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A DESCRIPTION OF LINEARLY ADDITIVE METRICS ON $\mathbb{R}^n$

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Abstract. There is an integral-geometric approach, proposed by Busemann, for building linearly additive metrics on $\mathbb{R}^n$ (it uses hyperplanes). Hilbert's Fourth Problem was solved with the help of this construction. In this article, we present a new description (using straight lines) of linearly additive metrics on $\mathbb{R}^n$, generated by a norm. There is a link between this description and the sine transform.

1. Introduction and posing of the questions

Hilbert's Fourth Problem consists of describing the geometries in $\mathbb{R}^n$ for which the shortest geodesic between any two points is a straight line. G. Hamel (see [12]) proved that this problem is equivalent to describing the continuous linearly additive metrics on open convex subsets of $\mathbb{R}^n$.

A metric d on $\mathbb{R}^n$ is said to be linearly additive if any three points x, y, z lying on a straight line in this order, satisfy the equality

$$d(x,z) = d(x,y) + d(y,z).$$

We let $\mathbb{R}^n$ ($n \geq 2$) denote Euclidean n-dimensional space. Let $\mathbb{S}^{n-1}$ be the unit sphere of dimension $n-1$ in $\mathbb{R}^n$ with centre at the origin $O \in \mathbb{R}^n$, and let λ_{n-1} be the surface Lebesgue measure in $\mathbb{S}^{n-1}$ (σ_{n-1} is its total surface measure). We set $\mathbb{S}_\omega \subset \mathbb{S}^{n-1}$ to be a large $(n-2)$-dimensional sphere with pole at $\omega \in \mathbb{S}^{n-1}$, take $\mathbb{E}^n$ ($\mathbb{E}^3 = \mathbb{E}$) to be a space of hyperplanes of $\mathbb{R}^n$, and $[x]$ a bundle of hyperplanes passing through a point $x \in \mathbb{R}^n$.

H. Busemann (see [8]) suggested the following construction of linearly additive metrics.

Let μ be a bundleless measure in $\mathbb{E}^n$ ($\mu([x]) = 0$ for any point $x \in \mathbb{R}^n$) which also satisfies the condition:

$$0 < \mu(\{e \in \mathbb{E}^n \colon e \cap [x;y] \neq \emptyset\}) < \infty \quad \text{for} \quad x \neq y, \tag{1.1}$$

where $[x;y]$ is the segment with endpoints x and y.

A metric given by the equality

$$d(x,y) = \mu(\{e \in \mathbb{E}^n \colon e \cap [x;y] \neq \emptyset\}), \tag{1.2}$$

for $x, y \in \mathbb{R}^n$, is a linearly additive metric.

It turns out that in $\mathbb{R}^2$ every continuous linearly additive metric can be obtained by means of (1.2) with measure in the space of lines on the surface (see [14], [1] and [4]). This solves Hilbert's Fourth Problem in $\mathbb{R}^2$ (see also [5]).

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It also turns out that for $n \geq 3$ not every continuous linearly additive on $\mathbb{R}^n$ can be obtained by means of (1.2) with some measure μ in $\mathbb{E}^n$ (see [15]). The problem is related to the application of the cosine transform $T\colon \mathscr{C}_c^\infty \to \mathscr{C}_c^\infty$ which is defined by

$$H(\xi) = Th(\xi) = \int_{\mathbb{S}^{n-1}} |\langle \omega, \xi \rangle|\, h(\omega) \lambda_{n-1}(d\omega), \quad \xi \in \mathbb{S}^{n-1}; \tag{1.3}$$

here and below, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

Indeed, suppose that the linearly additive metric $d(x, y) = \|x - y\|$, generated by the norm $\| \cdot \|$ on $\mathbb{R}^n$, is obtained by (1.2) with translation invariant measure μ on $\mathbb{E}^n$.

