AN ABSTRACT THEORY OF SINGULAR OPERATORS

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Abstract. We introduce a class of operators on abstract measure spaces, which unifies the Calderón-Zygmund operators on spaces of homogeneous type, the maximal functions, the martingale transforms and Carleson operators. We prove that such operators can be dominated by simple sparse operators with a definite form of the domination constant. Applying these estimates, we improve several results obtained by different authors in recent years.

1. INTRODUCTION

The study of weighted inequalities in Harmonic Analysis started in early 1970's. In 1972 Muckenhaupt [27] proved that the maximal function is bounded on $L^p(w)$ for $1 < p < \infty$ if and only if the weight w satisfies the A_p condition

$$[w]_{A_p} = \sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\right)^{p-1} < \infty. \tag{1.1}$$

One year later Hunt, Muckenhaupt and Wheeden [9] established the same property for Hilbert transform. For the general Calderón-Zygmund operators weighted $L^p(w)$ bound was first proved by Coifman and Fefferman [4].

In 1993 Buckley [3] discovered that the maximal function M has the sharp estimate

$$\|Mf\|_{L^p(w)\to L^p(w)} \le C\|w\|_{A_p}^{1/(p-1)}, \tag{1.2}$$

arising a similar problem for Calderón-Zygmund operators. Last fifteen years there has been an activity in the investigation of this problem. For the general Calderón-Zygmund operators the conjecture was the bound

$$\|Tf\|_{L^p(w)\to L^p(w)} \le C\|w\|_{A_p}^{\max\{1,1/(p-1)\}}. \tag{1.3}$$

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An extrapolation theorem proved in [7] reduced the conjecture to the case $p = 2$ so the inequality (1.3) became known as A_2 conjecture. After being established for several particular operators first ([33], [31], [32], [30], [13]), in 2010 Hytönen [11] proved A_2 conjecture for the general Calderón-Zygmund operators.

Series of recent works are motivated on domination of Calderón-Zygmund operators by very simple sparse operators ([22]-[25], [5], [21], [12]-[17]). From such results in particular follows (1.3), since the A_2 bound is very easy to establish for the sparse operators ([24], [14], [28], [29]). A domination of the classical Calderón-Zygmund operators on $\mathbb{R}^n$ by sparse operators was discovered by Lerner [24]. Applying this domination he gave a simplified proof of Hytönen A_2 theorem. Lacey [21] proved a pointwise domination theorem for more general ω-Calderón-Zygmund operators (see (7.17)-(7.20)) with ω satisfying the Dini condition

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty, \tag{1.4}$$

deriving weighted bound (1.3) for such operators too. Lacey's result was a stronger version of another pointwise bound independently proved by Conde-Alonso, Rey [5] and Lerner-Nazarov [26]. Moreover, Lacey's inequality only assumes the Dini condition, while prior approaches [5, 26] require $1/t$ in the Dini integral be replaced by $(\log 2/t)/t$. Hytönen, Roncal and Tapiola [17] elaborated the proof of Lacey [21] to get a precise linear dependence of the domination constant on the characteristic numbers of the operator. Lerner [25] gave a simple proof of Lacey-Hytönen-Roncal-Tapiola theorem.

Anderson and Vagharshakyan [1] proved A_2 theorem for the Calderón-Zygmund operators in general spaces of homogeneous type with modulus of continuity $\omega(t) = t^\alpha$, $\alpha > 0$.

In late 1970's, several authors considered also martingale analogs of the A_p theory. For instance, Izumisawa-Kazamaki [18], proved a variant of the Muckenhoupt maximal function result [27] in this setting. When it came to martingale transforms, the distinction between the homogeneous and non-homogeneous cases was already recognized by these authors. Nevertheless, norm inequalities for martingale transforms were proved by Bonami-Lépingle [2]. The A_2 theorem for martingale transform was proved recently by Thiele, Treil and Volberg [34]. Lacey [21] gave a self-contained, short and elementary proof of this theorem. Moreover, he established a pointwise domination theorem for martingale transforms too.

Grafakos-Martell-Soria [8] and Di Plinio-Lerner [6] considered weighted estimates for maximal modulations of Calderón-Zygmund operators on $\mathbb{R}^n$, in particular, for Carleson or Walsh-Carleson operators. The paper [8] establishes weighted norm control of the Carleson operators by the maximal function. In [6] the authors proved weighted norm estimates of Carleson and Walsh-Carleson operators with explicit dependence of the constants on A_p-characteristics of the weight.

In this paper we introduce so called BO (bounded oscillation) operators on abstract measure spaces. Those operators unify the Calderón-Zygmund operators and maximal functions in general homogeneous spaces, martingale transforms (non-homogeneous case) and Carleson operators. The definition of BO operators is motivated by the papers [19], [20], where some exponential estimates for Calderón-Zygmund and other related operators were proved. We shall prove that those operators have pointwise domination by sparse operators and then satisfy the bound (1.3). We derive variety of other properties of BO operators significant for their further investigations.

To define BO operators we introduce a concept of ball-basis for an abstract measure space, which is a family of measurable sets holding some common properties of d-dimensional balls on $\mathbb{R}^d$ and their analogues in related theories (martingales, dyadic analysis).

Definition 1.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. A family of sets $\mathfrak{B} \subset \mathfrak{M}$ is said to be a ball-basis if it satisfies the following conditions:*

B1) $0 < \mu(B) < \infty$ *for any ball* $B \in \mathfrak{B}$.

B2) For any points $x, y \in X$ there exists a ball $B \ni x, y$.

B3) If $E \in \mathfrak{M}$, then for any $\varepsilon > 0$ there exists a finite or infinite sequence of balls B_k, $k = 1, 2, \dots$, such that

$$\mu\left(E \triangle \bigcup_k B_k\right) < \varepsilon.$$

B4) For any $B \in \mathfrak{B}$ there is a ball $B^* \in \mathfrak{B}$ (called hull *of B) satisfying the conditions*

$$\bigcup_{A\in\mathfrak{B}:\, \mu(A)\le 2\mu(B),\, A\cap B\neq\varnothing} A \subset B^*, \tag{1.5}$$

$$\mu(B^*) \le \mathcal{K}\mu(B), \tag{1.6}$$

where $\mathcal{K}$ is a positive constant.

One can easily check that the family of Euclidean balls in $\mathbb{R}^n$ forms a ball-basis. Moreover, we will see below (see Theorem 7.1) that if the family of metric balls in spaces of homogeneous type satisfies the density condition (see Definition 3.1), then it is a ball-basis too. The martingale basis considered in Section 8 is an example of ball-bases having non-doubling property.

Definition 1.2. *Let $1 \le r < \infty$, $(X, \mathfrak{M}, \mu)$ be a measure space and $L^0(X)$ be the linear space of functions (include non-measurable functions) on X. An operator*

$$T : L^r(X) \to L^0(X)$$

is said to be subadditive if

$$|T(\lambda \cdot f)(x)| = |\lambda| \cdot |Tf(x)|, \quad \lambda \in \mathbb{R},$$
$$|T(f+g)(x)| \le |Tf(x)| + |Tg(x)|.$$

Remark 1.1. *As we will see below in the definitions of some operators (maximal function, T^*) some non-measurable functions can appear. To apply the results of the paper for such operators we allow non-measurably of the images in the definition of general subadditive operators. The definitions of L^p (weak-L^p) norms and some standard inequalities that we need for non-measurable functions will be stated in the next section.*

Let $1 \le r < \infty$ be a fixed number in Sections 1-4. For $f \in L^r(X)$ and $B \in \mathfrak{B}$ we set

$$\langle f \rangle_{B,r} = \left(\frac{1}{\mu(B)} \int_B |f|^r \right)^{1/r}, \quad \langle f \rangle^*_{B,r} = \sup_{A \in \mathfrak{B}: A \supset B} \langle f \rangle_{A,r}.$$

In the case $r = 1$ for those quantities it will be used the notations $\langle f \rangle_B$ and $\langle f \rangle^*_B$ respectively (Sections 5-8).

