ON THE DIVERGENCE OF WALSH AND HAAR SERIES BY SECTORIAL AND TRIANGULAR REGIONS

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Abstract. We study almost everywhere divergence problems of the triangular and sectorial partial sums of the double Fourier series in Walsh and Haar orthonormal systems. In particular, we construct an example of bounded function on the unit square, which double Walsh-Fourier series diverges almost everywhere by an increasing sequence of triangular regions.

1. Introduction

Almost everywhere (a.e.) convergence and divergence problems of Fourier series in different classical orthonormal systems is one of the basic fields in Harmonic analysis. Carleson proved in [4] that the partial sums of the trigonometric Fourier series of a function $f \in L^2(0, 2\pi)$ converge a.e.. This fundamental theorem became a basis in the further study of a.e. convergence properties of the trigonometric and Walsh series. Hunt, Sjölin and Antonov [1] established a.e. convergence of Fourier series of functions from wider classes than L^2. For the Walsh system analogous problems were studied in [3], [22], [2], [23]. Convergence a.e. of the cubical partial sums of the trigonometric and Walsh Fourier series were investigated in [5], [21], [24]. In particular, Sjölin [21] proved that such partial sums of trigonometric Fourier series of a function from $L^p(0, 2\pi)$, $p > 1$, converge a.e.. In the case of Walsh system the analogous is known only for the functions from L^2 (Tevzadze [24]). The problem of a.e. convergence of cubical partial sums of the Fourier-Walsh series of a function $f \in L^p$ with $p > 2$ is still open. Ch. Fefferman [6] constructed a continuous function, which double trigonometric Fourier series diverges everywhere by cubes. An analogous example for Walsh system is constructed by Getzadze [8]. The a.e. convergence of Cezaro means of the isosceles triangular sums of double Fourier-Walsh series considered in the papers [25], [9], [7], [20].

In the present paper we consider a.e. divergence problems for the sectorial and arbitrary triangular partial sums of the double Fourier series in Haar and Walsh systems. Let $\phi = \{\phi_n(x),\, n \in \mathbb{N}\}$ be an orthonormal system. For a given region $G \subset \mathbb{N}^2$ denote by

$$S_G(x, y, \phi, f) = \sum_{(n,m)\in G} a_{nm}\phi_n(x)\phi_m(y), \quad a_{nm} = \int_0^1 f(t, s)\phi_n(t)\phi_m(s)dtds,$$

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the partial sum of double Fourier series of a function $f \in L^1(\mathbb{R}^2)$ corresponding to the region G. We shall consider the sectorial and triangular regions

$$V(\alpha,\beta)=\left\{(n,m):\, n,m\in\bar{\mathbb{N}},\quad \frac{m}{n}\in(\tan\alpha,\tan\beta)\right\},\quad 0\le\alpha<\beta\le\frac{\pi}{2},$$
$$\Delta(u,v)=\left\{n,m\in\bar{\mathbb{N}}:\frac{n}{v}+\frac{m}{u}\le 1\right\},\quad u,v>0,$$

where $\bar{\mathbb{N}}=\mathbb{N}$ in the case of Haar system and $\bar{\mathbb{N}}=\mathbb{N}\cup\{0\}$, while Walsh system is considered. We say an increasing sequence of regions G_k is complete, if $\cup_k G_k=\bar{\mathbb{N}}^2$. We denote by $\mathbb{I}_F(x,y)$ the indicator function of a set $F\subset(0,1)^2$. Haar and Walsh systems will be defined below and denoted correspondingly by $\chi=\{\chi_n(x):\, n=1,2,\ldots\}$ and $w=\{w_n(x):\, n=0,1,\ldots\}$. The following theorem shows, that the double Fourier-Haar series of a bounded function can diverge almost everywhere. Moreover, we prove

Theorem 1. *If V_k is a complete increasing sequence of sectors, then there exists a measurable set $F\subset(0,1)^2$ such that*

$$\limsup_{k\to\infty}|S_{V_k}(x,y,\chi,\mathbb{I}_F)|=\infty \text{ a.e. on } (0,1)^2.$$

In the next two theorems we establish analogous theorems for the double series in Walsh system for the sectorial and triangular partial sums.

