SINGULAR OPERATORS FOR BEGINNERS

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Contents

1. Maximal operators

1.1. Definition of Hardy-Littlewood maximal operator and simple properties.

Denote by $L^1_{loc}(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which

$$\int_a^b |f| < \infty$$

for any bounded interval $(a.b) \subset \mathbb{R}$. Given a function $f \in L_{loc}(\mathbb{R})$ denote

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f|,$$

where $\sup$ is taken over the intervals $I = (a, b) \ni x$. The function Mf is said to be the Hardy-Littlewood maximal function of f (or just maximal operator). The following properties of the maximal operator are easy to check:

1) For any $\lambda > 0$ and $f \in L_{loc}(\mathbb{R})$ the set $\{Mf > \lambda\}$ is an open set.
2) $Mf(x) \geq 0$ for any $f \in L_{loc}(\mathbb{R})$ and $x \in \mathbb{R}$,
3) For any $f, g \in L_{loc}(\mathbb{R})$ we have
$$M(f + g)(x) \leq Mf(x) + Mg(x),$$
4) $M(\lambda f)(x) = |\lambda| Mf(x)$ for any $f \in L_{loc}(\mathbb{R})$ and $\lambda \in \mathbb{R}$.
5) $\|Mf\|_\infty \leq \|f\|_\infty$.

Property 1) follows from the continuity property of Lebesgue integral and it implies that Mf is a measurable function. The conditions 3) and 4) say that maximal function M defines a sublinear operator and by 5) it is of strong (∞, ∞) type. We shall see below that the maximal operator satisfies weak-L^1 and strong-L^p, $1 < p \leq \infty$ bounds. We shall need the following

1.2. A covering lemma.

For an in interval I and a number $r > 0$ we denote by rI the interval, which has the same center as I and $|rI| = r|I|$.

Lemma 1.1. *If $E \subset \mathbb{R}$ is bounded and $\mathcal{G}$ is a family of intervals with*

$$E \subset \bigcup_{I \in \mathcal{G}} I,$$

then there exists a finite or infinite sequence of pairwise disjoint intervals $I_k \in \mathcal{G}$ such that

$$E \subset \bigcup_k 5I_k. \tag{1.1}$$

Proof. Suppose we have $E \subset \Delta = [a, b]$. If there is an interval $I \in \mathcal{G}$ so that $I \cap [a, b] \neq \varnothing$ and $|I| > b - a$, then we have $E \subset B \subset 3I$. Thus the desired sequence can be formed by a single element I. Hence we can suppose that any element $I \in \mathcal{G}$ satisfies the conditions $G \cap [a, b] \neq \varnothing$ and $|I| \leq b - a$. Therefore we get

$$\bigcup_{G \in \mathcal{G}} G \subset 3\Delta.$$

Take I_1 to be a ball from $\mathcal{G}$ satisfying

$$|I_1| > \frac{1}{2} \sup_{I \in \mathcal{G}} |I|.$$

Then, suppose by induction we have already chosen elements $I_1, \ldots, I_k$ from $\mathcal{G}$. Take $I_{k+1} \in \mathcal{G}$ disjoint with the intervals $I_1, \ldots, I_k$ and satisfying

$$|I_{k+1}| > \frac{1}{2} \sup_{I \in \mathcal{G}\,:\, I \cap I_j = \varnothing,\, j=1,\ldots,k} |I|.$$

If for some n we will not be able to determine I_{n+1} the process will stop and we will get a finite sequence $I_1, I_2, \ldots, I_n$. Otherwise our sequence will be infinite. We shall consider the infinite case of the sequence (the finite case can be done similarly). Since the balls I_n are pairwise disjoint and $I_n \subset 3\Delta$, we have $|I_n| \to 0$. Take an arbitrary $I \in \mathcal{G}$ such that $I \neq I_k$, $k = 1, 2, \ldots$. Let m be the smallest integer such that

$$|I| > \frac{1}{2}|I_{m+1}|.$$

Observe that we have

$$I \cap I_j \neq \varnothing$$

for some of $1 \leq j \leq m$, since otherwise I had to be chosen instead of I_{m+1}. Besides, we have $|I| \leq 2|I_j|$, which implies $I \subset 5I_j$. Since $I \in \mathcal{G}$ was taken arbitrarily, we get (1.1). ☐

1.3. **Weak-L^1 and strong-L^p properties.**

Theorem 1.2. *The maximal operator* (1.4) *satisfies weak-L^1 and strong-L^p, $1 < p \leq \infty$, inequalities:*

$$|\{M(f) > \lambda\}| \leq \frac{c \cdot \|f\|_1}{\lambda} \tag{1.2}$$

$$\|M(f)\|_p \leq c_p \|f\|_p. \tag{1.3}$$

Proof. Denote

$$E = \{x \in X : \, Mf(x) > \lambda\}.$$

By the definition of the maximal function for any $x \in E$ there exists an interval $I(x) \ni x$ such that

$$\frac{1}{|I(x)|} \int_{I(x)} |f| > \lambda.$$

We have $E = \cup_{x \in E} I(x)$. Given interval $\Delta = (-A, A)$ consider the collection of intervals $\mathcal{G} = \{I(x) : \, x \in E \cap \Delta\}$. Applying Lemma 1.1, we find a sequence of pairwise disjoint subcollection $\{I_k\} \subset \mathcal{G}$ such that

$$E \cap \Delta \subset \bigcup_k 5I_k.$$

Thus we get

$$|E \cap (-A, A)| \le 5 \sum_k |I_k| \le \frac{5}{\lambda} \sum_k \int_{I_k} |f(t)|dt \le \frac{5}{\lambda} \int_{\mathbb{R}} |f(t)|dt.$$

Since A can be taken arbitrarily big, we get

$$|E| = |\{x \in X : Mf(x) > \lambda\}| \lesssim \frac{1}{\lambda} \int_{\mathbb{R}} |f(t)|dt$$

and so (1.2). The inequality (1.3) follows from the Marcinkiewicz interpolation theorem, since M satisfies weak-L^1 and strong-L^∞ inequalities. □

Remark 1.1. *The following example shows that the maximal operator does not satisfy strong-L^1 inequality. For the function $f(x) = \mathbb{I}_{[0,1]}(x)$ we have*

$$Mf(x) = \begin{cases} 1 \text{ if } & x \in [0,1], \\ \frac{1}{1-x} \text{ if } & x < 0, \\ \frac{1}{x} \text{ if } & x > 1, \end{cases}$$

and so $f \in L^1$, but $Mf \notin L^1$. Thus M is not of strong L^1 type.

1.4. **Some other maximal operators.** Consider the following extension of the maximal operator. For $L^r_{loc}(\mathbb{R})$ define

$$M_r f(x) = \sup_{I \ni x} \left(\frac{1}{|I|} \int_I |f(t)|^r dt \right)^{1/r}. \tag{1.4}$$

Note that the case $r = 1$ coincides with the Hardy-Littlewood maximal operator M and obviously we have

$$M(f) \le M_r(f) = (M(|f|^r))^{1/r}.$$

Thus Theorem 1.3 immediately yields the following.

Theorem 1.3. *The maximal operator (1.4) satisfies weak-L^r and strong L^p, $r < p \le \infty$, inequalities:*

$$|\{M_r(f) > \lambda\}| \le \frac{c \cdot \|f\|_r^r}{\lambda^r}$$
$$\|M_r(f)\|_p \le c_p \|f\|_p.$$

Now consider the operators

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|, \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|.$$

Obviously we have

$$M^\pm f(x) \le Mf(x),$$

so they both satisfy the weak-L^1 and strong-L^p inequalities for $1 < p < \infty$., but the following assertion is common only for those operators.

Theorem 1.4. *Let $f \in L^1(\mathbb{R})$ and $\lambda > 0$. Then the set $G_\lambda = \{M^+f(x) > \lambda\}$ is either empty or*

$$G_\lambda = \cup_k (a_k, b_k), \tag{1.5}$$

where (a_k, b_k) are pairwise disjoint intervals and

$$\int_{a_k}^{b_k} |f| = \lambda(b_k - a_k). \tag{1.6}$$

The same statement is valid also for the operator M^-.