Definition 1.3. *We say that a subadditive operator T is a bounded oscillation operator with respect to a ball-basis $\mathfrak{B}$ if*

T1) (Localization) for every $B \in \mathfrak{B}$ we have

$$\sup_{x, x' \in B,\, f \in L^r(X)} \frac{|T(f \cdot \mathbb{I}_{X \setminus B^*})(x) - T(f \cdot \mathbb{I}_{X \setminus B^*})(x')|}{\langle f \rangle^*_{B,r}} \le \mathcal{L}_1 = \mathcal{L}_1(T) < \infty, \tag{1.7}$$

T2) (Connectivity) for any $A \in \mathfrak{B}$ ($A^* \ne X$) there exists a ball $B \supsetneq A$ (i.e. $B \supset A$, $B \ne A$) such that

$$\sup_{x \in A,\, f \in L^r(X)} \frac{|T(f \cdot \mathbb{I}_{B^* \setminus A^*})(x)|}{\langle f \rangle_{B^*,r}} \le \mathcal{L}_2 = \mathcal{L}_2(T) < \infty, \tag{1.8}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the least constants satisfying (1.7) and (1.8) respectively. The family of all bounded oscillation operators with respect to a ball-basis $\mathfrak{B}$ will be denoted by $\mathrm{BO}_{\mathfrak{B}}$ or simply BO.

It will be proved below (Theorem 4.4) that if the ball-basis $\mathfrak{B}$ satisfies the doubling condition, then the localization property implies connectivity.

Definition 1.4. *A collection of balls $\mathcal{S} \subset \mathfrak{B}$ is said to be sparse or γ-sparse if for any $B \in \mathcal{S}$ there is a set $E_B \subset B$ such that $\mu(E_B) \geq \gamma\mu(B)$ and the sets $\{E_B : B \in \mathcal{S}\}$ are pairwise disjoint, where $0 < \gamma < 1$ is a constant.*

Given family of balls $\mathcal{S}$ we associate the operator

$$\mathcal{A}_{\mathcal{S},r} f(x) = \sum_{A \in \mathcal{S}} \langle f \rangle_{A,r} \cdot \mathbb{I}_A(x).$$

If $\mathcal{S}$ is a sparse collection of balls, then we say $\mathcal{A}_{\mathcal{S},r}$ is a sparse operator. In the case $r = 1$ we will simply write $\mathcal{A}_{\mathcal{S}}$. Further, positive constants depending on $\mathcal{K}$ (see (1.6)) will be called admissible constants and the relation $a \lesssim b$ will stand for $a \leq c \cdot b$, where $c > 0$ is admissible. We write $a \sim b$ if the relations $a \lesssim b$ and $b \lesssim a$ hold at the same time.

Theorem 1.1. *Let an operator $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfy weak-L^r inequality. Then for any function $f \in L^r(X)$ and a ball $B \in \mathfrak{B}$ there exists a family of balls $\mathcal{S}$, which is a union of two γ-sparse collections and*

$$|Tf(x)| \lesssim (\mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r \to L^{r,\infty}}) \cdot \mathcal{A}_{\mathcal{S},r} f(x), \text{ a.e. } x \in B, \tag{1.9}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the constants (1.7), (1.8) and $0 < \gamma < 1$ is an admissible constant.

Definition 1.5. *For $T \in \mathrm{BO}_{\mathfrak{B}}$ define*

$$T^* f(x) = \sup_{B \in \mathfrak{B} :\, x \in B} |T(f \cdot \mathbb{I}_{X \setminus B^*})(x)|.$$

We shall prove below (Theorem 4.3) that if $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfies weak-L^r estimate, then $T^* \in \mathrm{BO}_{\mathfrak{B}}$ and satisfies weak-L^r bound too. So from Theorem 1.1 we will immediately get

Theorem 1.2. *If $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfies weak-L^r inequality, then for any function $f \in L^r(X)$ and a ball $B \in \mathfrak{B}$ there exists a family of balls $\mathcal{S}$, which is a union of two γ-sparse collections and*

$$|T^* f(x)| \lesssim (\mathcal{L}_1(T) + \mathcal{L}_2(T) + \|T\|_{L^r \to L^{r,\infty}}) \cdot \mathcal{A}_{\mathcal{S},r} f(x), \text{ a.e. } x \in B.$$

Theorem 8.3, Theorem 8.1 and Theorem 7.5 below prove that the ω-Calderón-Zygmund operators on spaces of homogeneous type, the martingale transforms and the maximal functions are BO operators and so they all satisfy the estimate (1.9). Moreover, combining (1.9) with the weighted bounds for sparse operators, we obtain A_2 theorems for all these operators. Hence Theorem 1.1 and Theorem 1.2 cover the above stated results concerning the weighted bounds and the domination by sparse operators. Lacey-Hytönen-Roncal-Tapiola ([21], [17]) theorem is a version of Theorem 1.2 for the ω-Calderón-Zygmund operators on $\mathbb{R}^n$ with the Dini condition. Lacey [21] domination theorem for martingale transforms is another case of inequality (1.9). In $\mathbb{R}^n$ and in general spaces of homogeneous type, where a doubling condition holds, the proofs of such dominations are based on dyadic decomposition theorems ([14], [10]). In the case of martingale transforms ([34]) instead of dyadic decomposition the properties of martingale basis is essentially used. Since a general ball-basis does not always satisfy the doubling condition (the typical example is the martingale basis), there is no dyadic decomposition in general. So our method of proof of Theorem 1.1 is different. It is direct and partially based on the papers [19] and [20].

In the last sections we prove several weighted bounds for sparse operators to get A_2 theorems for some BO operators. The method of proofs of such theorems are based on a duality argument developed in the papers [25], [5], [14], [21].

Note that Theorem 1.1 also imply exponential integrability results of our papers [19], [20] proved for the Calderón-Zygmund operators on Euclidean spaces and for the partial sums of Walsh-Fourier series.

Applying Theorem 1.1 for the maximal function corresponding to a general ball-basis, we do not get the full weighted estimate (1.2), which is known to be optimal for the maximal function in Euclidean spaces ([3]). In the general case the optimality only occurs when $1 < p \le 2$. So (1.9) do not cover some estimates that has the maximal function. The reason is that the maximal function has some extra properties that the general BO operators do not have. An example of such a property is L^∞-bound.

In the last section we prove that the maximal modulation of a BO operator is also BO operator, deriving pointwise sparse domination for maximal modulated BO operators too.