Theorem 2. *For an arbitrary sequence of sectors V_k there exists a set $F\subset(0,1)^2$ such that*

$$\limsup_{k\to\infty}|S_{V_k}(x,y,w,\mathbb{I}_F)|=\infty \text{ a.e. on } (0,1)^2.$$

Theorem 3. *There exists a function $f\in L^\infty(0,1)^2$ and an increasing sequence of triangular regions Δ_k such that*

$$\limsup_{k\to\infty}|S_{\Delta_k}(x,y,w,f)|=\infty \text{ a.e. on } (0,1)^2.$$

In the proofs of these theorems we use a technique of divergent rearrangements of Haar series. We say, that a functional series $\sum_{n=1}^\infty f_n(x)$ unconditionally converges a.e. on E, if a.e. convergence holds on E for any rearrangements of the terms of series. Nikishin and Ulyanov [15] (see also [14] p. 104) established, that for the Haar series a.e. unconditionally convergence is equivalent to a.e. absolute convergence. Ulyanov in [17] constructed an example of function from $L^2[0,1]$, which Fourier-Haar series diverges a.e. after a suitable rearrangement of the terms. Olevskii in [18], [19] extended this result for arbitrary complete orthonormal systems, additionally guaranteeing continuity of the constructed function. In [16] it is proved

Theorem A. *There exists a measurable set $E\subset[0,1]$ such that*

$$\sum_{k=0}^\infty|a_k(\mathbb{I}_E)\chi_k(x)|=\infty \text{ a.e.}$$

on $[0,1]$.

In the proofs of the theorems we use also different Haar type systems, which spectrums are in some sectorial or triangular regions. For these constructions we apply technique, which previously was used in [12], [13].

2. Definitions of Haar and Walsh systems

Dyadic intervals are the intervals of the form

$$\Delta_n = \Delta_k^i = \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right),$$

where $n = 2^k + i$, $1 \le i \le 2^k$, $k = 0, 1, 2 \ldots$. The first Haar function is defined by $\chi_1(x) \equiv 1$. For $n \ge 2$ we define

$$\chi_n(x) = \begin{cases} 2^{k/2} & \text{if} \quad x \in (\Delta_n)^-, \\ -2^{k/2} & \text{if} \quad x \in (\Delta_n)^+, \\ 0 & \text{if} \quad x \notin \bar{\Delta}_n, \end{cases}$$

where $(\Delta_n)^-$ and $(\Delta_n)^+$ are left and right halves of Δ_n. We do not need to define the Haar functions at the points of discontinuity, since the present paper studies only a.e. behavior of the Haar series. We shall use also the Haar system normalized in L^∞. We denote these functions by

$$\tilde{\chi}_n(x) = 2^{-k/2}\chi_n(x), \quad n = 1, 2, \ldots.$$

Recall also the definitions of the Rademacher and Walsh systems (see [10] or [20]). Consider a function

$$r_0(x) = \begin{cases} 1, & \text{if} \quad x \in [0, 1/2), \\ -1, & \text{if} \quad x \in [1/2, 1), \end{cases}$$

periodically continued over the real line. The Rademacher functions are defined by $r_k(x) = r_0(2^k x)$, $k = 0, 1, 2, \ldots$. The Walsh functions are defined by the products of Rademacher functions. We set $w_0(x) \equiv 1$. To define $w_n(x)$ as $n \ge 1$ we write n in the dyadic form

$$n = \sum_{j=0}^{k} \varepsilon_j 2^j,$$

where $\varepsilon_k = 1$ and $\varepsilon_j = 0$ or 1 if $j = 0, 1, \ldots, k-1$, and denote

$$w_n(x) = \prod_{j=0}^{k} (r_j(x))^{\varepsilon_j}.$$

The dyadic addition for the numbers $x, y > 0$ with the dyadic decompositions

$$x = \sum_{k=-\infty}^{\infty} \theta_k(x) 2^{-k}, \quad y = \sum_{k=-\infty}^{\infty} \theta_k(y) 2^{-k},$$

is defined by

$$x \oplus y = \sum_{k=-\infty}^{\infty} |\theta_k(x) - \theta_k(y)| 2^{-k}.$$

3. Auxiliary lemmas

Recall the definition of the Haar type system on $(0,1)^2$ by [14]. Given a family of measurable sets

$$E_n = E_k^i \subset (0,1)^2, \quad i = 1,2,\dots,2^k, \quad k = 0,1,\dots,$$

where $n = 2^k + i$, $1 \le i \le 2^k$, $k = 0,1,2,\dots$, and

$$\begin{aligned} &|E_k^i| = 2^{-k}, \\ &E_k^i = E_{k+1}^{2i-1} \cup E_{k+1}^{2i}, \\ &E_k^i \cap E_k^j = \varnothing, \text{ if } i \ne j. \end{aligned} \tag{1}$$