Proof. We will consider only the operator M^+, since for M^- it can be proved analogously. First of all observe that the continuity property of integral implies that the set $G_\lambda = \{M^+f(x) > \lambda\}$ is open and has unique representation (1.5). The inequality

$$\int_{a_k}^{b_k} |f| > \lambda(b_k - a_k)$$

is not possible, since it implies

$$M^+(a_k) \ge \frac{1}{b_k - a_k} \int_{a_k}^{b_k} |f| > \lambda$$

and so $a_k \in G_\lambda$, which is not true. Now suppose to the contrary we have

$$\int_{a_k}^{b_k} |f| < \lambda(b_k - a_k)$$

for some I_k. Thus by continuity for some $a_k < a < b_k$ (close to a_k) we will have

$$\int_{a}^{b_k} |f| < \lambda(b_k - a). \tag{1.7}$$

Since

$$0 \le \frac{1}{u-a} \int_a^u |f| \le \frac{\|f\|_1}{u-a} \to 0 \text{ as } u \to +\infty,$$

and $a \in G_\lambda$, the value of

$$b = \sup\left\{u > a : \int_a^u |f| > \lambda(u-a)\right\}$$

is finite and

$$\int_a^b |f| = \lambda(b-a). \tag{1.8}$$

Inequality $b > b_k$ is not possible, since in that case from (1.7) and (1.8) one can get

$$\int_{b_k}^b |f| = \int_a^b |f| - \int_a^{b_k} |f| > \lambda(b-a) - \lambda(b_k - a) = \lambda(b - b_k), \tag{1.9}$$

which means $b_k \in G_\lambda$ that is not true. If $b < b_k$, then we will have $b \in (a_k, b_k) \in G_\lambda$. So there is $b' > b$ such that

$$\int_b^{b'} |f| > \lambda(b' - b). \tag{1.10}$$

Likewise (1.9) from (1.8) and (1.10) we get

$$\int_a^{b'} |f| > \lambda(b' - a)$$

that is a contradiction by the definition of number b. Thus we conclude $b = b_k$, which is also not possible by (1.7) and (1.8). □

2. Sequences of general convolution operators

2.1. Convolution of two functions.

Definition 2.1. *The convolution of given functions $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ is defined by*

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt = \int_{\mathbb{R}} f(t)g(x-t)dt.$$

Theorem 2.2. *If $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $1 \le p < \infty$, then $(f * g)(x)$ exists almost everywhere and*

$$\|f * g\|_p \le \|f\|_1 \|g\|_p. \tag{2.1}$$

Proof. Without loss oof generality we can suppose that $f, g \ge 0$. Using the Fubini's theorem for positive functions (Tornelli theorem) for a given positive function $h \in L^q$ of norm one we obtain

$$\begin{aligned}
\int_{\mathbb{R}} (f * g)(x)h(x)dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)g(x-t)dt \cdot h(x)dx \\
&= \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} g(x-t)h(x)dxdt \\
&\le \int_{\mathbb{R}} f(t)\|g\|_p\|h\|_q dt \\
&\le \|f\|_1\|g\|_p.
\end{aligned}$$

□

Theorem 2.3. *For any functions $f, g \in L^2(\mathbb{R})$ we have*

$$\widehat{f \star g} = \hat{f} \cdot \hat{g}.$$

Proof. Given $a > 0$ consider the functions $f_a = f\cdot\mathbb{I}_{(-a,a)}$ and $g_a = g\cdot\mathbb{I}_{(-a,a)}$. Clearly,$f_a, g_a \in L^1(\mathbb{R})$. Then applying Fubini's theorem, it follows that

$$\begin{aligned}(\widehat{f_a \star g_a})(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_a(u-t)g_a(t)dt\cdot e^{-ixu}du\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_a(u-t)g_a(t)\cdot e^{-ixu}dtdu\\ &= \int_{\mathbb{R}} g_a(t)e^{-ixt}\int_{\mathbb{R}} f_a(u-t)e^{-ix(u-t)}dudt\\ &= \hat{f}_a(x)\cdot\hat{g}_a(x).\end{aligned}$$

Letting $a \to \infty$ and applying the Plancherel theorem we complete the proof. □

2.2. **Operators associated with an approximation of identity and initial convergence properties.** Given function $\phi \in L^\infty(\mathbb{R})$ we denote

$$\phi^*(x) = \|\phi\cdot\mathbb{I}_{\{t:\,|t|>|x|\}}\|_\infty.$$

One can easily to check that

- $\phi^*(x)$ is even function,
- $\phi^*(x)$ is increasing on $(-\infty, 0]$ (and decreasing on $[0,\infty)$),
- $|\phi(x)| \le \phi^*(x)$.

Definition 2.4. *A sequence of functions $\phi_n \in L_1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $n = 1, 2, \ldots,$ is said to be approximation of identity if it satisfies the relations*

$$\begin{aligned}&1)\int_{\mathbb{R}}\phi_n \to 1 \text{ as } n\to\infty,\\ &2)\sup_n\|\phi_n^*\|_1 < \infty,\\ &3)\int_\delta^\infty \phi_n^* \to 0,\ \text{ as } n\to\infty,\ \text{ for any } \delta > 0.\end{aligned}$$

For any $f \in L^p(\mathbb{R})$, $1 \le p < \infty$ we denote

$$\Phi_n f(x) = \int_{\mathbb{R}} f(x-t)\phi_n(t)dt. \tag{2.2}$$

By Theorem 2.2 Φ_n defines a bounded linear operator on $L^p(\mathbb{R})$. Moreover,

$$\|\Phi_n\|_{L^p\to L^p} \le \|\phi_n\|_1 < \infty.$$

Theorem 2.5. *Let ϕ_n be an approximation of identity, then the operators (2.2) satisfy the properties*

(i) *If $f \in C_K(\mathbb{R})$, then $\Phi_n f$ uniformly converges to f,*
(ii) *If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then $\|\Phi_n f - f\|_p \to 0$ as $n \to \infty$.*

Proof. Let $\delta > 0$. Using properties of approximation of identity, we get

$$\begin{aligned}\limsup_{n\to\infty}\left|\int_{-\delta}^{\delta}\phi_n(t)dt - 1\right| &= \limsup_{n\to\infty}\left|\int_{-\delta}^{\delta}\phi_n(t)dt - \int_{\mathbb{R}}\phi_n(t)dt\right| \\ &= \limsup_{n\to\infty}\left|\int_{|t|>\delta}\phi_n(t)dt\right| \\ &\le \lim_{n\to\infty}\int_{|t|>\delta}\phi_n^*(t)dt = 0\end{aligned}$$

Thus $\gamma_n = \int_{-\delta}^{\delta}\phi_n(t)dt - 1 \to 0$ as $n \to \infty$ and

$$\begin{aligned}\Phi_n f(x) - f(x) &= \int_{\mathbb{R}} f(x-t)\phi_n(t)dt - f(x)\int_{-\delta}^{\delta}\phi_n(t)dt + f(x)\left(\int_{-\delta}^{\delta}\phi_n(t)dt - 1\right) \\ &= \int_{-\delta}^{\delta}(f(x-t) - f(x))\phi_n(t)dt + \int_{|t|>\delta} f(x-t)\phi_n(t)dt + \gamma_n f(x) \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{2.3}$$

From this we conclude

$$|\Phi_n f(x) - f(x)| \le \omega(\delta, f)\|\phi_n\|_1 + \|f\|_C\int_{|t|>\delta}|\phi_n(t)|dt + |\gamma_n|\|f\|_C$$

that immediately implies (i). To proof the second part of theorem we use again (2.3). Applying Hölder's inequality, we get

$$\begin{aligned}\|I_1\|_p^p &= \int_{\mathbb{R}}\left|\int_{-\delta}^{\delta}(f(x-t) - f(x))\phi_n(t)dt\right|^p dx \\ &\le \int_{\mathbb{R}}\left|\int_{-\delta}^{\delta}|f(x-t) - f(x)||\phi_n(t)|^{1/p}|\phi_n(t)|^{1/q}dt\right|^p dx \\ &\le \left(\int_{-\delta}^{\delta}|\phi_n(t)|dt\right)^{p-1}\int_{\mathbb{R}}\int_{-\delta}^{\delta}|f(x-t) - f(x)|^p|\phi_n(t)|dtdx \\ &= \left(\int_{-\delta}^{\delta}|\phi_n(t)|dt\right)^{p-1}\int_{-\delta}^{\delta}|\phi_n(t)|\int_{\mathbb{R}}|f(x-t) - f(x)|^p dxdt \\ &\le (\omega_p(\delta, f))^p\left(\int_{-\delta}^{\delta}|\phi_n|\right)^p \\ &\lesssim (\omega_p(\delta, f))^p.\end{aligned}$$