2. Outer measure and L^p-norms of non-measurable functions

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Define the outer measure of a set $E \subset X$ by

$$\mu^*(E) = \inf_{F \in \mathfrak{M}:\, F \supset E} \mu(F).$$

We say two sets $A, B \subset X$ (not necessarily measurable) satisfy the relation $A \sim B$ if

$$\mu^*(A \triangle B) = 0.$$

Denote by $\bar{\mathfrak{M}}$ the family of sets in X, which are equivalent to a measurable set. It is clear that $\bar{\mathfrak{M}}$ is σ-algebra. For a given set $E \subset X$ we define

$$\bar{E} = \bigcap_{F \in \mathfrak{M}:\, F \supset E} F.$$

Observe that

$$\bar{E} \in \bar{\mathfrak{M}}, \quad \mu^*(\bar{E}) = \mu^*(E).$$

For any function $f \in L^0(X)$ we denote

$$G_f(t) = \{x \in X : |f(x)| > t\},$$
$$\lambda_f(t) = \mu^*\left(G_f(t)\right).$$

Observe that the function

$$\bar{f}(x) = \inf\{t \geq 0 : x \in \overline{G_f(t)}\}$$

is positive, $\bar{\mathfrak{M}}$-measurable and $|f(x)| \leq \bar{f}(x)$. Besides, $\bar{f}$ is the smallest $\bar{\mathfrak{M}}$-measurable positive function dominating $|f|$. Namely, if $g(x)$ is $\bar{\mathfrak{M}}$-measurable and satisfies $|f(x)| \leq g(x)$, then $\bar{f}(x) \leq g(x)$. For arbitrary $f \in L^0(X)$ we define

$$\|f\|_{L^p} = \|\bar{f}\|_{L^p} = \left(p \int_0^\infty t^{p-1} \lambda_f(t)dt\right)^{1/p},$$
$$\|f\|_{L^p \to L^{p,\infty}} = \|\bar{f}\|_{L^p \to L^{p,\infty}} = \sup_{t>0} t(\lambda_f(t))^{1/p}.$$

Definition 2.1. *We say a subadditive operator T satisfies weak-L^p or strong-L^p estimate if*

$$\|T\|_{L^p \to L^{p,\infty}} = \sup_{t>0,\, f \in L^p(X)} \frac{t \cdot (\lambda_{Tf}(t))^{1/p}}{\|f\|_{L^p}} < \infty,$$
$$\|T\|_{L^p} = \sup_{f \in L^p(X)} \frac{\|Tf\|_{L^p}}{\|f\|_{L^p}} < \infty,$$

respectively.

One can easily check that the standard triangle and Hölder inequalities as well as the Marcinkiewicz interpolation theorem hold in such setting of L^p norms. We will need the following case of the interpolation theorem.

Theorem 2.1 (Marcinkiewicz interpolation theorem, [36], ch. 12.4)**.** *If a subadditive operator T satisfies the weak-L^1 estimate and the strong-L^∞ estimate, then for $1 < p < \infty$ it holds*

$$\|T\|_{L^p} \le c_p(\|T\|_{L^1\to L^{1,\infty}})^{1/p} \times (\|T\|_{L^\infty})^{1/q}.$$

3. Some properties of ball-basis

Let $\mathfrak{B}$ be a ball-basis for the measure space $(X, \mathfrak{M}, \mu)$. From B4) condition it follows that if balls A, B satisfy $\mu(A) \le 2\mu(B)$, then $A \subset B^*$. This property will be called two balls relation. Hull levels of a given ball $B \in \mathfrak{B}$ will be denoted by

$$B^{[0]} = B, \quad B^{[n+1]} = \left(B^{[n]}\right)^*.$$

By property B4) we have $\mu(B^{[n+1]}) \le \mathcal{K}\mu(B^{[n]})$. Applying this inequality consecutively, we get

$$\mu(B^{[n]}) \le \mathcal{K}^n \mu(B), \quad n \ge 0. \tag{3.1}$$

We say a set $E \subset X$ is bounded if $E \subset B$ for some $B \in \mathfrak{M}$.

Lemma 3.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and the family of sets $\mathfrak{B} \subset \mathfrak{M}$ satisfy B4)-condition. If $E \subset X$ is bounded and $\mathcal{G}$ is a family of balls with*

$$E \subset \bigcup_{G\in\mathcal{G}} G,$$

then there exists a finite or infinite sequence of pairwise disjoint balls $G_k \in \mathcal{G}$ such that

$$E \subset \bigcup_k G_k^{[1]}. \tag{3.2}$$

Proof. The boundedness implies $E \subset B$ for some $B \in \mathfrak{B}$. If there is a ball $G \in \mathcal{G}$ so that $G \cap B \ne \varnothing$ and $\mu(G) > \mu(B)$, then by two balls relation we have $E \subset B \subset G^{[1]}$. Thus the desired sequence can be formed by a single element G. Hence we can suppose that any element $G \in \mathcal{G}$ satisfies the conditions $G \cap B \ne \varnothing$ and $\mu(G) \le \mu(B)$. Therefore we get

$$\bigcup_{G\in\mathcal{G}} G \subset B^{[1]}.$$

Take G_1 to be a ball from $\mathcal{G}$ satisfying

$$\mu(G_1) > \frac{1}{2}\sup_{G\in\mathcal{G}} \mu(G).$$

Then, suppose by induction we have already chosen elements $G_1,\ldots,G_k$ from $\mathcal{G}$. Take $G_{k+1}\in\mathcal{G}$ disjoint with the balls $G_1,\ldots,G_k$ and satisfying

$$\mu(G_{k+1}) > \frac{1}{2}\sup_{G\in\mathcal{G}\,:\,G\cap G_j=\varnothing,\,j=1,\ldots,k} \mu(G).$$

If for some n we will not be able to determine G_{n+1} the process will stop and we will get a finite sequence $G_1,G_2,\ldots,G_n$. Otherwise our sequence will be infinite. We shall consider the infinite case of the sequence (the finite case can be done similarly). Since the balls G_n are pairwise disjoint and $G_n\subset B^{[1]}$, we have $\mu(G_n)\to 0$. Take an arbitrary $G\in\mathcal{G}$ such that $G\neq G_k$, $k=1,2,\ldots$. Let m be the smallest integer such that

$$\mu(G) > \frac{1}{2}\mu(G_{m+1}).$$

Observe that we have

$$G\cap G_j\neq\varnothing$$

for some minimal $1\le j\le m$, since otherwise G had to be chosen instead of G_{m+1}. Besides, we have $\mu(G)\le 2\mu(G_j)$, which implies $G\subset G_j^{[1]}$. Since $G\in\mathcal{G}$ was taken arbitrarily, we get (3.2). □

Lemma 3.2. *Let $(X,\mathfrak{M},\mu)$ be a measure space with a ball-basis $\mathfrak{B}$. If balls $B\in\mathfrak{B}$, $G_k\in\mathfrak{B}$, $k=1,2,\ldots$, satisfy the relation*

$$G_k\cap B\neq\varnothing,\quad \mu(G_k)\to r=\sup_{A\in\mathfrak{B}}\mu(A), \tag{3.3}$$

then

$$X\subset\bigcup_k G_k^{[1]}.$$

Moreover, for any ball $A\in\mathfrak{B}$ we have $A\subset G_k^{[1]}$ for some $k\ge k_0$.