Denote

$$\xi_1(x,y) = 1,$$
$$\xi_n(x,y) = \begin{cases} 2^{k/2}, & \text{if} \quad (x,y) \in E_{k+1}^{2i-1}, \\ -2^{k/2}, & \text{if} \quad (x,y) \in E_{k+1}^{2i}, \\ 0, & \text{if} \quad (x,y) \notin E_k^i. \end{cases}$$

The system $\{\xi_n(x,y)\}_{n=1}^\infty$ is said to be Haar type system. If

$$n = 2^k + j, \quad 1 \le j \le 2^k, \tag{2}$$

then we denote

$$\bar{n} = 2^{k-1} + \left[\frac{j+1}{2}\right], \tag{3}$$

where $[\cdot]$ denotes the integer part of a number. It is easy to observe that the number $\bar{n}$ may be equivalently defined by the relations

$$E_n \subset E_{\bar{n}}, \quad |E_n| = |E_{\bar{n}}|/2,$$

where E_n are the sets, defined in (1). This remark immediately implies

Lemma 1. *For the functions $\xi_n(x,y)$, $n = 1,2,\dots$, defined on $(0,1)^2$, to form Haar type system, it is necessary and sufficient to satisfy the conditions*

$$\begin{aligned} &|\operatorname{supp} \xi_n| = 2^{-k}, \\ &|\{\xi_n(x,y) = 2^{k/2}\}| = |\{\xi_n(x,y) = -2^{k/2}\}| = 2^{-k-1}, \\ &\operatorname{supp} \xi_n \subset \{(-1)^{j+1} \cdot \xi_{\bar{n}} > 0\}, \end{aligned}$$

where k and j are defined in (2).

Lemma 2. *If $0 < \alpha < \beta/8$, $\beta < \pi/4$, then for an arbitrary number $M > 0$ there exist natural numbers $l, m > M$ such that*

$$[2^l, 2^{l+1}] \times [2^m, 2^{m+1}] \subset V(\alpha,\beta). \tag{4}$$

Proof. It is clear, that for any $l \in \mathbb{N}$ there exists a number $m \in \mathbb{N}$ such that

$$2^{m-1} \le \tan\alpha \cdot 2^{l+1} < 2^m. \tag{5}$$

It is easy to observe that

$$2\tan(x/2) < \tan x = \frac{2\tan(x/2)}{1-\tan^2(x/2)}, \quad 0 < x < \pi/2,$$

which implies also $8\tan(x/8) < \tan x$. From this inequality and (5), using the hypothesis of the lemma, we obtain

$$2^{m+1} \le 4\tan\alpha \cdot 2^{l+1} < \tan\beta \cdot 2^l. \tag{6}$$

The inequalities (5) and (6) imply

$$\tan\alpha < \frac{2^m}{2^{l+1}}, \quad \frac{2^{m+1}}{2^l} < \tan\beta.$$

Thus we conclude, each vertex of the rectangle (4) is in the sector $V(\alpha,\beta)$, which implies (4). ☐

Lemma 3. *If $U_k = V(\alpha_{k+1},\alpha_k)$ is an arbitrary sequence of sectors with $0 < \alpha_{k+1} < \alpha_k/8$, then there exists a Haar type system $\xi_n(x,y)$, $n = 1,2,\dots,$ such that*

$$\xi_k(x,y) = \sum_{(p,q)\in D_k} b_{ij}\chi_p(x)\chi_q(y), \quad D_k \subset U_k, \quad k = 2,3,\dots. \tag{7}$$

Proof. We construct $\xi_n(x,y)$ by induction. Applying Lemma 2, we find natural numbers l_2 and m_2, satisfying

$$[2^{l_2}, 2^{l_2+1}] \times [2^{m_2}, 2^{m_2+1}] \subset U_2. \tag{8}$$

We set

$$\xi_2(x,y) = \sum_{i=2^{l_2}+1}^{2^{l_2+1}} \sum_{j=2^{m_2}+1}^{2^{m_2+1}} \tilde{\chi}_i(x)\tilde{\chi}_j(y).$$