Therefore we have $\|I_1\|_p \to 0$ as $\delta \to 0$. The integral I_2 is the convolution of f and the function $\phi_n \cdot \mathbb{I}_{\{|t|>\delta\}}$. So applying convolution norm inequality (2.1) we obtain

$$\|I_2\|_p \le \|f\|_p\int_{|t|>\delta}|\phi_n| \le \|f\|_p\int_{|t|>\delta}\phi_n^* \to 0 \text{ as } n \to \infty.$$

The relation $\|I_3\|_p \to 0$ as $n \to \infty$ is trivial. Thus we get

$$\|\Phi_n f - f\|_p \to 0 \text{ as } n \to \infty.$$

□

2.3. **Lemma-estimation by maximal function.**

Lemma 2.6. *Let the positive function $\phi \in L_1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$. Then for any $f \in L^1(\mathbb{R})$ it holds the inequality*

$$\left| \int_{\mathbb{R}} f(t)\phi(t)dt \right| \le \|\phi\|_1 Mf(0), \tag{2.4}$$

where $Mf(0)$ is the value of maximal function of f at 0.

Proof. Given positive integer n consider the intervals $I_k = [a_k, b_k]$, $k = 1, 2, \ldots, n-1$ where

$$a_k = \inf\left\{x \le 0 : \phi(x) \ge \frac{k\|\phi\|_\infty}{n}\right\}, \quad b_k = \sup\left\{x \ge 0 : \phi(x) \ge \frac{k\|\phi\|_\infty}{n}\right\}$$

It is easy to see that $I_1 \supset I_2 \supset \ldots \supset I_{n-1} \ni 0$ and

$$\phi_n(x) = \frac{1}{n}\sum_{k=1}^{n-1} \mathbb{I}_{I_n}(x) \le \phi(x) \le \phi_n(x) + \frac{1}{n}.$$

Thus we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(t)\phi(t)dt \right| &\le \int_{\mathbb{R}} |f(t)|\phi(t)dt \le \frac{1}{n}\sum_{k=1}^{n-1}\int_{I_n} |f(t)|dt + \frac{\|f\|_1}{n} \\
&= \frac{1}{n}\sum_{k=1}^{n-1} |I_n| \cdot \frac{1}{|I_n|}\int_{I_n} |f(t)|dt + \frac{\|f\|_1}{n} \\
&\le Mf(0) \cdot \frac{1}{n}\sum_{k=1}^{n-1} |I_n| + \frac{\|f\|_1}{n} \\
&= Mf(0) \cdot \int_{\mathbb{R}} \phi_n(t)dt + \frac{\|f\|_1}{n} \\
&\le Mf(0)\|\phi\|_1 + \frac{\|f\|_1}{n}.
\end{aligned}$$

Since n can be arbitrary large we get (2.4). □

2.4. **Maximal convolution operators and basic properties.**

Let ϕ_n be an AI sequence. Consider the operator

$$\Phi f(x) = \sup_n |\Phi_n f(x)|$$

where Φ_n are the operators in (2.2). One can easily see that Φ is a sublinear operator.

Theorem 2.7. *The operator Φ is of weak- L^1 and strong-L^p type for $1 < p \leq \infty$. That is*

$$|\{x \in \mathbb{R} : \Phi f(x) > \lambda\}| \leq \frac{c}{\lambda}\|f\|_1,$$

$$\|\Phi f\|_p \leq c_p \|f\|_p. \tag{2.5}$$

Proof. From Lemma 2.6 and properties of approximation of identity it follows that

$$|\Phi_n f(x)| \leq \int_{\mathbb{R}} |f(x-t)||\phi_n(t)|dt \leq \int_{\mathbb{R}} |f(x-t)|\phi_n^*(t)dt \leq \|\phi_n^*\|_1 Mf(x)$$

and so we get

$$\Phi f(x) \leq C \cdot Mf(x).$$

Since the maximal function M satisfies the weak L^1, so we have for Φ. On he other hand Φ satisfies also (∞, ∞) inequality, since for $f \in L^\infty$ we have

$$|\Phi_n f(x)| \leq \int_{\mathbb{R}} |\Phi_n(t)||f(x-t)|dt \leq \|f\|_\infty \int_{\mathbb{R}} |\phi_n(t)|dt \leq C\|f\|_\infty.$$

Applying Marcinkiewicz interpolation theorem we obtain also (2.5). □

Corollary 2.8. *If $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, then $\Phi_n f(x) \to f(x)$ almost everywhere.*

Proof. Approximating $f \in L^p(\mathbb{R})$ by a function $g \in C_K(\mathbb{R})$, for a given $\varepsilon > 0$ we may have decomposition $f = g + h$ such that $\|h\|_p < \varepsilon$. Chose a number $\lambda > 0$. Applying the first part of Theorem 2.5, we get $\Phi_n g(x) \to g(x)$ at any point x and so

$$\begin{aligned} E_\lambda &= \{x \in \mathbb{R} : \limsup_{n\to\infty} |\Phi_n f(x) - f(x)| > \lambda\} \\ &= \{x \in \mathbb{R} : \limsup_{n\to\infty} |\Phi_n h(x) - h(x)| > \lambda\}. \end{aligned}$$

According to Theorem 2.7 the operator Φ satisfies weakL^p inequality. Thus, applying also Chebyshev's inequality, in the case $1 \leq p < \infty$ we get

$$\begin{aligned} |E_\lambda| &\leq |\{x \in \mathbb{R} : \sup_n |\Phi_n h(x)| + |h(x)| > \lambda\}| \\ &\leq |\{x \in \mathbb{R} : \sup_n |\Phi_n h(x)| > \lambda/2\}| + |\{x \in \mathbb{R} : |h(x)| > \lambda/2\}| \\ &\leq |\{x \in \mathbb{R} : \Phi h(x) > \lambda/2\}| + \left(\frac{2}{\lambda}\right)^p \|h\|_p^p \\ &\leq c\left(\frac{2}{\lambda}\right)^p \|h\|_p^p + \left(\frac{2}{\lambda}\right)^p \|h\|_p^p \\ &\leq (c+1)\left(\frac{2}{\lambda}\right)^p \varepsilon^p, \end{aligned}$$

that means $|E_\lambda| = 0$ for any $\lambda > 0$, since ε can be arbitrarily small. Thus we get $\Phi_n f(x) \to f(x)$ a.e.. Now suppose $p = \infty$. Take $f \in L^\infty(\mathbb{R})$ and denote $f_a = f \cdot \mathbb{I}_{(-a,a)}$, where $a > 0$. Obviously, $f \in L^1(\mathbb{R})$ and so by the first part of the theorem we have

$\Phi_n f_a(x) \to f_a(x)$ a.e.. On the other hand if $x \in (-a, a)$, then $\delta = \min\{a-x, x+a\} > 0$, $f(x) = f_a(x)$ and

$$\begin{aligned} |\Phi_n f(x) - \Phi_n f_a(x)| &= \left| \int_{|t|\geq a} \phi_n(x-t) f(t) dt \right| \\ &\leq \|f\|_\infty \int_{|t|\geq a} |\phi_n(x-t)| dt \\ &\leq \|f\|_\infty \int_{|t|\geq \delta} |\phi_n^*(t)| dt \to 0, \end{aligned}$$

as $n \to \infty$. Hence for almost all $x \in (-a, a)$ we get

$$\lim_{n\to\infty} \Phi_n f(x) = \lim_{n\to\infty} \Phi_n f_a(x) = f_a(x) = f(x).$$