Proof. Since by B2)-condition every point $x\in X$ is in some ball, it is enough to prove the second part of the theorem. So let $A\in\mathfrak{B}$. Chose points $x\in A$, $y\in B$. According to B2)-condition there is $C\in\mathfrak{B}$ such that $x,y\in C$. Let G be one of the balls A,B and C, which has a biggest measure. Applying two ball relation twice, one can easily check that

$$A\cup B\cup C\subset G^{[2]}.$$

From (3.3) we find an integer k_0 such that $\mu(G_k)>\mu(G^{[2]})/2$ for $k>k_0$. Therefore, since $G_k\cap G^{[2]}\neq\varnothing$, we get $A\subset G^{[2]}\subset G_k^{[1]}$, $k>k_0$. □

Definition 3.1. *For a set $E \in \mathfrak{M}$ a point $x \in E$ is said to be density point if for any $\varepsilon > 0$ there exists a ball B such that*

$$\mu(B \cap E) > (1 - \varepsilon)\mu(B).$$

We say a measure space $(X, \mathfrak{M}, \mu)$ satisfies the density property if for any measurable set E almost all points $x \in E$ are density points.

Lemma 3.3. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. If a family of measurable sets $\mathfrak{B}$ satisfies the density property and B4)-condition, then for any bounded measurable set E, $\mu(E) > 0$, and $\varepsilon > 0$ there is a sequence of balls $\{B_k\}$ such that*

$$\mu\left(\bigcup_k B_k \setminus E\right) < \varepsilon, \quad \mu\left(E \setminus \bigcup_k B_k\right) < \alpha\mu(E) \tag{3.4}$$

where $0 < \alpha < 1$ is an admissible constant.

Proof. Applying the density property, one can find a family of balls $\mathfrak{B}$ satisfying

$$\begin{aligned} &E \subset \bigcup_{B \in \mathfrak{B}} B \text{ a.s.}, \\ &\mu(B \cap E) > (1 - \delta)\mu(B), \quad B \in \mathfrak{B}, \end{aligned}$$

where

$$\delta = \min\left\{\frac{\varepsilon}{2\mu(E)}, 1/2\right\}.$$

Then, apply Lemma 3.1 we get a subfamily of pairwise disjoint balls B_k with (3.2). Thus we have

$$\begin{aligned} \mu\left(\bigcup_k B_k \setminus E\right) &= \sum_k \mu(B_k \setminus E) \\ &< \frac{\delta}{1 - \delta} \sum_k \mu(B_k \cap E) \le 2\delta\mu(E) \le \varepsilon. \end{aligned}$$

On the other hand

$$\begin{aligned}
\mu\left(E\setminus\bigcup_k B_k\right) &= \mu(E)-\sum_k \mu\left(E\cap B_k\right)\\
&\le \mu(E)-(1-\delta)\sum_k \mu\left(B_k\right)\\
&\le \mu(E)-\frac{1-\delta}{\mathcal{K}}\mu\left(\bigcup_k B_k^{[1]}\right)\\
&\le \mu(E)-\frac{1}{2\mathcal{K}}\mu\left(\bigcup_{B\in\mathfrak{B}} B\right)\\
&\le \mu(E)-\frac{1}{2\mathcal{K}}\mu(E)\\
&= \left(1-\frac{1}{2\mathcal{K}}\right)\mu(E).
\end{aligned}$$

So conditions (3.4) are satisfied with a constant $\alpha = 1-1/2\mathcal{K}$. □

Observe that if B3)-condition holds for the bounded measurable sets, then it holds for all the measurable sets. Indeed, according Lemma 3.2, there is a sequence of balls G_k such that $X=\cup_k G_k$. This implies that any measurable set E can be written as a countable union of bounded measurable sets $E=\cup_k E_k$. Apply B3)-condition to each set E_k with an approximation number $\varepsilon/2^k$. The union of all obtained approximating balls will give an ε-approximation of E.

Lemma 3.4. *Let $(X,\mathfrak{M},\mu)$ be a measurable set. If a family of measurable sets $\mathfrak{B}$ satisfies B4)-condition, then it will satisfy B3)-condition if and only if the density condition holds.*

Proof. Let $\mathfrak{B}$ satisfy B4) and density conditions and E be a measurable set. The remark stated before the lemma allows us to suppose that E is bounded. Applying Lemma 3.3 consecutively, we can find sequences of balls $\mathfrak{B}_k$, $k=1,2,\ldots$, such that

$$\mu\left(\bigcup_{B\in\mathfrak{B}_k} B\setminus E\right) < \frac{\varepsilon}{2^k},\quad k\ge 1, \tag{3.5}$$

$$\mu\left(E\setminus\bigcup_{B\in\cup_{j=1}^{k}\mathfrak{B}_j} B\right) < \alpha\mu\left(E\setminus\bigcup_{B\in\cup_{j=1}^{k-1}\mathfrak{B}_j} B\right),\quad k\ge 1, \tag{3.6}$$

where in the case $k=1$ the right hand side of (3.6) is assumed to be E. Then we denote $\mathfrak{B}=\cup_{j=1}^{\infty}\mathfrak{B}_j$. From (3.6) it easily follows that

$$E\subset\bigcup_{B\in\mathfrak{B}}B \text{ a.s.},$$

while from (3.5) we obtain

$$\mu\left(\left(\bigcup_{B\in\mathfrak{B}}B\right)\setminus E\right)\le\sum_{k=1}^{\infty}\mu\left(\left(\bigcup_{B\in\mathfrak{B}_k}B\right)\setminus E\right)<\varepsilon.$$

To prove the second part of lemma let $\mathfrak{B}$ satisfy B4) and B3) conditions. Suppose to the contrary $\mathfrak{B}$ does not have the density property. That is, there exists a number α, $0<\alpha<1$, a set $E\in\mathfrak{M}$ together with its subset $F\subset E$, $\mu^*(F)>0$, such that for any $x\in F$ and $B\in\mathfrak{B}$ with $x\in B$ we have

$$\mu(B\setminus E)>\alpha\mu(B). \tag{3.7}$$

By B3)-condition for any $\varepsilon>0$ it can be found a sequence of balls B_k, $k=1,2,\ldots$, such that

$$\mu\left(\bar{F}\bigtriangleup\bigcup_k B_k\right)<\varepsilon. \tag{3.8}$$

We can suppose that $\mu(B_k\cap\bar{F})>0$ for each B_k. Observe that it implies $B_k\cap F\neq\varnothing$. Indeed, suppose to the contrary $B_{k_0}\cap F=\varnothing$. Then we get $F\subset\bar{F}\setminus B_{k_0}$ and so a contradiction $\mu^*(F)\le\mu^*(\bar{F}\setminus B_{k_0})<\mu^*(\bar{F})$. Thus by (3.7) we get

$$\mu(B_k\setminus\bar{F})\ge\mu(B_k\setminus E)>\alpha\mu(B_k),\quad k=1,2,\ldots. \tag{3.9}$$

Applying Lemma 3.3, we find a subsequence of pairwise disjoint balls $\tilde{B}_k$, $k=1,2,\ldots$, such that

$$\mu\left(\bar{F}\setminus\bigcup_k\tilde{B}_k^{[1]}\right)<\varepsilon.$$

Thus, applying B4)-condition, (3.8) and (3.9), we obtain

$$\begin{aligned}\mu^*(\bar{F})&\le\mu\left(\bigcup_k\tilde{B}_k^{[1]}\right)+\varepsilon\le\mathcal{K}\sum_k\mu(\tilde{B}_k)+\varepsilon\le\frac{\mathcal{K}}{\alpha}\sum_k\mu(\tilde{B}_k\setminus\bar{F})+\varepsilon\\&=\frac{\mathcal{K}}{\alpha}\mu\left(\bar{F}\bigtriangleup\bigcup_k B_k\right)+\varepsilon<\varepsilon\left(\frac{\mathcal{K}}{\alpha}+1\right).\end{aligned}$$

Choosing enough small ε, we get $\mu^*(F)=\mu^*(\bar{F})=0$ and so a contradiction.