Obviously we have (7) if $k = 2$. Then we suppose, that we have already constructed the functions $\xi_k(x,y)$, $k = 1,2,\dots,n-1$, satisfying (7). Since each of these functions is a Haar polynomial, they are constant on the dyadic rectangles

$$\left(\frac{i-1}{2^{l_n}}, \frac{i}{2^{l_n}}\right) \times \left(\frac{j-1}{2^{m_n}}, \frac{j}{2^{m_n}}\right), \quad i = 1,2,\dots,2^{l_n}, \quad j = 1,2,\dots,2^{m_n},$$

if we take $l_n, m_n \in \mathbb{N}$ to be sufficiently large. Using Lemma 2, we may additionally provide

$$[2^{l_n}, 2^{l_n+1}] \times [2^{m_n}, 2^{m_n+1}] \subset U_n.$$

Define

$$\xi_n(x,y) = 2^{k/2}\left(\sum_{i=2^{l_n}+1}^{2^{l_n+1}} \sum_{j=2^{m_n}+1}^{2^{m_n+1}} \tilde{\chi}_i(x)\tilde{\chi}_j(y)\right) \cdot \mathbb{I}_E(x,y),$$

where

$$E = \{(-1)^{j+1} \cdot \xi_{\bar{n}}(x,y) > 0\}, \tag{9}$$

and the number $\bar{n}$ is defined in (2) and (3). We note, that $\bar{n} < n$, and so the function $\xi_{\bar{n}}(x,y)$ is defined according the assumption of the induction. By Lemma 1 it is clear, that the system obtained in this way satisfies the conditions of the lemma. □

A similar lemma for arbitrary sectors can be proved also for Walsh system .

Lemma 4. *For any sequence of sectors U_k there exists a Haar type system $\xi_n(x,y)$, $n = 1,2,\dots$, such that*

$$\xi_k(x,y) = \sum_{(p,q)\in D_k} b_{ij} w_p(x) w_q(y), \quad D_k \subset U_k, \quad k = 2,3,\dots.$$

Proof. The system $\xi_k(x,y)$ again will be constructed by the induction. We define $\xi_2(x,y)$ to be an arbitrary function of the double Walsh system $w_n(x)w_m(y)$ with indexes $(n,m) \in U_2$. Then we suppose the functions $\xi_k(x,y)$, $k = 1,2,\dots,n-1$, have been already constructed. Since each of these functions is a Haar polynomial, they are constant on the dyadic rectangles

$$\left(\frac{i-1}{2^{l_n}}, \frac{i}{2^{l_n}}\right) \times \left(\frac{j-1}{2^{m_n}}, \frac{j}{2^{m_n}}\right), \quad i = 1,2,\dots,2^{l_n}, \quad j = 1,2,\dots,2^{m_n},$$

for large enough numbers $l_n, m_n \in \mathbb{N}$. Such that that sectors U_k are arbitrary, we can not provide (8) always. We define the function $\xi_n(x,y)$ by

$$\xi_n(x,y) = 2^{k/2} \left(\sum_{i=2^{l_n}+1}^{2^{l_n+1}} \sum_{j=2^{m_n}+1}^{2^{m_n+1}} \tilde{\chi}_i(x)\tilde{\chi}_j(y) \right) \cdot \mathbb{I}_E(x,y) \cdot w_p(x) w_q(x),$$

where the set E is defined like (9). It is clear, that this function is a polynomial in double Walsh system and its spectrum is in the sector U_n for suitable choices of p and q. Obviously the obtained system satisfies the conditions of the lemma. □

Lemma 5. *Let $L, n \in \mathbb{N}$, $n > 2$, and σ is a rearrangement of the numbers $2,3\dots,n$. Then there exists an increasing sequence of triangles Δ_k, $k = 1,2,\dots,n$ which sides are bigger than L and a Haar type system $\xi_k(x,y)$ such that*

$$\xi_{\sigma(k)}(x,y) = w_{2^l-1}(x) \sum_{(p,q)\in B_k} b_{pq} w_p(x) w_q(y), \quad B_k \subset \Delta_k \setminus \Delta_{k-1}, \quad k = 2,3,\dots,n. \tag{10}$$

where $l \in \mathbb{N}$ is an integer.