It is well known (see [2]) that a translation invariant measure μ can be decomposed:

$$d\mu = m(d\omega) \cdot dp,$$

where m is a finite even measure given on $\mathbb{S}^{n-1}$ (here (p, ω) is the usual parametrisation of a hyperplane e: p is the distance of e from the origin O; $\omega \in \mathbb{S}^{n-1}$ is the direction normal to e). Now supposing that $m(d\omega) = h(\omega) d\omega$, we obtain

$$d(x, 0) = \int_{\mathbb{S}^{n-1}} |(x, \omega)|\, m(d\omega) = \int_{\mathbb{S}^{n-1}} |(x, \omega)|\, h(\omega) \lambda_{n-1}(d\omega). \tag{1.4}$$

W. Blaschke ([9], see also [17]) proved that if the even function $d(x, 0)$ can be differentiated enough times, then (1.4) has a unique continuous even solution that is not necessarily positive. If h is a solution to (1.4), then we can define a translation invariant measure $d\mu = h(\omega) d\omega \cdot dp$ in $\mathbb{E}^n$ that satisfies (1.2).

This solves Hilbert's Fourth Problem in $\mathbb{R}^n$, $n \geq 3$, for metrics generated by a norm (Minkowsky space). Thus: *for every metric on* $\mathbb{R}^n$ generated by a sufficiently smooth norm *there exists a translation invariant measure* $d\mu = m(d\omega) \cdot dp$ *on* $\mathbb{E}^n$ *with even alternating measure* m *on* $\mathbb{S}^{n-1}$ *such that the metric is generated by* (1.2) via Busemann's construction.

In the case of a general sufficiently smooth linearly additive metric, the situation is as follows.

A. V. Pogorelov [14] considered the Finsler metrics on $\mathbb{R}^n$, since a sufficiently smooth linearly additive metric is a Finsler metric (see also [15]).

According to the definition of a Finsler metric on $\mathbb{R}^n$, it is a continuous function $H\colon \mathbb{R}^n \times \mathbb{R}^n \to [0; \infty)$ with the property that $H(x, \cdot)$ is a norm on $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$. Let μ be a locally finite alternating measure on $\mathbb{E}$, which has density with respect to the standard invariant measure (with respect to Euclidean motion, see [3])

$$\mu(de) = h(e) de = h(e)\, dp\, \lambda_{n-1}(d\omega), \tag{1.5}$$

where de is an element of the standard invariant measure. We consider the restriction of h onto $[x]$ as a function on the hemisphere. Then we extend this restriction to $\mathbb{S}^{n-1}$ by symmetry, since the direction completely determines the plane from $[x]$. Thus, on $\mathbb{S}^2$ we define the following function h_x:

$$h_x(\omega) = h(e_{x,\omega}) \quad \text{for} \quad \omega \in \mathbb{S}^{n-1},$$

where $e_{x,\omega} \in [x]$ is the hyperplane containing x and normal to ω. Below, we call h_x the *restriction* of h to $[x]$.

Theorem (Pogorelov's Theorem [14]). *Let H be a smooth linearly additive Finsler metric on $\mathbb{R}^n$ $(H\colon \mathbb{R}^n \times \mathbb{R}^n \to [0; \infty))$. Then there exists a unique locally finite alternating measure μ on $\mathbb{E}^n$, with continuous density, such that for $x \in \mathbb{R}^n$*

$$H(x, \xi) = \int_{\mathbb{S}^{n-1}} |(\omega, \xi)|\, h_x(\omega)\, \lambda_{n-1}(d\omega), \qquad \xi \in \mathbb{S}^{n-1}. \tag{1.6}$$

Here, h_x is the restriction of h onto $[x]$, and $d\omega$ denotes the surface Lebesgue measure on $\mathbb{S}^{n-1}$.