□

We say a measurable set E is almost surely subset of a measurable set F if $\mu(E \setminus F) = 0$. We denote this relation by

$$E \subset F \text{ a.s..}$$

Lemma 3.5. *For any bounded measurable set $E \in \mathfrak{M}$ there exists a sequence of balls B_k, $k = 1, 2, \ldots$, such that*

$$E \subset \bigcup_k B_k \textit{ a.s.} \tag{3.10}$$

$$\sum_k \mu(B_k) \leq 2\mathcal{K}\mu(E). \tag{3.11}$$

Proof. Then observe that applying B3)-condition consecutively, for a given measurable set E and $\varepsilon > 0$ one can find a countable family of balls $\mathfrak{A}$ such that

$$E \subset \bigcup_{A \in \mathfrak{A}} A \text{ a.s.} \tag{3.12}$$

$$\mu\left(\bigcup_{A \in \mathfrak{A}} A\right) < (1 + \varepsilon)\mu(E). \tag{3.13}$$

Applying Lemma 3.1, we find a pairwise disjoint collection $\{A_j\} \subset \mathfrak{A}$ such that

$$E \subset \bigcup_j {A_j}^{[1]} \text{ a.s. } .$$

The B4)-condition and (3.13) yield

$$\begin{aligned}
\sum_j \mu\left({A_j}^{[1]}\right) &\leq \mathcal{K} \sum_j \mu\left(A_j\right) = \mathcal{K}\mu\left(\bigcup_j A_j\right) \\
&\leq \mathcal{K}\mu\left(\bigcup_{A \in \mathfrak{A}} A\right) \\
&< 2\mathcal{K}\mu(E).
\end{aligned}$$

Define $B_k = A_k^{[1]}$, one can easily check that (3.10) and (3.11) are satisfied. □

We denote by $\#A$ the cardinality of a finite set A.

Lemma 3.6. *Let $A \in \mathfrak{B}$ and $\mathcal{G}$ be a family of pairwise disjoint balls such that each $G \in \mathcal{G}$ satisfies the relations*

$$G^{[1]} \cap A \neq \varnothing, \tag{3.14}$$

$$0 < c_1 \leq \mu(G) \leq c_2, \tag{3.15}$$

with some positive constants c_1, c_2. Then the number of elements in $\mathcal{G}$ is finite and satisfies the bound

$$\#\mathcal{G} \leq \frac{\min\{\mathcal{K}^3 c_2, \mathcal{K}\mu(A)\}}{c_1}.$$

Remark 3.1. *One can easily check that this lemma implies a similar lemma with the condition $G^{[2]} \cap A \neq \varnothing$ instead of (3.14).*

Proof. Suppose $G_1, G_2, \ldots, G_N$ are some elements from $\mathcal{G}$. We can assume that

$$\mu(G_1^{[1]}) \geq \mu(G_i^{[1]}) \tag{3.16}$$

for each $1 \leq j \leq N$. If $\mu(A) \geq \mu(G_1^{[1]})$, then from (3.14) and B4)-condition we get

$$\bigcup_{1\leq j\leq N} G_k \subset \bigcup_{1\leq j\leq N} G_j^{[1]} \subset A^{[1]}.$$

Thus, since G_k are pairwise disjoint, from (3.15) we obtain

$$N \cdot c_1 \leq \mu\left(\bigcup_{1\leq j\leq N} G_k\right) \leq \mu(A^{[1]}) \leq \mathcal{K}\mu(A)$$

that is

$$N \leq \frac{\mathcal{K}\mu(A)}{c_1}. \tag{3.17}$$

In the case $\mu(A) < \mu(G_1^{[1]})$ we get $A \subset G_1^{[2]}$ and therefore by (3.14),

$$G_j^{[1]} \cap G_1^{[2]} \neq \varnothing, \quad 1 \leq j \leq N.$$

Thus, applying two balls relation and (3.16), we obtain

$$\bigcup_{1\leq j\leq N} G_j^{[1]} \subset G_1^{[3]}.$$

Then, again using (3.15) and (3.1), we get

$$N \cdot c_1 \leq \mu\left(\bigcup_k G_k\right) \leq \mu(G_1^{[3]}) \leq \mathcal{K}^3\mu(G_1) \leq \mathcal{K}^3 c_2. \tag{3.18}$$

Combination of (3.17) and (3.18) completes the proof of lemma. □

4. Preliminary properties of bounded oscillation operators

In this section we derive some preliminary properties of BO operators. Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$ and $1 \le r < \infty$. We will need weak-L^r inequality of the maximal operator

$$M_r f(x) = \sup_{B \in \mathfrak{B}:\, x \in B} \left(\frac{1}{\mu(B)} \int_B |f(t)|^r d\mu(t) \right)^{1/r} \tag{4.1}$$

associated with a ball-basis $\mathfrak{B}$. The maximal operator corresponding to $r = 1$ will be denoted by M.

Theorem 4.1. *The maximal operator (4.1) satisfies weak-L^r inequality. Moreover, we have*

$$\|M_r\|_{L^r \to L^{r,\infty}} \le \mathcal{K}^{1/r}. \tag{4.2}$$

Proof of Theorem 4.1. Denote

$$E = \{x \in X : \, M_r f(x) > \lambda\}.$$

Note that E can be non-measurable. For any $x \in E$ there exists a ball $B(x) \subset X$ such that

$$x \in B(x), \quad \frac{1}{\mu(B(x))} \int_{B(x)} |f|^r > \lambda^r.$$

We have $E = \cup_{x \in E} B(x)$. Given $B \in \mathfrak{B}$ consider the collection of balls $\{B(x) : \, x \in E \cap B\}$. Applying Lemma 3.1, we find a sequence of pairwise disjoint balls $\{B_k\}$ taken from this collection such that

$$E \cap B \subset \bigcup_k B_k^{[1]} = Q(B).$$

We have $Q(B)$ is measurable and

$$\begin{aligned} \mu(Q(B)) &\le \sum_k \mu(B_k^{[1]}) \\ &\le \mathcal{K} \sum_k \mu(B_k) \\ &\le \frac{\mathcal{K}}{\lambda^r} \sum_k \int_{B_k} |f(t)|^r dt \\ &\le \frac{\mathcal{K}}{\lambda^r} \int_X |f(t)|^r dt. \end{aligned} \tag{4.3}$$

According to Lemma 3.2 there is a sequence of balls G_k such that

$$X = \bigcup_{n\ge 1}\bigcap_{k\ge n} G_k,$$

and so we get

$$E \subset \bigcup_{n\ge 1}\bigcap_{k\ge n} Q(G_k).$$

Hence we obtain

$$\mu^*(E) = \mu\left(\bigcup_{n\ge 1}\bigcap_{k\ge n} Q(G_k)\right) \le \frac{\mathcal{K}}{\lambda^r}\int_X |f(t)|^r dt$$

and so (4.2). □

Theorem 4.2. *If a subadditive operator T satisfies T1)-condition and the weak-L^r inequality, then T^* satisfies weak-L^r inequality too. Moreover, we have*

$$\|T^*\|_{L^r\to L^{r,\infty}} \lesssim \mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}.$$