Since a is arbitrary, we conclude $\Phi_n f(x) \to f(x)$ a.e. on $\mathbb{R}$. □

3. Almost everywhere convergence of sequences of general operators

3.1. **A lemma on approximation of kernels.** We denote by $\mathrm{BV}(\mathbb{R})$ the right continuous functions of bounded variation on $\mathbb{R}$. We say that the given approximation of identity $\{\varphi_n(x)\}$ is regular if each $\varphi_n(x)$ is positive, decreasing on $[0, \infty]$ and increasing on $[-\infty, 0]$. In the regular case ϕ_n coincides with ϕ_n^* and for any $\delta > 0$ we have

$$\delta \cdot \phi_n(2\delta) \leq \int_\delta^{2\delta} \phi_n \to 0.$$

Thus we can conclude

$$\phi_n(x) \to 0 \text{ whenever } |x| \neq 0. \tag{3.1}$$

Lemma 3.1. *Let $\{\phi_n(x)\}$ be a regular AI. Then there exists a another AI of the form*

$$\psi_n(x) = c_n \sum_{k=1}^{m_n} \mathbb{I}_{\Delta_k^{(n)}}(x), \quad \Delta_i^{(n)} = [a_i^{(n)}, b_i^{(n)}),$$

such that
1) $0 \in \bar{\Delta}_{m_n}^{(n)}$, $\Delta_1^{(n)} \supset \Delta_2^{(n)} \supset \dots \Delta_{m_n}^{(n)}$, $|\Delta_1^{(n)}| \to 0$ *as* $n \to \infty$,
2) $\gamma_n = \sup_{x\in\mathbb{R}} |\phi_n(x) - \psi_n(x)| \to 0$ *as* $n \to \infty$.

Proof. We claim there exists a sequence $\alpha_n \searrow 0$ such that

$$\int_{|t|>\alpha_n} \phi_n(t) dt \to 0 \text{ as } n \to \infty,$$
$$\phi_n(\alpha_n) + \phi_n(-\alpha_n) \to 0 \text{ as } n \to \infty.$$

Using (3.1) and the property 3) of ϕ_n, we may fix a sequence of integers $1 = N_1 < N_2 < \dots$ such that

$$\int_{|t|>1/k} \phi_n(t)dt < \frac{1}{k}, \quad n \geq N_k,$$
$$\phi_n(1/k) + \phi_n(-1/k) < \frac{1}{k}.$$

Then we define

$$\alpha_n = \frac{1}{k} \text{ if } N_k \leq n < N_{k+1}.$$

Now take m_n arbitrarily satisfying $m_n \geq n\phi_n(0)$, and define $c_n = \phi_n(0)/m_n$. Set

$$a_k^{(n)} = \inf\{-\alpha_n \leq x < 0 : \phi_n(x) \geq kc_n\},$$
$$b_k^{(n)} = \sup\{0 < x \leq \alpha_n : \phi_n(x) \geq kc_n\}$$

If $|x| > \alpha_n$, then we have $\psi_n(x) = 0$, $\phi_n(x) \leq \max\{\phi_n(\alpha_n), \phi(-\alpha_n)\}$ and therefore we get

$$|\phi_n(x) - \psi_n(x)| \leq \max\{\phi_n(\alpha_n), \phi(-\alpha_n)\}, \quad |x| > \alpha_n. \tag{3.2}$$

If $|x| \leq \alpha_n$, then we have $x \in \Delta_k^{(n)} \setminus \Delta_{k+1}^{(n)}$ for some $k = 0, 1, \dots$ where $\Delta_0^{(n)} = [-\alpha_n, \alpha_n]$. This implies

$$\psi_n(x) = kc_n, \quad kc_n \leq \phi_n(x) < (k+1)c_n$$

and therefore $|\phi_n(x) - \psi_n(x)| \leq c_n$. This together with (3.2) gives us the condition 2) of lemma. □

3.2. **Almost everywhere simple convergence of sequences of general operators.**

Theorem 3.2. *If μ is a bounded generalized measure on $\mathbb{R}$ (a function of bounded variation) and $\mu'(x_0)$ exists, then $\Phi_n(x, d\mu) \to \mu'(x_0)$ as $n \to \infty$.*

Proof. We may suppose $x_0 = 0$. Let $\psi_n(x)$ be the sequence obtained from lemma. We have

$$\int_{\mathbb{R}} \psi_n(t)dt = c_n \sum_{k=1}^{m_n} (b_k^{(n)} - a_k^{(n)}) \to 1 \text{ as } n \to \infty.$$

Then we have

$$|\Phi_n(0, d\mu) - \Psi_n(0, d\mu)| \leq \int_{\mathbb{R}} |\phi_n(t) - \psi_n(t)| d|\mu|(t) \leq \gamma_n \cdot \|\mu\| \to 0.$$

On the other hand

$$\Psi_n(0, d\mu) = c_n \sum_{k=1}^{m_n} (\mu(b_k^{(n)}) - \mu(a_k^{(n)})) = c_n \sum_{k=1}^{m_n} (b_k^{(n)} - a_k^{(n)}) \frac{\mu(b_k^{(n)}) - \mu(a_k^{(n)})}{b_k^{(n)} - a_k^{(n)}}$$

Since $|\Delta_1^{(n)}| \to 0$ we get

$$\delta_n = \sup_{1\le k\le m_n} \left| \frac{\mu(b_k^{(n)}) - \mu(a_k^{(n)})}{b_k^{(n)} - a_k^{(n)}} - \mu'(0) \right| \to 0.$$

Thus we get

$$\Psi_n(0, d\mu) = c_n \mu'(0) \sum_{k=1}^{m_n} (b_k^{(n)} - a_k^{(n)}) + o(1) \to \mu'(0).$$

□

3.3. **Almost everywhere λ_n-convergence of sequences of general operators.**

Theorem 3.3. *If μ is a bounded generalized measure on $\mathbb{R}$ (a function of bounded variation) and $\mu'(x_0)$ exists, then*

$$\sup_{|\theta|\le\lambda_n} |\Phi_n(x+\theta, d\mu) \to \mu'(x_0)| \to 0 \text{ as } n \to \infty,$$

where $\lambda_n = c/\phi_n(0)$.

Proof. It is enough to proof that for any sequence θ_n with $|\theta_n| \le \lambda_n$ we have

$$\lim_{n\to\infty} \Phi_n(x_0 + \theta_n, d\mu) = \mu'(x_0).$$

We may suppose $x_0 = 0$ and $\theta_n \ge 0$. So our claim is

$$\int_{\mathbb{R}} \phi_n(\theta_n + t) d\mu(t) \to \mu'(0).$$

Introduce the kernels

$$u_n(x) = \begin{cases} \phi_n(x), & \text{if} \quad x \notin [-\theta_n, 0], \\ \phi_n(0), & \text{if} \quad x \in [-\theta_n, 0], \end{cases}$$

$$v_n(x) = \begin{cases} 0, & \text{if} \quad x \notin [-\theta_n, 0], \\ \phi_n(0) - \phi_n(x), & \text{if} \quad x \in [-\theta_n, 0], \end{cases}$$

It is clear $\phi_n(\theta_n + x) = u_n(x) - v_n(x)$. Observe that

$$\|v_n\|_1 \le M, \quad \|\phi_n\|_1 \le \|u_n\|_1 \le M, \quad \|u_n\|_1 - \|v_n\|_1 \to 1.$$

Thus we get that the sequences

$$U_n(x) = \frac{u_n(x)}{\|u_n\|}, \quad V_n(x) = \frac{v_n(x)}{\|v_n\|},$$

form regular AI. Indeed take an arbitrary $\delta > 0$. We will have $\theta_n \le \delta$ for $n \ge N$. Hence for such n we obtain

$$\int_{|t|>\delta} U_n(t) dt = \frac{1}{\|u_n\|_1} \int_{|t|>\delta} \phi_n(t) dt \le \frac{1}{\|\phi_n\|_1} \int_{|t|>\delta} \phi_n(t) dt \to 0 \text{ as } n \to \infty.$$

Thus, according to the previous theorem, we get

$$\begin{aligned}\int_{\mathbb{R}} \phi_n(\theta_n+t)d\mu(t) &= \|u_n\|_1 \int_{\mathbb{R}} U_n(t)d\mu(t) - \|v_n\|_1 \int_{\mathbb{R}} V_n(t)d\mu(t) \\ &= \|u_n\|_1(\mu'(0)+o(1)) - \|v_n\|_1(\mu'(0)+o(1)) \\ &= (\|u_n\|_1 - \|v_n\|_1)\mu'(0) + o(1) \to \mu'(0).\end{aligned}$$