2. A description of continuous and linearly additive metrics by means of straight lines

In this article, we propose a new approach to the description of continuous linearly additive metrics on $\mathbb{R}^n$. We denote the space of straight lines in $\mathbb{R}^n$ by $\mathbb{G}^n$. We use the usual parametrisation of straight lines $\gamma = (P, \Omega)$, where Ω is the direction of γ and P is the point of intersection of γ with the hyperplane $e_{O,\Omega}$ (passing through the origin perpendicular to Ω). Let η be a translation invariant (invariant with respect to the group of Euclidean motions in $\mathbb{R}^n$) measure on $\mathbb{G}^n$ (assumed locally finite). It is known that η can be decomposed up to a constant factor: there exists a finite even measure v on $\mathbb{S}^{n-1}$ such that

$$\eta(d\gamma) = v(d\Omega) \cdot dP,$$

where $d\gamma$ is an element of the invariant measure on $\mathbb{G}^n$, and dP is an element of the Lebesgue measure on $e_{O,\Omega}$ (see [2]). Since $(P, \Omega) = (P, -\Omega)$, we can assume that the measure v is even ($v(A) = v(-A)$ for the Borel set $A \subset \mathbb{S}^{n-1}$).

Let $s = [x; y]$ be a segment with endpoints $x, y \in \mathbb{R}^n$, and $\xi \in \mathbb{S}^{n-1}$ the direction of s. Let $\mathscr{C}$ be a regular circular cylinder with axis $s = [x; y]$ and base $\mathscr{C}$ — this is the $(n-2)$-dimensional ball $B_{n-2}(r, \xi)$ with radius r and normal $\xi \in \mathbb{S}^{n-1}$.

For $\eta([\mathscr{C}])$, where $[\mathscr{C}] = \{\gamma \in \mathbb{G}^n : \gamma \cap \mathscr{C} \neq \emptyset\}$, we have

$$\eta([\mathscr{C}]) = \int_{[\mathscr{C}]} v(d\Omega)\, dP = V_{n-2}(r)|x-y| \int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi, \Omega}) v(d\Omega) + o(r^{n-2}). \tag{2.1}$$

Here and below we denote the angle between the two directions $\xi, \Omega \in \mathbb{S}^{n-1}$ by $(\widehat{\xi, \Omega})$, the Euclidean distance between $x, y \in \mathbb{R}^n$ by $|x-y|$, and the $(n-2)$-dimensional volume $B_{n-2}(r, \xi)$ by $V_{n-2}(r)$.

We define

$$d(x, y) = \lim_{r \to 0} \frac{\eta([\mathscr{C}])}{r^{n-2}} = \frac{|x-y|\, \pi^{(n-2)/2}}{\Gamma(\frac{n}{2})} \int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi, \Omega}) v(d\Omega) \tag{2.2}$$

for $x, y \in \mathbb{R}^n$. It is known that

$$V_{n-2}(r) = V_{n-2}(B_{n-2}(r, \xi)) = \frac{(r\sqrt{\pi})^{n-2}}{\Gamma(\frac{n}{2})}.$$

Theorem 1. *Let $\eta = v(d\Omega) \cdot dP$ be a translation invariant measure on $\mathbb{G}^n$, and let $s = [x; y]$ be the segment with endpoints $x, y \in \mathbb{R}^n$ and direction $\xi \in \mathbb{S}^{n-1}$. If we define d by* (2.2), *then d is a continuous and linearly additive metric.*

Proof. It follows from the definition of d that it is linearly additive (trivially true). The triangle inequality follows from the fact that for each $\Omega \in \mathbb{S}^{n-1}$ the function

$$d(x, y) = |x - y| \sin(\widehat{\xi, \Omega})$$

(where $\xi \in \mathbb{S}^{n-1}$ is the direction of the segment $[x; y]$) is the length of the projection of the segment $[x; y]$ onto the hyperplane with normal Ω. We note also that integration with respect to v preserves convexity. □

Now we consider the question: does this construct all continuous and linearly additive metrics on $\mathbb{R}^n$ that are generated by a norm? In $\mathbb{R}^2$, the answer is yes (see [14], [1] and

[4]). In $\mathbb{R}^n$ for $n \geq 3$, our investigations lead us to the sine transform $Q\colon \mathscr{C}_c^\infty \to \mathscr{C}_c^\infty$, defined by the formula:

$$Qf(\xi) = \int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, f(\Omega)\lambda_{n-1}(d\Omega), \quad \text{for} \ \ \xi \in \mathbb{S}^{n-1}. \tag{2.3}$$