Proof. Given $\lambda > 0$ consider the set

$$E = \{x \in X : T^*f(x) > \lambda\},$$

which can be non-measurable. For any $x \in E$ there is a ball $B(x) \in \mathfrak{B}$ such that

$$x \in B(x), \quad |T(f\cdot \mathbb{I}_{X\setminus B^{[1]}(x)})(x)| > \lambda. \tag{4.4}$$

One can check that $E = \cup_{x\in E} B(x)$. Given ball B apply Lemma 3.1, we find a sequence $x_k \in E$ such that the balls $\{B_k = B(x_k)\}$ are pairwise disjoint and

$$E \cap B \subset \bigcup_k B_k^{[1]} = Q(B). \tag{4.5}$$

Since T satisfies T1)-condition, we have

$$|T(f\cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x_k) - T(f\cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x)| \le \mathcal{L}_1 \cdot \langle f\rangle^*_{B_k,r}, \quad x \in B_k.$$

Thus, one can easily conclude from (4.4) that

$$\begin{aligned} |T(f\cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x)| &\ge |T(f\cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x_k)| \\ &\quad - |T(f\cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x_k) - T(f\cdot \mathbb{I}_{X\setminus B_k^{[1]}})(x)| \\ &\ge \lambda - \mathcal{L}_1 \cdot \langle f\rangle^*_{B_k,r}, \quad x \in B_k. \end{aligned} \tag{4.6}$$

Given $\beta > 0$ define

$$\tilde{B}_k = \{x \in B_k : |T(f\cdot \mathbb{I}_{B_k^{[1]}})(x)| < \beta \cdot \langle f\rangle^*_{B_k,r}\}. \tag{4.7}$$

Using weak-L^r inequality of the operator T, the measure of the complement of $\tilde{B}_k$ is estimated by

$$\mu^*(\tilde{B}_k^c) \le \frac{\|T\|^r_{L^r\to L^{r,\infty}}}{(\beta\cdot\langle f\rangle^*_{B_k,r})^r}\cdot\int_{B_k^{(1)}}|f|^r \le \left(\frac{\|T\|_{L^r\to L^{r,\infty}}}{\beta}\right)^r\mu(B_k^{(1)})$$
$$\lesssim \left(\frac{\|T\|_{L^r\to L^{r,\infty}}}{\beta}\right)^r\mu(B_k)$$

and so for an appropriate constant $\beta\sim\|T\|_{L^r\to L^{r,\infty}}$ we have

$$\mu^*(\tilde{B}_k)\ge\mu(B_k)-\mu^*(B_k\setminus\tilde{B}_k)\ge\mu(B_k)-\mu^*(\tilde{B}_k^c)\ge\frac{1}{2}\mu(B_k).$$

If

$$x\in\tilde{B}_k\setminus\{M_rf(x)>\delta\lambda\},$$

then, using subadditivity of T together with relations (4.7), (4.6), we obtain

$$\begin{aligned}|Tf(x)|&\ge|T(f\cdot\mathbb{I}_{X\setminus B_k^{[1]}})(x)|-|T(f\cdot\mathbb{I}_{B_k^{[1]}})(x)|\\&\ge\lambda-\mathcal{L}_1\cdot\langle f\rangle^*_{B_k,r}-\beta\cdot\langle f\rangle^*_{B_k,r}\\&\ge\lambda-(\mathcal{L}_1+\beta)\cdot M_rf(x)\\&\ge\lambda-(\mathcal{L}_1+\beta)\delta\lambda\\&\ge\lambda/2,\end{aligned}$$

where the last inequality can be satisfied for

$$\delta=1/2(\mathcal{L}_1+\beta)\sim(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})^{-1}. \tag{4.8}$$

Hence we conclude

$$\bigcup_k\tilde{B}_k\subset\{M_rf(x)>\delta\lambda\}\bigcup\{|Tf(x)|>\lambda/2\}. \tag{4.9}$$

Since the maximal function M_r and the operator T satisfy weak-L^r bound, from (4.5), (4.8) and (4.9) we get

$$\begin{aligned}\mu(Q(B))&\le\sum_k\mu(B_k^{[1]})\\&\le\mathcal{K}\cdot\sum_k\mu(B_k)\\&\le2\mathcal{K}\cdot\sum_k\mu^*(\tilde{B}_k)\\&\lesssim\mu^*\{M_rf(x)>\delta\lambda\}+\mu^*\{|Tf(x)|>\lambda/2\}\\&\lesssim(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})^r\frac{1}{\lambda^r}\int_X|f|^r.\end{aligned}$$

The same argument used at the end of the proof of Theorem 4.1 implies

$$\mu^*(E) \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}})^r \frac{1}{\lambda^r}\int_X |f|^r,$$

and so the theorem is proved. □

Definition 4.1. *Let T be a subadditive operator. Given balls A and B with $A \subset B$ we denote*

$$\Delta(A,B) = \Delta_T(A,B) = \sup_{x\in A,\, f\in L^r(X)} \frac{|T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)|}{\langle f\rangle_{B^{[1]},r}}.$$

Notice that T2)-condition for a subadditive operator T means that for any $A \in \mathfrak{B}$ there exists a ball B such that $A \subsetneq B$ and $\Delta(A,B) \le \mathcal{L}_2$.

Lemma 4.1. *If T is an arbitrary subadditive operator, then for any balls A, B and C satisfying $A \subset B \subset C$ we have*

$$\Delta(A,B) \le \Delta(A,C). \tag{4.10}$$

Proof. Given function $f \in L^r(X)$ denote

$$g(x) = f(x)\cdot \mathbb{I}_{B^{[1]}}(x).$$

Then we get the estimate

$$\begin{aligned}\sup_{x\in A}|T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)| &= \sup_{x\in A}|T(g\cdot \mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x)|\\ &\le \Delta(A,C)\cdot \langle g\rangle_{C^{[1]},r}\\ &= \Delta(A,C)\cdot\left(\frac{1}{\mu(C^{[1]})}\int_{B^{[1]}}|f|^r\right)^{1/r}\\ &\le \Delta(A,C)\cdot \langle f\rangle_{B^{[1]},r},\end{aligned}$$

which implies (4.10). □

Lemma 4.2. *Let a subadditive operator T satisfy T1)-condition and the weak-L^r bound. Then for any balls A, B satisfying $A \subset B$ we have*

$$\Delta(A,B) \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}})\left(\frac{\mu(B)}{\mu(A)}\right)^{1/r}. \tag{4.11}$$

Proof. Applying the weak-L^r estimate, we get

$$\mu^*\left\{x\in A:\, |T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)| > \|T\|_{L^r\to L^{r,\infty}}\left(\frac{2}{\mu(A)}\int_{B^{[1]}}|f|^r\right)^{1/r}\right\} \le \frac{\mu(A)}{2}$$

and so we find a point $x_0 \in A$ such that

$$|T(f \cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})\}(x_0)| \le \|T\|_{L^r\to L^{r,\infty}} \left(\frac{2}{\mu(A)}\int_{B^{[1]}} |f|^r\right)^{1/r} \lesssim \|T\|_{L^r\to L^{r,\infty}} \cdot \left(\frac{\mu(B)}{\mu(A)}\right)^{1/r} \cdot \langle f\rangle_{B^{[1]},r}. \tag{4.12}$$

