f) = o(N^{-q-3}),\ N \to \infty$$

inside of $(-1, 1)$.

Now consider the case $p = 1$ in more details.

4 The Case $p = 1$.

Theorem 3 *Let $f \in C^{q+4}[-1,1]$, $q \geq 0$, $f^{(q+5)} \in L_1[-1,1]$ and $A_j(f) = 0$ for $j = 0, \cdots, q-1$. If*

$$\theta_{\pm 1} = 1 - \frac{q+1}{N} + \frac{a_{\pm 1}}{N^2}, \tag{18}$$

then for $|x| < 1$ the following holds

$$\begin{aligned} N^{q+3} R^{\pm}_{1,N}(\theta, f) = \\ = A_q(f) \frac{(-1)^{N+1}\left(a_{\pm 1} - \frac{q(q+1)}{2}\right)}{2(\pm i\pi)^{q+1}} \frac{e^{\pm i\pi(N+1)x}}{(1+e^{\pm i\pi x})^2} + \\ + A_q(f) \frac{(-1)^N (q+1)}{2(\pm i\pi)^{q+1}} \frac{e^{\pm i\pi(N+1)x}}{(1+e^{\pm i\pi x})^3} + \\ + A_{q+1}(f) \frac{(-1)^{N+1}}{2(\pm i\pi)^{q+2}} \frac{e^{\pm i\pi(N+1)x}}{(1+e^{\pm i\pi x})^2} + o(1), \ N \to \infty. \end{aligned} \tag{19}$$

Proof. We apply twice Abel transformation to $R^{+}_{1,N}(\theta, f)$ and obtain

$$\begin{aligned} R^{+}_{1,N}(\theta, f) = \\ - \frac{\Delta_N(\theta)}{(1+\theta_1 e^{i\pi x})(1+e^{i\pi x})} e^{i\pi(N+1)x} - \frac{\Delta_N(\theta) + \Delta_{N-1}(\theta)}{(1+\theta_1 e^{i\pi x})(1+e^{i\pi x})^2} e^{i\pi(N+1)x} + \\ + \frac{1}{(1+\theta_1 e^{i\pi x})(1+e^{i\pi x})^2} \sum_{n=N+1}^{\infty} (\Delta_n(\theta) + 2\Delta_{n-1}(\theta) + \Delta_{n-2}(\theta)) e^{i\pi n x}. \end{aligned} \tag{20}$$

By Lemma 1, the last term in (20) is of order $O(N^{-q-4})$, $N \to \infty$. Now we need more precise asymptotic estimates for $\Delta_N(\theta) + \Delta_{N-1}(\theta)$ and $\Delta_N(\theta)$ rather than in Lemma 1. Taking into account the well-known asymptotic expansion of Fourier coefficients ($n \neq 0$)

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{m} \frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^{m+1}} \int_{-1}^{1} f^{(m+1)}(x) e^{-i\pi n x} dx \tag{21}$$

by direct calculations we get

$$\Delta_N(\theta) = A_q(f) \frac{(-1)^N \left(a_1 - \frac{q(q+1)}{2}\right)}{2(i\pi)^{q+1} N^{q+3}} +$$

$$+A_{q+1}(f)\frac{(-1)^N}{2(i\pi)^{q+2}N^{q+3}}+O(N^{-q-4}),\ N\to\infty, \tag{22}$$

$$\Delta_N(\theta)+\Delta_{N-1}(\theta)=\Delta_N^2-\frac{a_1}{N}\Delta_{N-1}+\frac{a_1}{N^2}\Delta_{N-1}=$$

$$=A_q(f)\frac{(-1)^{N+1}(q+1)}{2(i\pi)^{q+!}N^{q+3}}+O(N^{-q-4}),\ N\to\infty. \tag{23}$$

Substituting these formulae in (20) and tending N to infinity we obtain the required result as the same arguments are valid for $R_{1,N}^{-}(\theta,f)$. •

From Theorem 3 it follows that if $A_q(f)\neq 0$ then by the choice

$$a_{\pm 1}=\frac{q(q+1)}{2}\pm\frac{1}{i\pi}\frac{A_{q+1}(f)}{A_q(f)}+\frac{q+1}{1+e^{\pm i\pi x}}. \tag{24}$$

and otherwise ($A_q(f)=0$) by the choice

$$a_{\pm 1}=\frac{q(q+1)}{2}+\frac{q+1}{1+e^{\pm i\pi x}} \tag{25}$$

we derive approximation of the order

$$R_{1,N}(\theta,f)=o(N^{-q-3}),\ N\to\infty$$

inside of $(-1,1)$.

Note that in the first case $S_{1,N}(\theta,f)$ is nonlinear as Fourier-Pade approximation and in the second case it is linear approximation.

Now we represent a typical numerical example. Consider the following simple function

$$f(x)=(1-x^2)\sin(x-1). \tag{26}$$

It is trivial to check that $A_0(f)=0, A_1(f)\neq 0$. In Fig. 1 graphics of the errors are represented while approximating (26) by Fourier-Pade approximation **(a)** and by $S_{1,N}(\theta,f)$ with $\theta_{\pm 1}$ as in Theorem 3 with $a_{\pm 1}$ from (24). Here $N=32$ and $|x|\leq 0.2$. As we see approximation $S_{1,N}(\theta,f)$ is 10 times more precise than Fourier-Pade approximation.