The sine transform plays an important role later on. Indeed, let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. We suppose now that the linearly additive metric $d(x,y) = \|x-y\|$ is generated by (2.2) by means of η on $\mathbb{G}^n$. We show below that η is unique. We also note that η is translation invariant, since d is translation invariant ($\eta(d\gamma) = v(d\Omega)\cdot dP = f(\Omega)\lambda_{n-1}(d\Omega)\cdot dP$). We have

$$d(x,y) = d(x-y,0) = |x-y|\cdot H(\xi), \tag{2.4}$$

where H is the support function of the dual unit ball, and ξ is the direction of the segment $[y;x]$. Comparing (2.2) and (2.4), we see that for the continuous linearly additive metric d, generated by (2.2), we have

$$H(\xi) = \frac{\pi^{(n-2)/2}}{\Gamma(\frac{n}{2})} \int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi,\Omega}) f(\Omega)\, \lambda_{n-1}(d\Omega). \tag{2.5}$$

3. Inversion of the sine transform

The sine transform is invertible, which is equivalent to the following theorem.

Theorem 2. *Suppose that an alternating measure v on $\mathbb{S}^{n-1}$ satisfies the condition*

$$\int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, v(d\Omega) = 0 \tag{3.1}$$

for any $\xi \in \mathbb{S}^{n-1}$. Then $v \equiv 0$.

Proof. Let $Y_d(\xi)$ be a spherical function of order d. We multiply both sides of (3.1) by $Y_d(\xi)$ and integrate. We have

$$\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, Y_d(\xi)\lambda_{n-1}(d\xi) \right) v(d\Omega) = 0. \tag{3.2}$$

It follows from the Funk–Hecke formula (see [11]) that

$$\int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, Y_d(\xi)\lambda_{n-1}(d\xi) = c_d Y_d(\Omega), \tag{3.3}$$

where c_d depends only on d and $c_d \neq 0$ if d is even. Consequently, for all spherical functions of order d we have

$$\int_{\mathbb{S}^{n-1}} Y_d(\Omega)\, v(d\Omega) = 0. \tag{3.4}$$

This also holds if d is odd. Using the uniform approximation of a continuous function by linear combinations of spherical functions on $\mathbb{S}^{n-1}$, we obtain

$$\int_{\mathbb{S}^{n-1}} g(\Omega)\, v(d\Omega) = 0, \tag{3.5}$$

for any continuous g. Hence it follows that $v \equiv 0$. □

In [6], we found an inversion formula for the sine transform. In this article, we find an inversion formula using the relation between the sine and cosine transforms.

We denote the spherical Radon transform (Funk transform) by R. It is defined by

$$(RH)(\xi) = \frac{1}{\sigma_{n-2}} \int_{\mathbb{S}_\xi} H(\Omega)\, \lambda_{n-2}(d\Omega), \quad \xi \in \mathbb{S}^{n-1}, \tag{3.6}$$

where H is a function given on $\mathbb{S}^{n-1}$. For $n \geq 3$ the inversion formula for R was found by Helgason in [13] (for $n = 3$, the inversion formula was given by Minkowsky and Blaschke [9]; see also [10]). In [7], the generalised spherical Radon transform was considered and an inversion formula found.

There exists the following connection between the sine and cosine transforms.

Theorem 3. *Let $Q\colon \mathscr{C}_c^\infty \to \mathscr{C}_c^\infty$ be a sine transform and $T : \mathscr{C}_c^\infty \to \mathscr{C}_c^\infty$ be a cosine transform. We have*

$$Q = \frac{(n-2)\sigma_{n-2}}{2\sigma_{n-3}} RT. \tag{3.7}$$

Proof. For $f \in \mathscr{C}_c^\infty$ we have

$$(RTf)(\xi) = \frac{1}{\sigma_{n-2}} \int_{\mathbb{S}_\xi} \left(\int_{\mathbb{S}^{n-1}} |\langle \omega, \Omega \rangle| f(\omega) \lambda_{n-1}(d\omega) \right) \lambda_{n-2}(d\Omega). \tag{3.8}$$