According to T1)-condition, for any $x \in A$ we have

$$|T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})\}(x) - T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})\}(x_0)| \le \mathcal{L}_1 \cdot \langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle^*_{A,r}. \tag{4.13}$$

By the definition of $\langle f\rangle^*_{A,r}$ there is a ball $C \supset A$ such that

$$\langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle^*_{A,r} = \langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C,r}.$$

If $\mu(C) \le \mu(B^{[1]})$, then we have $C \subset B^{[2]}$ and therefore

$$\begin{aligned}\langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C,r} &\le \left(\frac{\mu(B^{[2]})}{\mu(C)}\right)^{1/r} \langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{B^{[2]},r}\\ &= \left(\frac{\mu(B^{[2]})}{\mu(C)}\right)^{1/r} \langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{B^{[1]},r}\\ &\lesssim \left(\frac{\mu(B)}{\mu(A)}\right)^{1/r} \cdot \langle f\rangle_{B^{[1]},r}.\end{aligned} \tag{4.14}$$

In the case of $\mu(C) > \mu(B^{[1]})$, we have $B^{[1]} \subset C^{[1]}$. Hence we get

$$\begin{aligned}\langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C,r} &\le \left(\frac{\mu(C^{[1]})}{\mu(C)}\right)^{1/r} \langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C^{[1]},r}\\ &\lesssim \langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{C^{[1]},r}\\ &\le \langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle_{B^{[1]},r}\\ &\le \langle f\rangle_{B^{[1]},r}.\end{aligned} \tag{4.15}$$

The estimates (4.14) and (4.15) imply the inequality

$$\langle f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}\rangle^*_{A,r} \lesssim \left(\frac{\mu(B)}{\mu(A)}\right)^{1/r} \cdot \langle f\rangle_{B^{[1]},r},$$

which together with (4.12) and (4.13) gives

$$|T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})\}(x)| \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \left(\frac{\mu(B)}{\mu(A)}\right)^{1/r} \langle f\rangle_{B^{[1]},r}, \quad x \in A.$$

The last inequality completes the proof of lemma. □

Lemma 4.3. *If a subadditive operator T satisfies T1)-condition and the weak-L^r bound, then for any balls A, B and C satisfying $A \subset B \subset C$ we have*

$$\Delta(A,C) \lesssim \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \cdot (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(A,B)). \tag{4.16}$$

Proof. According to Lemma 4.2 we have

$$\Delta(B,C) \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r}.$$

Thus, taking $f \in L^r(X)$ and $x \in A$, we obtain

$$\begin{aligned}
&|T(f\cdot \mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x)| \\
&\qquad \le |T(f\cdot \mathbb{I}_{C^{[1]}\setminus B^{[1]}})(x)| + |T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)| \\
&\qquad \le \Delta(B,C)\langle f\rangle_{C^{[1]},r} + \Delta(A,B)\langle f\rangle_{B^{[1]},r} \\
&\qquad \lesssim (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \langle f\rangle_{C^{[1]},r} + \Delta(A,B)\langle f\rangle_{B^{[1]},r} \\
&\qquad \le (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}}) \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \langle f\rangle_{C^{[1]},r} \\
&\qquad\qquad + \Delta(A,B) \left(\frac{\mu(C^{[1]})}{\mu(B^{[1]})}\right)^{1/r} \langle f\rangle_{C^{[1]},r} \\
&\qquad \lesssim \langle f\rangle_{C^{[1]},r} \left(\frac{\mu(C)}{\mu(B)}\right)^{1/r} \cdot (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(A,B))
\end{aligned}$$

and so we get (4.16). □

Inequality (3.1) and Lemma 4.3 immediately yield

Lemma 4.4. *Let an operator $T \in \mathrm{BO}_{\mathfrak{B}}$ and satisfy the weak-L^r bound. Then for any balls $A, B \in \mathfrak{B}$ with $A \subset B$ we have*

$$\Delta(A, B^{[n]}) \lesssim \mathcal{K}^{n/r}(\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(A,B)).$$

Theorem 4.3. *If an operator $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfies weak-L^r estimate, then the operator $T^* \in \mathrm{BO}_{\mathfrak{B}}$ and satisfies weak-L^r inequality. Moreover, we have*

$$\begin{aligned}
&\mathcal{L}_1(T^*) \lesssim \mathcal{L}_1(T) + \|T\|_{L^r\to L^{r,\infty}}, \\
&\mathcal{L}_2(T^*) = \mathcal{L}_2(T).
\end{aligned}$$

Proof. One can easily check that for any balls A, B satisfying $A \subset B$ we have the equality

$$\Delta_{T^*}(A,B) = \Delta_T(A,B).$$

If balls A and B satisfy T2)-condition for the operator T, then the same conditions will hold also for T^* with $\mathcal{L}_2(T^*) = \mathcal{L}_2(T)$. To prove T1)-condition, let $B \in \mathfrak{B}$ and $f \in L^r(X)$ satisfy

$$\operatorname{supp} f \in X \setminus B^{[1]}. \tag{4.17}$$

Take arbitrary points $x, x' \in B$ and estimate $|T^*f(x) - T^*f(x')|$. If $T^*f(x) = T^*f(x')$, then the estimation is trivial. So we can suppose that $T^*f(x) > T^*f(x')$. Using the definition of $T^*f(x)$, we find a ball $A \in \mathfrak{B}$ with $x \in A$ such that

$$\frac{T^*f(x) + T^*f(x')}{2} < |T(f \cdot \mathbb{I}_{X \setminus A^{[1]}})(x)|. \tag{4.18}$$

Denote

$$A' = \begin{cases} B & \text{if } \mu(A) < \mu(B), \\ A^{[1]} & \text{if } \mu(A) \ge \mu(B). \end{cases} \tag{4.19}$$

Since $x, x' \in B$, from Lemma 4.2 and (4.17) it follows that

(4.20)

$$\begin{aligned} |T(f \cdot \mathbb{I}_{B^{[2]} \setminus A^{[1]}})(x)| &= |T(f \cdot \mathbb{I}_{B^{[2]} \setminus A^{[1]}} \cdot \mathbb{I}_{B^{[2]} \setminus B^{[1]}}))(x)| \\ &\le \Delta(B, B^{[1]}) \langle f \cdot \mathbb{I}_{B^{[2]} \setminus A^{[1]}} \rangle_{B^{[2]}, r} \\ &\le \Delta(B, B^{[1]}) \langle f \rangle_{B^{[2]}, r} \\ &\lesssim \left(\frac{\mu(B^{[1]})}{\mu(B)} \right)^{1/r} (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}}) \langle f \rangle^*_{B,r}, \\ &\lesssim (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}}) \langle f \rangle^*_{B,r}, \end{aligned}$$

and similarly

$$|T(f \cdot \mathbb{I}_{B^{[2]} \setminus A'^{[1]}}(x')| \lesssim (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}}) \langle f \rangle^*_{B,r}. \tag{4.21}$$