□

4. Hilbert transform

4.1. **Definition of Hilbert transform and Privalov-Zygmund theorem.** Given $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, we denote

$$H_\varepsilon f(x) = \int_{|t|>\varepsilon} \frac{f(x-t)}{t}dt = \int_\varepsilon^\infty \frac{f(x-t)-f(x+t)}{t}dt. \tag{4.1}$$

This integral can be considered as a convolution of f with the kernel function

$$h_\varepsilon(x) = \frac{\mathbb{I}_{\{|t|>\varepsilon\}}(x)}{x}.$$

Observe that $h_\varepsilon \in L^q(\mathbb{R})$ for any $1 < q \le \infty$. Indeed, $\|h_\varepsilon\|_\infty = \varepsilon^{-1} < \infty$, and if $1 < q < \infty$, then

$$\|h_\varepsilon\|_q^q = 2\int_\varepsilon^\infty \frac{dt}{t^q} = 2\varepsilon^{-q+1} < \infty$$

Thus, by Hölder's inequality the integral (4.1) is well defined at any point $x \in \mathbb{R}$. We will study different convergence properties of $H_\varepsilon f(x)$ as $\varepsilon \to 0$. The limit function $H_\varepsilon f(x)$ will be denoted by $Hf(x)$, which is said to be the Hilbert transform of f. Denote by $\Lambda_\alpha(\mathbb{R})$ the Lipschitz class of function, that are the functions satisfying $|f(x)-f(y)| \le C|x-y|^\alpha$ with constant $C > 0$.

Theorem 4.1. *If $f \in \Lambda_\alpha(\mathbb{R}) \cap C_K(\mathbb{R})$, $0 < \alpha < 1$, then*
1) $H_\varepsilon f(x)$ uniformly converges as $\varepsilon \to 0$,
2) $\|H_\varepsilon(f) - H(f)\|_p \to 0$ as $\varepsilon \to 0$, $1 < p < \infty$,
3) $H(f) \in \Lambda_\alpha(\mathbb{R})$.

Proof. 1) For $0 < \varepsilon_1 < \varepsilon_2$ we have

$$\begin{aligned}|H_{\varepsilon_2} f(x) - H_{\varepsilon_1} f(x)| &\le \int_{\varepsilon_2>|t|>\varepsilon_1} \frac{|f(x-t)-f(x)|}{|t|}dt \\ &\le 2\int_{\varepsilon_1}^{\varepsilon_2} \frac{t^\alpha}{t}dt = \frac{2}{\alpha}((\varepsilon_2)^\alpha - (\varepsilon_1)^\alpha) \to 0.\end{aligned} \tag{4.2}$$

as $\varepsilon_2 \to 0$. Thus $H_\varepsilon f(x)$ uniformly converges and so $Hf(x)$ is defined at any point $x \in \mathbb{R}$..

2) It is enough to show that $H_\varepsilon f \in L^p$ and the satisfactory of Cauchy principle. We can suppose,

$$\operatorname{supp} f \subset (-A, A). \tag{4.3}$$

For $0<\varepsilon<1$ we have

$$H_\varepsilon f(x)=\int_{|t|>\varepsilon}\frac{f(x-t)}{t}dt=\int_{|t|\ge 1}\frac{f(x-t)}{t}dt+\int_{1>|t|>\varepsilon}\frac{f(x-t)-f(x)}{t}dt.$$

The first integral is the convolution of functions $f\in C_K(\mathbb{R})\subset L^1$ and $h_1(x)\in L^p$. So by Theorem 2.2 it belongs to $L^p(\mathbb{R})$. Observe that the second integral as a function on x is supported in the interval $(-(A+1), A+1)$, since for $|t|<1$ and $|x|\ge A+1$ according to (4.3) we have $f(x-t)=f(x)=0$. Clearly, it is also a continuous function. Thus the second integral is from $L^p(\mathbb{R})$ too. Hence we get $H_\varepsilon f\in L^p(\mathbb{R})$. On the other hand for $0<\varepsilon_1<\varepsilon_2<1$ the integral

$$H_{\varepsilon_2} f(x)-H_{\varepsilon_1} f(x)=\int_{\varepsilon_2>|t|>\varepsilon_1}\frac{f(x-t)-f(x)}{t}dt$$

is a function supported in $(-(A+1), A+1)$, so by (4.2) we get

$$\|H_{\varepsilon_2} f(x)-H_{\varepsilon_1} f\|_p\le (2A+2)\cdot\frac{2}{\alpha}((\varepsilon_2)^\alpha-(\varepsilon_1)^\alpha)\to 0.$$

3) Given $h>0$, using oddness of the kernels K_ε, observe that

$$\begin{aligned}Hf(x)&=\text{p.v.}\int_{\mathbb{R}}\frac{f(x-t)}{t}dt=\text{p.v.}\int_{\mathbb{R}}\frac{f(x-t)-f(x)}{t}dt\\&=\int_{|t|>2h}\frac{f(x-t)-f(x)}{t}dt+\text{p.v.}\int_{|t|\le 2h}\frac{f(x-t)-f(x)}{t}dt\\Hf(x+h)&=\int_{|t|>2h}\frac{f(x+h-t)-f(x)}{t}dt+\text{p.v.}\int_{|t|\le 2h}\frac{f(x+h-t)-f(x+h)}{t}dt\end{aligned}$$

For the second integrals in the representations of $Hf(x)$ and $Hf(x+h)$ we have

$$\begin{aligned}\left|\text{p.v.}\int_{|t|\le 2h}\frac{f(x-t)-f(x)}{t}dt\right|&\le 2\int_0^{2h}\frac{t^\alpha}{t}dt\le C\cdot h^\alpha,\\\left|\text{p.v.}\int_{|t|\le 2h}\frac{f(x+h-t)-f(x+h)}{t}dt\right|&\le 2\int_0^{2h}\frac{t^\alpha}{t}dt\le C\cdot h^\alpha.\end{aligned}$$

Thus we conclude

$$|Hf(x+h)-Hf(x)|\le\left|\int_{|t|>2h}\frac{f(x+h-t)}{t}dt-\int_{|t|>2h}\frac{f(x-t)}{t}dt\right|+O(h^\alpha)$$

On the other hand we have

$$\begin{aligned}
&\left|\int_{|t|>2h} \frac{f(x+h-t)}{t}dt - \int_{|t|>2h} \frac{f(x-t)}{t}dt\right| \\
&\quad = \left|\int_{|t|>2h} \frac{f(x+h-t)-f(x)}{t}dt - \int_{|t|>2h} \frac{f(x-t)-f(x)}{t}dt\right| \\
&\quad = \left|\int_{|t|>2h} \frac{f(x-t)-f(x)}{t+h}dt - \int_{|t|>2h} \frac{f(x-t)-f(x)}{t}dt\right| + O(h^\alpha) \\
&\quad \leq \left|\int_{|t|>2h} \frac{h|f(x-t)-f(x)|}{|t(t+h)|}dt\right| + O(h^\alpha) \\
&\quad \leq C\cdot h \int_{2h<|t|<\infty} \frac{t^\alpha}{t^2}dt \\
&\quad \leq C\cdot h\cdot h^{\alpha-1} = Ch^\alpha.
\end{aligned}$$

Thus we get $|Hf(x+h) - Hf(x)| \leq C\cdot h^\alpha$, that means $Hf(x) \in \Lambda_\alpha(\mathbb{R})$. □

4.2. L^2-bound of H_ε.

Theorem 4.2. *For any $f \in L^2(\mathbb{R})$ we have*

$$\|H_\varepsilon f\| \leq c\|f\|_2 \tag{4.4}$$

where c is an absolute constant.