Proof. We consider the sequence of sectors

$$V_k = V\left(0, \frac{\pi}{4} - \frac{1}{k}\right), \quad k = 1,2,\dots,n,$$

and let $U_k = V_k \setminus V_{k-1}$, $k = 2, 3, \ldots, n$. Applying the lemma 4, we find a Haar type system of the form

$$\xi_k(x,y) = \sum_{(p,q)\in D_k} b_{pq} w_p(x) w_q(y), \quad D_k \subset U_{\sigma^{-1}(k)}, \quad k = 2,3,\ldots,n.$$

We note, that the last can be written in the form

$$\xi_{\sigma(k)}(x,y) = \sum_{(p,q)\in D_{\sigma(k)}} b_{pq} w_p(x) w_q(y), \quad D_{\sigma(k)} \subset U_k, \quad k = 2,3,\ldots,n.$$

We take the number l such that

$$\cup_{k=2}^n D_k \subset [0, 2^l)^2.$$

Denote

$$B_k = \left\{(p,q) \in \mathbb{N}^2 : \, (2^l - p - 1, q) \in D_{\sigma(k)}\right\}, \quad k = 1,2,\ldots,n,$$
$$\Delta_k = \left\{(p,q) \in \mathbb{N}^2 : \, 1 \le p, q < 2^l, \quad (2^l - p - 1, q) \in V_k\right\}, \quad k = 1,2,\ldots,n.$$

It is clear, that Δ_k is an increasing sequence of triangular regions and their sides can be bigger than given number L, if l is sufficiently big. Thus, using the relation $D_{\sigma(k)} \subset U_k$, we obtain

$$B_k \subset \Delta_k \setminus \Delta_{k-1}, \quad k = 2,3,\ldots,n. \tag{11}$$

Considering the dyadic decomposition, we easily get

$$2^l - p - 1 = p \oplus (2^l - 1)$$

for any $0 \le p < 2^l$. This implies

$$\begin{aligned} w_{2^l-1}(x)\xi_{\sigma(k)}(x,y) & \\ &= \sum_{(p,q)\in D_{\sigma(k)}} b_{pq} w_{p\oplus(2^l-1)}(x) w_q(y) \\ &= \sum_{(p,q)\in D_{\sigma(k)}} b_{pq} w_{2^l-p-1}(x) w_q(y) \\ &= \sum_{(p,q)\in B_k} \tilde{b}_{pq} w_p(x) w_q(y), \quad k = 2,3,\ldots,n. \end{aligned}$$

The last equality together with (11) gives (10).

□

Lemma 6 ([14], p. 105)**.** *For any Haar polynomial*

$$\sum_{n=N}^{M} b_n \chi_n(x)$$

there exists a rearrangement $\sigma(n)$ of the numbers $N, N+1, ..., M$, such that

$$\max_{N<p\leq q\leq M}\left|\sum_{n=p}^{q} b_{\sigma(n)}\chi_{\sigma(n)}(x)\right| \geq \frac{1}{4}\sum_{n=N}^{M}|b_n\chi_n(x)|,$$

for any $x\in[0,1]$.

From Theorem A it easily follows

Lemma 7. *For any natural number N there exists a Haar polynomial*

$$Q_N(x)=\sum_{i=N}^{c(N)} a_i\chi_i(x),$$

which satisfies the conditions

$$\|Q_N\|_\infty \leq 1,$$
$$\left|\left\{x\in(0,1): \sum_{i=N}^{c(N)}|a_i\chi_i(x)|>N\right\}\right| > 1-\frac{1}{N}.$$

4. Proof of the Theorems

Proof of Theorem 1. Without loss of generality we may suppose that

$$V_k = V(\alpha_k, \pi/2), \quad \alpha_{k+1} < \alpha_k/8, \quad k=1,2,\ldots.$$

Then we consider the sectors

$$U_1 = V_1, \quad U_k = V_k\setminus V_{k-1} = V(\alpha_{k-1},\alpha_k), \quad k=2,3,\ldots. \tag{12}$$

According to Theorem A, there exists a series in Haar system

$$\sum_{k=1}^{\infty} c_k\chi_k(x),$$

which is an indicator function of a measurable set and diverges a.e. after some rearrangement σ. According to the nature of Haar type system, the series

$$\sum_{k=1}^{\infty} c_k\xi_k(x,y) \tag{13}$$

with the same coefficients converges in L^1 norm to an indicator function on $(0,1)^2$, while, the series

$$\sum_{k=1}^{\infty} c_{\sigma(k)}\xi_{\sigma(k)}(x,y), \tag{14}$$

where σ is the same rearrangement, diverges a.e. on $(0,1)^2$. By Lemma 3, there exists a the Haar type system

$$\xi_k(x,y) = \sum_{(p,q)\in D_k} b_{ij}\chi_p(x)\chi_q(y),$$

with the condition

$$D_k \subset U_{\sigma^{-1}(k)}, \quad k = 2,3,\dots, \tag{15}$$

where U_k is defined in (12), and σ is the rearrangement from (14). According to (13), the series

$$\sum_{k=1}^{\infty} c_k\xi_k(x,y) = \sum_{k=1}^{\infty} c_k \sum_{(p,q)\in D_k} b_{ij}\chi_p(x)\chi_q(y)$$

can be considered as a Fourier series of some indicator function $\mathbb{I}_E(x,y)$ in double Haar system. In addition, a.e. divergence of (14) implies the same for the series

$$\sum_{k=1}^{\infty} c_{\sigma(k)} \sum_{(p,q)\in D_{\sigma(k)}} b_{ij}\chi_p(x)\chi_q(y).$$