Changing the order of integration in (3.8) (Fubini's Theorem), we obtain

$$(RTf)(\xi) = \frac{1}{\sigma_{n-2}} \int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}_\xi} |\langle \omega, \Omega \rangle| \lambda_{n-2}(d\Omega) \right) f(\omega) \lambda_{n-1}(d\omega). \tag{3.9}$$

We use the spherical coordinates $\omega = (\nu, \phi)$, where $\nu = (\widehat{\xi, \omega})$ is the polar angle measured from ξ (zenith direction), and $\phi \in \mathbb{S}_\xi$. Applying the spherical cosine rule, we find

$$|\langle \omega, \Omega \rangle| = |\langle \widehat{\phi, \omega} \rangle| \cdot |\langle \widehat{\phi, \Omega} \rangle| = \sin(\widehat{\xi, \omega})\, |\langle \widehat{\phi, \Omega} \rangle|. \tag{3.10}$$

Substituting (3.10) into (3.9) and considering that for any $\phi \in \mathbb{S}^{n-2}$, as can easily be calculated,

$$\int_{\mathbb{S}^{n-2}} |\cos(\widehat{\phi, \Omega})|\, \lambda_{n-2}(d\Omega) = \frac{2\sigma_{n-3}}{n-2},$$

we obtain

$$\begin{aligned} (RTf)(\xi) &= \frac{1}{\sigma_{n-2}} \int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi, \omega}) \left(\int_{\mathbb{S}_\xi} |\langle \widehat{\phi, \Omega} \rangle| \lambda_{n-2}(d\Omega) \right) f(\omega) \lambda_{n-1}(d\omega) \\ &= \frac{2\sigma_{n-3}}{\sigma_{n-2}(n-2)} \int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi, \omega}) f(\omega) \lambda_{n-1}(d\omega) = \frac{2\sigma_{n-3}}{\sigma_{n-2}(n-2)} (Qf)(\xi). \end{aligned} \tag{3.11}$$

□

We also use W to denote the transformation $W\colon \mathscr{C}_c^\infty \to \mathscr{C}_c^\infty$, defined by

$$W = \frac{((n-1) + \Delta)}{2\sigma_{n-2}}, \tag{3.12}$$

where Δ is the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$.

There exists a formula (linking the theory of Radon transforms and integral geometry), connecting T and R [10]:

$$WT = R. \tag{3.13}$$

It is known that R is invertible, therefore

$$T^{-1} = WR^{-1}. \tag{3.14}$$

The following theorem gives an inversion formula for the sine transform.

Theorem 4. *Let* $Q\colon \mathscr{C}_c^\infty \to \mathscr{C}_c^\infty$ *be the sine transform*

$$Qh(\xi) = \int_{\mathbb{S}^{n-1}} \sin(\widehat{\xi,\Omega})\, h(\Omega)\, \lambda_{n-1}(d\Omega), \quad \xi \in \mathbb{S}^{n-1}. \tag{3.15}$$

Then

$$Q^{-1} = \frac{2\sigma_{n-3}}{(n-2)\sigma_{n-2}} W\, R^{-2}. \tag{3.16}$$

The proof of Theorem 4 follows directly from (3.7) and (3.16).

We note that there exists a sufficiently smooth even function H, defined on $\mathbb{S}^{n-1}$ (such that $|x|H(\xi)$ is a norm, where ξ is the direction $\overrightarrow{Ox}$), for which the solution (2.5) also takes negative values (see [6]).

Thus, the following theorem is true.

Theorem 5. *For each metric d (generated by a sufficiently smooth norm) on $\mathbb{R}^n$, there exists a translation invariant measure $\eta = \upsilon(d\Omega)\cdot dP$ on $\mathbb{G}^n$ with even alternating measure υ on $\mathbb{S}^{n-1}$ such that d is generated by means of* (2.2).

Work on the case of general sufficiently smooth linearly additive metrics is underway.

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