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Stability Criterion for Systems of Two First-Order Linear Ordinary Differential Equations

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Abstract—The method of Riccati's equation is applied to find a stability criterion for systems of two first-order linear ordinary differential equations. The obtained result is compared for a particular example with results obtained by the Lyapunov and Bogdanov methods, by using estimates of solutions of systems in terms of the Losinskii logarithmic norms, and by the freezing method.

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1. INTRODUCTION

Stability theory, whose founder was Lyapunov (see [1]), arose from the needs of natural science (mechanics, physics, etc.) and is one of the most important areas of the qualitative theory of differential equations.

Let $a_{jk}(t)$, $j,k=1,2$, be continuous real-valued functions on $[t_0;+\infty)$. Consider the system of equations

$$\begin{cases}\phi'(t)=a_{11}(t)\phi(t)+a_{12}(t)\psi(t),\\ \psi'(t)=a_{21}(t)\phi(t)+a_{22}(t)\psi(t),\end{cases}\qquad t\ge t_0. \tag{1.1}$$

The stability of linear systems of differential equations, in particular, that of system (1.1) and second-order linear ordinary equations, is an important problem of the qualitative theory of differential equations; it has been considered in numerous works (see [2] and references therein; see also [3]–[11]). Bellman's fundamental theorem (see [12]) provides necessary and sufficient conditions for the boundedness of solutions of a nonlinear system (in a certain class of nonlinear systems) in terms of the right-hand side of the corresponding linear inhomogeneous system. Hence any stability criterion for a linear homogeneous system (in particular, for system (1.1)) allows us to narrow down, to a degree, the above-mentioned class of nonlinear systems, retaining systems with bounded solutions. Many problems of mechanics, physics, etc. can be reduced to the study of the stability of system (1.1) (see, e.g., [4] and [8]). One of the methods for establishing the stability or instability of a linear system of differential equations is the use of various methods for estimating its solutions. The main methods of this type are those that use Lyapunov, Ważewski, Bogdanov, and Losinskii estimates and the freezing method (see [3, pp. 40–98, 132–145]). An important method for establishing the stability of a linear system with periodic coefficients is the method of estimating the characteristic exponents of the system, which was proposed by Yakubovich (see [4]). The stability condition for the periodic system (1.1) was studied by estimating its characteristic exponents in [6]. The result was applied to obtain the conditions for the boundedness of the solutions of the equation

$$\phi''(t)+q(t)\phi(t)=0,\qquad t\ge t_0,$$

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in terms of the periodic coefficient $q(t)$. In [9], the method of the periodic boundary-value problem was used to find stability criteria for the equation

$$\phi''(t) + p_1(t)\phi'(t) + p_0(t)\phi(t) = 0, \qquad t \ge t_0,$$

with periodic coefficients ($p_j(t+\omega) = p_j(t)$, $t \ge t_0$, $\omega > 0$, $j = 0, 1$). In [10], the method of estimates of an "energy function" was used to obtain criteria for the boundedness and the convergence to zero as $t \to +\infty$ of a collection of functions $\phi(t)$, $\phi'(t)/\sqrt{q(t)}$, where $\phi(t)$ is an arbitrary solution of the equation

$$(p(t)\phi'(t))' + q(t)\phi(t) = 0, \qquad t \ge t_0.$$

All these and other methods allow us to determine wide classes of stable and unstable linear systems of ordinary differential equations (and of linear ordinary differential equations). However, all these methods together are far from solving the stability problem for linear systems of ordinary differential equations. In the present paper, we use an approach of [7], namely, the technique of Riccati's equation, to prove a coefficient stability criterion for system (1.1). We give an example for which we compare the result obtained in the paper with results obtained by the Lyapunov and Bogdanov methods, by using estimates of solutions of systems in terms of Losinskii logarithmic norms, and by the freezing method.

2. AUXILIARY ASSERTIONS

For a continuous function $x = x(t)$ on $[t_0; +\infty)$, we set

$$J_x(t_1; t) \equiv \exp\biggl\{\int_{t_1}^{t} x(s)\,ds\biggr\}, \qquad J_x(t) \equiv J_x(t_0; t), \quad t, t_1 \ge t_0.$$

Obviously,

$$J_x(t_1; t) = J_x(t_1; t_2) J_x(t_2; t), \qquad t, t_1, t_2 \ge t_0. \tag{2.1}$$

Let $a_0(t)$, $b_0(t)$, $c_0(t)$, $a_1(t)$, $b_1(t)$, and $c_1(t)$ be continuous real-valued functions on $[t_0; +\infty)$. Consider Riccati's equations

$$y'(t) + a_0(t)y^2(t) + b_0(t)y(t) + c_0(t) = 0, \qquad t \ge t_0, \tag{2.2}$$

$$y'(t) + a_1(t)y^2(t) + b_1(t)y(t) + c_1(t) = 0, \qquad t \ge t_0, \tag{2.3}$$

and the differential inequalities

$$\eta'(t) + a_0(t)\eta^2(t) + b_0(t)\eta(t) + c_0(t) \ge 0, \qquad t \ge t_0, \tag{2.4}$$

$$\eta'(t) + a_1(t)\eta^2(t) + b_1(t)\eta(t) + c_1(t) \ge 0, \qquad t \ge t_0. \tag{2.5}$$

For $a_0(t) \ge 0$ (for $a_1(t) \ge 0$), $t \ge t_0$, inequality (2.4) (respectively, (2.5)) has a solution on $[t_0; +\infty)$ satisfying any real initial condition (see [7]). In what follows, we assume that the solutions of the equations and systems of equations under consideration are real-valued.

Theorem 1. *Let $y_0(t)$ be a solution of Eq. (2.2) on $[t_0; +\infty)$, and let $\eta_0(t)$ and $\eta_1(t)$ be, respectively, solutions of inequalities (2.4) and (2.5) with $\eta_0(t_0) \ge y_0(t_0)$ and $\eta_1(t_0) \ge y_0(t_0)$. Suppose that $a_1(t) \ge 0$ and*

$$\begin{aligned} \eta_0(t_0) - y_0(t_0) + \int_{t_0}^{t} \exp\biggl\{\int_{t_0}^{\tau} [a_1(\xi)(\eta_0(\xi) + \eta_1(\xi)) + b_1(\xi)]\,d\xi\biggr\} \\ \times \bigl[(a_0(\tau) - a_1(\tau))y_0^2(\tau) + (b_0(\tau) - b_1(\tau))y_0(\tau) + c_0(\tau) - c_1(\tau)\bigr]\,d\tau \ge 0, \qquad t \ge t_0. \end{aligned}$$

Then Eq. (2.3) with any initial condition $y_1(t_0) \ge y_0(t_0)$ has a solution $y_1(t)$ on $[t_0; +\infty)$; moreover, $y_1(t) \ge y_0(t)$, $t \ge t_0$.

Proof. For a proof, see [13, Theorem 3.1]. □

Remark 1. We set $a_0(t) = a_1(t)$ for $t \ge t_0$. It immediately follows from Theorem 1 that if $a_1(t) \ge 0$ and $c_1(t) \le 0$ for $t \ge t_0$, then the solution $y_1(t)$ of Eq. (2.3) with $y_1(t_0) = 0$ exists on $[t_0; +\infty)$ and is nonnegative.