Proof. Since $C_K(\mathbb{R})$ is a dense subset of $L^1(\mathbb{R})$, without loss of generality we can suppose that $f \in C_K(\mathbb{R})$ and so the integral $H_\varepsilon f(x)$ is defined for all $x \in \mathbb{R}$. Also we have

$$\begin{aligned}
\hat{h}_\varepsilon(x) &= \int_{\mathbb{R}} h_\varepsilon(t)e^{-ixt}dt = \int_{|t|>\varepsilon} \frac{e^{-ixt}}{t}dt \\
&= 2\int_\varepsilon^\infty \frac{\sin xt}{t}dt \\
&= 2\operatorname{sign} x \int_{\varepsilon|x|}^\infty \frac{\sin t}{t}dt
\end{aligned}$$

that implies $\|\hat{h}_\varepsilon\|_\infty < \infty$. On the other hand applying Theorem 2.3, we have

$$\|H_\varepsilon f\|_2 = \left\|\widehat{H_\varepsilon f}\right\|_2 = \|\hat{f}\hat{h}_\varepsilon\|_2 \leq \|h_\varepsilon\|_\infty \|f\|_2.$$

Thus (4.4) is proved. □

4.3. $f_\lambda^\pm$ functions.

Lemma 4.3. *Let functions $f, g \in L^\infty[a,b]$ satisfy the relation*

$$\int_a^x f(t)dt \geq \int_a^x g(t)dt, \quad a < x \leq b,$$

and it holds equality if $x = b$. Then for any increasing function $h(t)$ on $[a, b]$ we have the inequality

$$\int_a^b f(t)h(t)dt \le \int_a^b g(t)h(t)dt. \tag{4.5}$$

Proof. Denote

$$R(x) = \int_a^x r(t)dt, \text{ where } r(t) = f(t) - g(t).$$

By the conditions of lemma it follows that $R(x) \ge 0$ and $R(b) = R(a) = 0$. Besides we have $R'(x) = r(x)$ a.e.. Thus the integration by part implies

$$\begin{aligned} \int_a^b f(t)h(t)dt - \int_a^b g(t)h(t)dt &= \int_a^x r(t)h(t)dt \\ &= R(b)h(b) - R(a)h(a) - \int_a^b R(t)dh(t) \\ &= -\int_a^b R(t)dh(t). \end{aligned} \tag{4.6}$$

Since $R(t) \ge 0$ and $h(t)$ is increasing, the right hand side of (4.6) is non-negative and we get (4.5). ☐

Let $f \in L^1(\mathbb{R})$ be a positive. Applying Theorem 1.4, we have

$$G_\lambda^+ = \{M^+ f(x) > \lambda\} = \cup_k (a_k^+, b_k^+),$$
$$G_\lambda^- = \{M^- f(x) > \lambda\} = \cup_k (a_k^-, b_k^-),$$

where the intervals $(a_k^\pm, b_k^\pm)$ satisfy (1.6). Define two functions

$$f_\lambda^\pm(x) = \begin{cases} \lambda & \text{if} \quad x \in G_\lambda^\pm, \\ f(x) & \text{if} \quad x \in \mathbb{R} \setminus G_\lambda^\pm. \end{cases}$$

Lemma 4.4. *There hold the relations*

$$\int_{\mathbb{R}} f_\lambda^\pm(x)dx = \int_{\mathbb{R}} f(x)dx, \tag{4.7}$$

$$0 \le f_\lambda^\pm(x) \le \lambda \text{ a.e .} \tag{4.8}$$

Proof. Indeed, applying (1.6), we obtain

$$\begin{aligned}R\int_{\mathbb{R}} f &= \int_{\mathbb{R}\setminus G^\pm} f + \int_{G^\pm} f = \int_{\mathbb{R}\setminus G^\pm} f + \sum_k \int_{a_k^\pm}^{b_k^\pm} f \\ &= \int_{\mathbb{R}\setminus G^\pm} f + \sum_k \lambda(b_k^\pm - a_k^\pm) \\ &= \int_{\mathbb{R}\setminus G^\pm} f_\lambda^\pm + \sum_k \int_{a_k^\pm}^{b_k^\pm} f_\lambda^\pm \\ &= \int_{\mathbb{R}} f_\lambda^\pm(x)dx,\end{aligned}$$

and (4.7) follows. To show (4.8) take $x \in \mathbb{R}$. If $x \in G_\lambda^\pm$, then $f_\lambda^\pm(x) = \lambda$ and (4.8) is immediate. Take a point $x \in \mathbb{R} \setminus G_\lambda^\pm$ and suppose that x is Lebesgue point for f. According to the definition of the set $G^\pm$ we have

$$f(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f \le \lambda.$$

Since a.e. point satisfies Lebesgue property we get (4.8). □

4.4. **Some estimates of $\bar{H}_\varepsilon$ operator.** Define the modification of kernel h_ε by

$$\begin{aligned}\bar{h}_\varepsilon(t) &= \frac{\mathbb{I}_{\{|t|>\varepsilon\}}(t)}{x} + \operatorname{sign} x \cdot \frac{\mathbb{I}_{\{|t|\le\varepsilon\}}(t)}{\varepsilon} \\ &= h_\varepsilon(t) + \frac{\mathbb{I}_{\{0\le t\le\varepsilon\}}(t)}{\varepsilon} - \frac{\mathbb{I}_{\{-\varepsilon\le t<0\}}(t)}{\varepsilon}\end{aligned}$$

and corresponding convolution operator

$$\bar{H}_\varepsilon f(x) = H_\varepsilon f(x) + \frac{1}{\varepsilon}\int_x^{x+\varepsilon} f - \frac{1}{\varepsilon}\int_{x-\varepsilon}^{x} f.$$

This relation obviously implies

$$|\bar{H}_\varepsilon f(x) - H_\varepsilon f(x)| \le Mf(x), \tag{4.9}$$

$$\lim_{h\to 0} \bar{H}_\varepsilon f(x) = \lim_{h\to 0} H_\varepsilon f(x) \text{ a.e.}, \tag{4.10}$$

where $Mf(x)$ is the maximal function of f and (4.10) holds at any Lebesgue point. Relations (4.9) and (4.10) show that the operators H_ε and $\bar{H}_\varepsilon$ have common boundedness and convergence properties and it follows from estimates of maximal operator (see Theorem 1.2).

Lemma 4.5. *For any function $f \in L^1$, $f \ge 0$, we have*

$$\begin{aligned}&\bar{H}_\varepsilon f(x) \ge \bar{H}_\varepsilon f_\lambda^+(x), \quad x \in \mathbb{R}\setminus G_\lambda^+, \\ &\bar{H}_\varepsilon f(x) \le \bar{H}_\varepsilon f_\lambda^-(x), \quad x \in \mathbb{R}\setminus G_\lambda^-.\end{aligned} \tag{4.11}$$

Proof. We shall prove the first inequality. Since f and f_λ^+ coincide on $\mathbb{R}\setminus G_\lambda^+$, it is enough to prove

$$D=\int_a^b h_\varepsilon(x-t)f(t)dt-\int_a^b h_\varepsilon(x-t)f_\lambda^+(t)dt\ge 0.$$

where (a,b) is one of the intervals (a_k^+,b_k^+). Since $a\notin G_\lambda^+$, we have

$$\int_a^x f\le \lambda(x-a)=\int_a^x f^+ \text{ for any } x>a.$$

This together with (1.6), implies that the functions f and f^+ satisfy the condition of Lemma 4.3. On the other hand the function $h(t)=h_\varepsilon(x-t)$ is increasing on (a,b) as a function on t for a fixed $x\in\mathbb{R}\setminus G_\lambda^+$. Thus, from Lemma 4.3 we conclude

$$\bar{H}_\varepsilon f(x)=\int_a^b f(t)h(t)dt\ge\int_a^b f^+(t)h(t)dt=\bar{H}_\varepsilon f_\lambda^+(x)$$

To prove (4.11) we suppose now (a,b) is one of the intervals (a_k^-,b_k^-). Using the relation $b\notin G_\lambda^-$, get the reverse inequality

$$\begin{aligned}\int_a^x f=\int_a^b f-\int_x^b f\ge\int_a^b f-\lambda(b-x)\\ =\lambda(b-a)-\lambda(b-x)=\lambda(x-a)=\int_a^x f^-.\end{aligned}$$

Likewise, applying Lemma 4.3, we will get (4.11). □

4.5. **Weak-L^1 and strong-L^p estimates of H_ε.**

Theorem 4.6. *For any $\varepsilon>0$ the operator H_ε satisfies weak-L^1 bound. Namely,*

$$|\{x\in\mathbb{R}:\,|H_\varepsilon f(x)|>\lambda\}|\le c\cdot\frac{\|f\|_1}{\lambda},\quad\lambda>0,\tag{4.12}$$

where $c>0$ is an absolute constant.