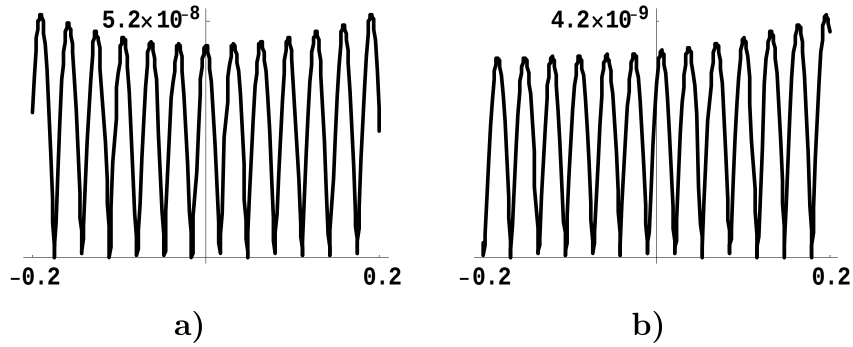

Fig.1. Graphics of the errors while approximating (26) by Fourier-Pade approximation **(a)** and $S_{1,N}(\theta, f)$ **(b)** for $N = 32$.

It is interesting to compare these approximations near the points of singularities $x = \pm 1$. In Fig. 2 we compare these approximations at the point $x = 1$ for $N = 32$. Note that approximation $S_{1,N}(\theta, f)$ is undefined at the points $x = \pm 1$. Hence by increasing the precision of approximation $S_{1,N}(\theta, f)$ inside of $(-1, 1)$ we simultaneously decrease the precision at the end points.

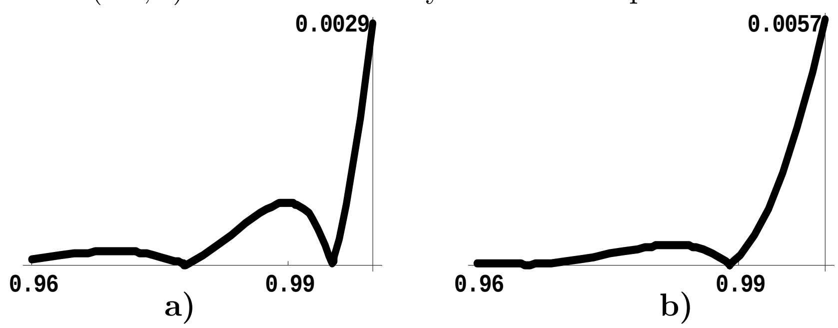

Fig.2. Graphics of the errors while approximating (26) by Fourier-Pade **(a)** and by $S_{1,N}(\theta, f)$ **(b)** for $N = 32$.

In [2] we show that

$$\theta_{\pm 1} = 1 - \frac{\tau}{N} \tag{27}$$

minimizes uniform or L_2 errors of $S_{1,N}(\theta, f)$ in the whole interval by appropriate choice of parameter τ. In Tables 1,2 we represent the corresponding optimal values of τ. Note that τ depends on the smoothness of f.

q	1	2	3	4	5
τ	1.17728	2.23568	3.24768	4.26805	5.27982

Table 1. Optimal values of τ that minimize L_2-error in the whole interval.

q	1	2	3	4	5	6
τ	1.3533	2.3199	3.3020	4.2915	5.2845	6.2795

Table 2. Optimal values of τ that minimize uniform error in the whole interval.

Hence, we have two different choice for parameters $\theta_{\pm 1}$. The first for approximation inside of (-1,1) and the second for approximation at the end points of the interval. Now we combine these two approaches and suggest, for example, the following

$$\theta_{\pm 1} = \sigma(x)\left(1 - \frac{q+1}{N} + \frac{a_{\pm 1}}{N^2}\right) + (1 - \sigma(x))\left(1 - \frac{\tau}{N}\right) \tag{28}$$

where

$$\sigma(x) = \frac{\cos^6 \frac{\pi}{2}x}{\cos^6 \frac{\pi}{2}x + \sin^6 \frac{\pi}{2}x}$$

and parameter τ can be taken from Tables 1,2 depending on the smoothness of f.

In Fig. 3 we represent graphics of the errors while approximating (26) by $S_{1,N}(\theta, f)$ with (28) and $\tau = 1.3533$ **(a)**, $\tau = 1.17728$ **(b)** for $N = 32$. For such choice of parameter $\theta_{\pm 1}$ convergence of $S_{1,N}(\theta, f)$ inside of $(-1, 1)$ preserves (see Fig.1 **b)**) and meanwhile the uniform error in the whole interval becomes 3 times and L_2-error 1.7 times less compared with Fourier-Pade approximation for $N = 32$.

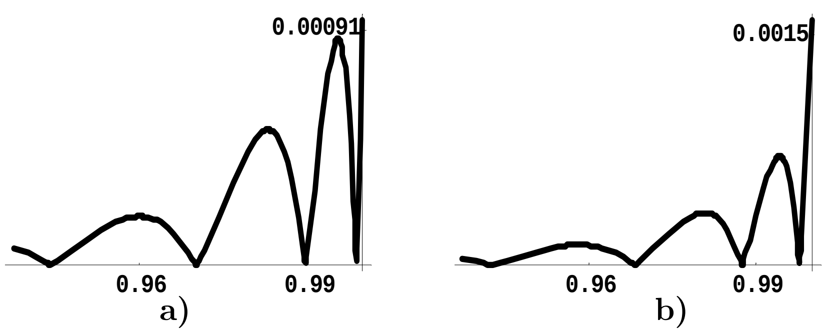

Fig.3. Graphics of the errors while approximating (26) by $S_{1,N}(\theta, f)$ with (28) and $\tau = 1.3533$ **(a)**, $\tau = 1.17728$ **(b)** for $N = 32$.

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