In view of $D_{\sigma(n)} \subset U_n$, coming from (15), it is easy to observe, that

$$S_{V_n}(x,y,\chi,\mathbb{I}_E) = \sum_{k=1}^{n} c_{\sigma(k)} \sum_{(p,q)\in D_{\sigma(k)}} b_{ij}\chi_p(x)\chi_q(y)$$

and these sums diverge a.e. as $n \to \infty$. $\square$

Proof of Theorem 2. To prove Theorem 2, we have just to repeat the proof of Theorem 1, using lemma 4 instead of 3. $\square$

Proof of Theorem 3. Applying Lemma 7, we can find a Haar polynomial

$$Q_k(x) = \sum_{i=n_k}^{m_k} a_i\chi_i(x),$$

and sets $E_k \subset (0,1)$, satisfying the conditions

$$\begin{aligned}
&|E_k| > 1 - 2^{-k},\\
&m_k < n_{k+1},\\
&\|Q_k\|_\infty \le 1,\\
&\sum_{i=n_k}^{m_k} |a_i\chi_i(x)| > 4^k, \quad x \in E_k.
\end{aligned}$$

Using Lemma 6, we get a rearrangement σ of the numbers $n_k, n_k+1, \ldots, m_k$, which satisfies the inequality

$$\sup_{n_k\le l\le m_k}\left|\sum_{i=n_k}^{l} a_{\sigma(i)}\chi_{\sigma(i)}(x)\right| > 4^{k-1}, \quad x\in E_k.$$

Since the intervals $[m_k, n_k]$ are pairwise disjoin, we will use a common notation σ for these rearrangements. Using Lemma 5 countable number of times, we will get an increasing sequence of triangular regions Δ_k, $k = 1, 2, \ldots$, and a Haar type system $\xi_k(x, y)$ such that

$$\varepsilon_k(x)\xi_{\sigma(j)}(x,y) = \sum_{(p,q)\in B_k} b_{pq}w_p(x)w_q(y), \tag{16}$$

$$B_j\subset \Delta_j\setminus\Delta_{j-1}, \quad n_k\le j\le m_k, \quad k=2,3,\ldots,$$

where $|\varepsilon_k(x)| \equiv 1$. Denote

$$\phi_j(x,y) = 2^{-k}\varepsilon_k(x)a_{\sigma(j)}\xi_{\sigma(j)}(x,y), \quad m_k\le j\le n_k,$$

and consider the function

$$f(x,y) = \sum_{j=1}^{\infty}\phi_j(x,y),$$

where the terms with indexes out of $\cup_k[n_k, m_k]$ are zero. It is obvious, that this series converges uniformly to a function $f \in L^\infty(0, 1)^2$. By (16) we have

$$S_{\Delta_l}(x,y,w,f) = \sum_{j=1}^{l}\phi_j(x,y).$$

Thus, for any $n_k \le l \le m_k$ we get

$$|S_{\Delta_l}(x,y,w,f) - S_{\Delta_{n_k}}(x,y,w,f)| = 2^{-k}\left|\sum_{j=n_k}^{l} a_{\sigma(j)}\xi_{\sigma(j)}(x,y)\right|,$$

and consequently

$$\sup_{n_k\le l\le m_k}|S_{\Delta_l}(x,y,w,f) - S_{\Delta_{n_k}}(x,y,w,f)| > 2^{k-2}, \quad (x,y)\in\tilde{E}_k,$$

where $\tilde{E}_k \subset (0, 1)^2$ is a set, obtained from E_k by the transformation, corresponding to the constructed Haar type system. Thus, we obtain

$$|\tilde{E}_k| = |E_k| > 1-2^{-k}.$$

Denoting

$$E = \cup_{k\ge1}\cap_{i\ge k}\tilde{E}_i,$$

we obviously get $|E| = 1$ and

$$\limsup_{l\to\infty}|S_{\Delta_l}(x,y,w,f)| = \infty, \quad x\in E,$$

which completes the proof of the theorem.

□

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