Proof. According to (4.9), we can consider $\bar{H}_\varepsilon$ instead of operator H_ε. From Lemma 4.3 it follows that

$$\bar{H}_\varepsilon f(x)\le\bar{H}_\varepsilon f_\lambda^+(x) \text{ whenever } x\in\mathbb{R}\setminus G_\lambda^+.$$

this implies

$$|\{x\in\mathbb{R}:\,\bar{H}_\varepsilon f(x)>\lambda\}|\le|G_\lambda^+|+|\{x\in\mathbb{R}\setminus G_\lambda^+:\,\bar{H}_\varepsilon f_\lambda^+(x)>\lambda\}|.$$

On the other hand using L^2 boundedness of H_ε (see (4.4)) and so $\bar{H}_\varepsilon$ along with (4.7) and (4.8), we get

$$\begin{aligned}|\{x\in\mathbb{R}\setminus G_\lambda^+:\,\bar{H}_\varepsilon f_\lambda^+(x)>\lambda\}|&\le|\{x\in\mathbb{R}:\,|\bar{H}_\varepsilon f_\lambda^+(x)|>\lambda\}|\\ &\le\frac{\|\bar{H}_\varepsilon(f_\lambda^+)\|_2^2}{\lambda^2}\lesssim\frac{\|f_\lambda^+\|_2^2}{\lambda^2}\\ &=\frac{1}{\lambda^2}\int_{\mathbb{R}}|f_\lambda^+|^2\le\frac{1}{\lambda}\int_{\mathbb{R}}|f_\lambda^+|=\frac{\|f\|_1}{\lambda}.\end{aligned}$$

Thus we get

$$|\{x \in \mathbb{R} : \bar{H}_\varepsilon f(x) > \lambda\}| \lesssim \frac{\|f\|_1}{\lambda}.$$

Similarly we can estimate $|\{x \in \mathbb{R} : \bar{H}_\varepsilon f(x) < -\lambda\}|$ that completes the proof of theorems. □

Theorem 4.7. *For any $1 < p < \infty$ and $f \in L^p(\mathbb{R})$ we have*

$$\|H_\varepsilon(f)\|_p \le c_p \|f\|_p. \tag{4.13}$$

Proof. Since we already know that H_ε satisfies weak-L^1 and strong L^2-inequalities with constants independent of ε, from Marzinkiewicz interpolation theorem it follows (4.13) in the case $1 < p \le 2$. Now suppose $p > 2$. Chose functions $f, g \in C_K(\mathbb{R})$. Observe that the function $h_\varepsilon(x-t)g(x)f(t)$ of two variables is integrable on $\mathbb{R}$ and so we can write

$$\begin{aligned}
\left|\int_{\mathbb{R}} H_\varepsilon f(x) \cdot g(x)dx\right| &= \left|\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h_\varepsilon(x-t)f(t)dt\right) g(x)dx\right| \\
&= \left|\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h_\varepsilon(x-t)g(x)dx\right) f(t)dt\right| \\
&\le \|H_\varepsilon g\|_q \cdot \|f\|_p \\
&\lesssim \|g\|_q \cdot \|f\|_p .
\end{aligned}$$

This implies (4.13) for the functions from $C_K(\mathbb{R})$. Since $C_K(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, we obtain (4.13) for arbitrary $f \in L^p(\mathbb{R})$, □

Corollary 4.8. *1) If $f \in L^1$, then $H_\varepsilon f$ converges to Hf in measure,*
2) If $f \in L^p$, $1 < p < \infty$, then $\|H_\varepsilon f - Hf\|_p \to 0$.

Proof. 1) Any $f \in L^1$ can be written in the form $f \in g + r$, where $g \in \Lambda_{1/2}(\mathbb{R}) \cap C_K(\mathbb{R})$ and $\|r\|_1 < \delta$. The applying (4.12), for $\lambda > 0$ we obtain

$$\begin{aligned}
&\lim_{\varepsilon,\varepsilon'\to 0} |\{|H_\varepsilon f(x) - H_{\varepsilon'} f(x)| > \lambda\}| \\
&\qquad = \lim_{\varepsilon\to 0} |\{|H_\varepsilon r(x) - H_{\varepsilon'} r(x)| > \lambda\}| \lesssim \frac{\|r\|_1}{\lambda} < \frac{\delta}{\lambda}.
\end{aligned} \tag{4.14}$$

Since δ can be arbitrary the left side of (4.14) is zero. This implies the convergence in measure of $H_\varepsilon f(x)$ and the limit as we know must be denoted by $Hf(x)$.

2) This part of the theorem can be proved by the same approximation argument. □

Taking limit in (4.12) and (4.13) as $\varepsilon \to 0$, one can deduce the following

Corollary 4.9. *The operator H satisfies weak-L^1 and strong-L^p inequalities if $1 < p < \infty$.*

5. Operator H^*

5.1. **Oscillation lemma for H_ε.** Consider the operator

$$H^*f(x)=\sup_{\varepsilon>0}|H_\varepsilon f(x)|.$$

Lemma 5.1. *If $x\cdot x'>0$, then*

$$|\bar h_\varepsilon(x)-\bar h_\varepsilon(x')|\le\frac{|x-x'|}{|x|\cdot|x'|}. \tag{5.1}$$

Proof. We can consider only the case $x,x'>0$. If $x,x'>\varepsilon$, then $\bar h_\varepsilon(x)=1/x$ and $\bar h_\varepsilon(x')=1/x'$ and we will obviously have equality in (5.1). If $0<x,x'\le\varepsilon$, then $\bar h_\varepsilon(x)=\bar h_\varepsilon(x')=1/\varepsilon$ and (5.1) trivially follows. Now consider the last case $0<x\le\varepsilon<x'$. So we have

$$|\bar h_\varepsilon(x)-\bar h_\varepsilon(x')|=\left|\frac{1}{\varepsilon}-\frac{1}{x'}\right|=\frac{x'-\varepsilon}{\varepsilon\cdot x'}\le\frac{x'-x}{x\cdot x'}$$

and (5.1) follows. □

Lemma 5.2. *Let $I\subset\mathbb{R}$ be an interval. Then for any function $f\in L^p(\mathbb{R})$, satisfying* $\operatorname{supp} f\subset\mathbb{R}\setminus 3I$, *it holds the inequality*

$$|H_\varepsilon f(x)-H_\varepsilon f(x')|\lesssim M_{I,p}(f),\quad x,x'\in I, \tag{5.2}$$

where

$$M_{I,p}(f)=\sup_{J\supset I}\left(\frac{1}{|J|}\int_J|f|^p\right)^{1/p}.$$

Proof. From (4.9) it follows that

$$\begin{aligned}|H_\varepsilon f(x)-H_\varepsilon f(x')|&\le|\bar H_\varepsilon f(x)-\bar H_\varepsilon f(x')|+Mf(x)+Mf(x')\\&\le|\bar H_\varepsilon f(x)-\bar H_\varepsilon f(x')|+Mf(x)+Mf(x')\\&\le|\bar H_\varepsilon f(x)-\bar H_\varepsilon f(x')|+M_pf(x)+M_pf(x')\\&\le|\bar H_\varepsilon f(x)-\bar H_\varepsilon f(x')|+2M_{I,p}f(x)\end{aligned}$$

so it is enough to prove (5.2) for the operator $\bar H_\varepsilon$. If $I=(c-\delta,c+\delta)$, $x,x'\in I$ and $t\in\mathbb{R}\setminus(3I)$, then on can check

$$|t-x|\ge|t-c|/2,\quad |t-x'|\ge|t-c|/2.$$

Then, applying Lemma 5.1, we obtain

$$|h_\varepsilon(x-t)-h_\varepsilon(x'-t)|=\frac{|x-x'|}{|t-x|\cdot|t-x'|}\le\frac{4|I|}{|t-c|^2}.$$

Thus we conclude

$$
\begin{aligned}
(5.3) \qquad |H_\varepsilon f(x) - H_\varepsilon f(x')| &\le \int_{\mathbb{R}\setminus(3I)} |f(t)||h_\varepsilon(x-t) - h_\varepsilon(x'-t)|dt \\
&\le |I| \int_{\mathbb{R}\setminus(3I)} \frac{|f(t)|}{|t-c|^2} dt \\
&\le 3\delta \int_{|t|>3\delta} \frac{|f(c-t)|}{|t|^2} dt \\
&\le \int_{\mathbb{R}} |f(c-t)|\phi(t)dt,
\end{aligned}
$$

where

$$\phi(t) = 3\delta \min\{\delta^{-2}, t^{-2}\}.$$

Notice that

$$\|\phi\|_1 = 18\delta^2 \cdot \delta^{-2} + 6\delta \int_{t>3\delta} \frac{dt}{t^2} = 20.$$

Thus, applying Lemma 2.6, from (5.3) we obtain

$$|H_\varepsilon f(x) - H_\varepsilon f(x')| \le 20Mf(c) \le 20M_p f(c) \le 20M_{I,p}(f)$$

completing the proof of lemma. □

5.2. **Weak-L^1 and strong-L^p estimate of H^*.**

Theorem 5.3. *The operator H^* satisfies weak-L^1 and strong L^p inequalities for $1 < p < \infty$.*

Proof. First we shall prove that H^* satisfies weak-L^p inequality for any $1 \le p < \infty$. Suppose $f \in L^p(\mathbb{R})$. Given $\lambda > 0$ and $A > 0$ consider the set

$$E = E_{\lambda,A} = \{x \in (-A, A) : \; H^* f(x) > \lambda\}.$$

For any $x \in E$ there is $\varepsilon(x) > 0$ such that

$$(5.4) \qquad |H_{\varepsilon(x)}(f)(x)| > \lambda.$$

Denote $I(x) = (x - \varepsilon(x), x + \varepsilon(x))$ and $J(x) = \frac{1}{5} I(x)$. We have $E \subset \cup_{x \in E} J(x)$. Applying covering Lemma 1.1, we find a sequence $x_k \in E$ such that the balls $\{J_k = J(x_k)\}$ are pairwise disjoint and

$$E \subset \bigcup_k I_k, \text{ where } I_k = I(x_k) = 5J_k.$$

According to Lemma 5.2, we have

$$|H(f \cdot \mathbb{I}_{\mathbb{R}\setminus I_k})(x_k) - H(f \cdot \mathbb{I}_{\mathbb{R}\setminus I_k})(x)| \le C \cdot M_{J_k,p}(f), \quad x \in J_k.$$

Thus, one can easily conclude from (5.4) that

$$
\begin{aligned}
(5.5)\qquad |H(f\cdot \mathbb{I}_{\mathbb{R}\setminus I_k})(x)| &\ge |H(f\cdot \mathbb{I}_{\mathbb{R}\setminus I_k})(x_k)|\\
&\qquad - |H(f\cdot \mathbb{I}_{\mathbb{R}\setminus I_k})(x_k) - H(f\cdot \mathbb{I}_{\mathbb{R}\setminus I_k})(x)|\\
&\ge \lambda - C\cdot M_{J_k,p}(f),\quad x\in J_k.
\end{aligned}
$$

For the constant

$$\beta = 10^{1/p}\|H\|_{L^p\to L^{p,\infty}}$$

we define

$$(5.6)\qquad \tilde{J}_k = \{x\in J_k :\, |H(f\cdot \mathbb{I}_{I_k})(x)| \le \beta\cdot M_{J_k,p}(f)\}.$$

Using the weak-L^p inequality for the operator H, we can write

$$
\begin{aligned}
|J_k\setminus \tilde{J}_k| &= |\{x\in J_k :\, |H(f\cdot \mathbb{I}_{I_k})(x)| > \beta\cdot M_{J_k,p}(f)\}|\\
&\le \frac{\|H\|^p_{L^r\to L^{p,\infty}}}{\beta^p\cdot (M_{J_k,p}(f))^p\}}\|f\cdot \mathbb{I}_{I_k}\|_p^p\\
&\le \frac{\|H\|^p_{L^r\to L^{p,\infty}}}{\beta^p\cdot \frac{1}{|I_k|}\int_{I_k}|f|^p}\int_{I_k}|f|^p\\
&= \frac{|I_k|}{10} = \frac{|J_k|}{2},
\end{aligned}
$$

and so we have

$$(5.7)\qquad |\tilde{J}_k| \ge |J_k| - |J_k\setminus \tilde{J}_k| \ge \frac{1}{2}|J_k| = \frac{1}{10}|I_k|.$$

Consider the constant

$$\delta = \frac{1}{2(C+\beta)}.$$

If

$$x\in \tilde{J}_k\setminus \{M_pf(x) > \delta\lambda\},$$

then, using subadditivity of H together with relations (5.6), (5.5), we obtain

$$
\begin{aligned}
|Hf(x)| &\ge |H(f\cdot \mathbb{I}_{\mathbb{R}\setminus I_k})(x)| - |H(f\cdot \mathbb{I}_{I_k})(x)|\\
&\ge \lambda - C\cdot M_{J_k,p}(f) - \beta\cdot M_{J_k,p}(f)\\
&\ge \lambda - (C+\beta)\cdot M_pf(x)\\
&\ge \lambda - (C+\beta)\delta\lambda\\
&= \lambda/2.
\end{aligned}
$$

Hence we conclude

$$\bigcup_k \tilde{J}_k \subset \{M_pf(x) > \delta\lambda\}\bigcup\{|Hf(x)| > \lambda/2\}.$$

Combining this with (5.7) and the weak-L^p boundedness of operators M_p and H, we obtain

$$\begin{aligned}|E_{\lambda,A}| &\leq \sum_k |I_k| \leq 10 \sum_k |\tilde{J}_k| \\ &\leq 10(|\{Mf(x) > \delta\lambda\}| + |\{|Hf(x)| > \lambda/2\}|) \\ &\lesssim \frac{\|f\|_p^p}{\lambda^p}.\end{aligned}$$

Since the estimate does not depend on A, we obtain

$$\{x \in \mathbb{R} : \ H^* f(x) > \lambda\} \lesssim \frac{\|f\|_p^p}{\lambda^p}$$

that is the weak-L^p inequality of H^*. To show the strong-L^p estimate for $1 < p < \infty$ set $p_1 = (p+1)/ < p < p_2 = p+1$. We have already proved that H^* satisfies weak L^{p_1} and weak-L^{p_2} inequalities. By Marcinkiewicz interpolation theorem so we obtain strong-L^p inequality H^*. Theorem is proved. □

Corollary 5.4. *If $f \in L^p(\mathbb{R})$, then $H_\varepsilon f(x)$ converges to $Hf(x)$ almost everywhere.*

Proof. Fix a $\lambda > 0$. For any $\delta > 0$ we can find a function $g \in \Lambda_{1/2}(\mathbb{R}) \cap C_K(\mathbb{R})$ such that $\|r\|_1 < \delta$, where $r = f - g$. According to Theorem 4.1 $H_\varepsilon g(x) \to Hg(x)$ a.e.. Denote

$$E_\lambda = \{x \in \mathbb{R} : \ \limsup_{\varepsilon \to 0} |H_\varepsilon f(x) - Hf(x)| > \lambda\}.$$

Thus, applying weak-L^p bound of operator H^*, we obtain

$$\begin{aligned}|E_\lambda| &= |\{x \in \mathbb{R} : \ \limsup_{\varepsilon \to 0} |H_\varepsilon r(x) - Hr(x)| > \lambda\}| \\ &\leq |\{x \in \mathbb{R} : \ 2H^* r(x) > \lambda\}| \\ &\lesssim \frac{\|r\|_1}{\lambda} \leq \frac{\delta}{\lambda}.\end{aligned}$$

Since $\delta > 0$ can be arbitrary small we obtain $|E_\lambda| = 0$ for any $\lambda > 0$ that means

$$\lim_{\varepsilon \to 0} |H_\varepsilon f(x) - Hf(x)| = 0 \text{ a.e. } .$$

This completes the proof. □