One can easily check that from (4.19) it follows that $B \subset A'$ and so $x' \in A'$. This implies

$$T^*f(x') \ge |T(f \cdot \mathbb{I}_{X \setminus A'^{[1]}})(x')|,$$

which together with (4.18) yields

(4.22)

$$\begin{aligned} |T^*f(x) - T^*f(x')| &= T^*f(x) - T^*f(x') \\ &\le 2|T(f \cdot \mathbb{I}_{X \setminus A^{[1]}})(x)| - 2T^*f(x') \\ &\le 2\Big(|T(f \cdot \mathbb{I}_{X \setminus A^{[1]}})(x)| - |T(f \cdot \mathbb{I}_{X \setminus A'^{[1]}})(x')| \Big). \end{aligned}$$

In the case $\mu(A) < \mu(B)$ we get $A \subset B^{[1]}$ and therefore

$$A'^{[1]} = B^{[1]} \subset B^{[2]}, \quad A^{[1]} \subset B^{[2]}.$$

Thus, applying T1)-condition for T, from (4.20), (4.21) and (4.22) we conclude

$$\begin{aligned} |T^*f(x) - T^*f(x')| &\le 2\bigg(|T(f \cdot \mathbb{I}_{X\setminus B^{[2]}})(x)| + |T(f \cdot \mathbb{I}_{B^{[2]}\setminus A^{[1]}})(x)| \\ &\qquad - |T(f \cdot \mathbb{I}_{X\setminus B^{[2]}})(x')| + |T(f \cdot \mathbb{I}_{B^{[2]}\setminus A'^{[1]}})(x')|\bigg) \\ &\lesssim |T(f \cdot \mathbb{I}_{X\setminus B^{[2]}})(x) - T(f \cdot \mathbb{I}_{X\setminus B^{[2]}})(x')| \\ &\qquad + (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}})\langle f\rangle^*_{B,r} \\ &\lesssim (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}})\langle f\rangle^*_{B,r}. \end{aligned}$$

If $\mu(A) \ge \mu(B)$, then by (4.19) we have $A' = A^{[1]}$ and so $x, x' \in B \subset A^{[1]}$, $A'^{[1]} = A^{[2]}$. Thus, applying Lemma 4.2 and (4.22), we get

$$\begin{aligned} |T^*f(x) - T^*f(x')| &\le 2\bigg(|T(f \cdot \mathbb{I}_{X\setminus A^{[1]}})(x)| - |T(f \cdot \mathbb{I}_{X\setminus A^{[2]}})(x')|\bigg) \\ &\le 2\bigg(|T(f \cdot \mathbb{I}_{X\setminus A^{[2]}})(x)| - |T(f \cdot \mathbb{I}_{X\setminus A^{[2]}})(x')|\bigg) \\ &\qquad + 2|T(f \cdot \mathbb{I}_{A^{[2]}\setminus A^{[1]}})(x)| \\ &\lesssim \mathcal{L}_1(T)\langle f\rangle^*_{A^{[1]},r} + (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}})\langle f\rangle_{A^{[2]},r} \\ &\le (\mathcal{L}_1(T) + \|T\|_{L^r \to L^{r,\infty}})\langle f\rangle^*_{B,r}, \end{aligned}$$

and finally, T1)-condition. Weak-L^r bound of T^* follows from Theorem 4.2. □

We say that a ball-basis $\mathfrak{B}$ in a measure space satisfies the doubling condition if there is a constant $\eta > 1$ such that for any ball $A \in \mathfrak{B}$, $A^{[1]} \ne X$, one can find a ball B satisfying

$$A \subsetneq B, \quad \mu(B) \le \eta \cdot \mu(A). \tag{4.23}$$

Theorem 4.4. *Let the ball-basis $\mathfrak{B}$ satisfy the doubling condition. If a subadditive operator T satisfies T1)-condition and the weak-L^r bound, then $T \in \mathrm{BO}_{\mathfrak{B}}$. Moreover, we have*

$$\mathcal{L}_2(T) \lesssim \eta^{1/r}(\mathcal{L}_1 + \|T\|_{L^r \to L^{r,\infty}}),$$

where η is the doubling constant from (4.23).

Proof. We need to check T2)-condition. If balls A and B satisfy conditions (4.23), then applying Lemma 4.2, we get

$$\Delta(A,B)\lesssim(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})\left(\frac{\mu(B)}{\mu(A)}\right)^{1/r}\le\eta^{1/r}(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}}).$$

Thus we get $\mathcal{L}_2(T)\lesssim\eta^{1/r}(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})<\infty$. □

5. Proof of main theorems

Lemma 5.1. *If a subadditive operator T satisfies T1)-condition and the weak-L^r bound, then for any $B\in\mathfrak{B}$ there exists a ball B' such that*

$$B^{[2]}\subset B', \tag{5.1}$$

$$\Delta(B^{[2]},B')\lesssim\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}}, \tag{5.2}$$

and we either have

$$B'^{[1]}=B' \text{ or } \mu(B')\ge 2\mu(B). \tag{5.3}$$

Proof. Letting

$$\mathfrak{A}=\{A\in\mathfrak{B}:\ A\cap B\neq\varnothing\},$$

we denote

$$a=\sup_{A\in\mathfrak{A}:\,\mu(B)\le\mu(A)\le2\mu(B)}\mu(A)\le2\mu(B), \tag{5.4}$$

$$b=\inf_{A\in\mathfrak{A}:\,\mu(A)>2\mu(B)}\mu(A)\ge2\mu(B). \tag{5.5}$$

Observe that there is no ball $A\in\mathfrak{A}$ with $a<\mu(A)<b$ and there exist balls $G_1,G_2\in\mathfrak{A}$ such that

$$\frac{a}{2}<\mu(G_1)\le a,$$
$$b\le\mu(G_2)<2b.$$

If $b\le\mathcal{K}^2a$, then we define $B'=G_2^{[3]}$. Since $B\cap G_2\neq\varnothing$ and $\mu(B)\le a\le b\le\mu(G_2)$, we get $B\subset G_2^{[1]}$ and therefore $B^{[2]}\subset G_2^{[3]}=B'$. Thus we get (5.1). Taking into account the inequalities

$$\mu(B')=\mu(G_2^{[3]})\le\mathcal{K}^3\mu(G_2)\le\mathcal{K}^3\cdot2b\le2a\mathcal{K}^5,$$
$$\mu(B^{[2]})\ge\mu(B)\ge\frac{a}{2},$$

from Lemma 4.2 it follows that

$$\Delta(B^{[2]},B')\lesssim\left(\frac{\mu(B')}{\mu(B^{[2]})}\right)^{1/r}(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})\lesssim\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}}.$$

Hence we obtain (5.2). Then by (5.5) we have

$$\mu(B')\ge \mu(G_2)\ge b\ge 2\mu(B) \tag{5.6}$$

and so we get the second relation in (5.3). Now suppose we have $b>\mathcal{K}^2a$. Define $B'=G_1^{[1]}\in\mathfrak{A}$. We have

$$\mu(B'^{[1]})\le \mathcal{K}^2\mu(G_1)\le a\mathcal{K}^2<b.$$

Since there is no ball from $\mathfrak{A}$ with a measure in the interval (a,b), we get $\mu(B'^{[1]})\le a$. Thus we get

$$B'^{[1]}\cap G_1\neq\varnothing,\quad \mu(B'^{[1]})\le 2\mu(G_1).$$

These relations imply $B'^{[1]}\subset G_1^{[1]}=B'$, that means $B'^{[1]}=B'$ and we get the first relation in (5.3). On the other hand by (5.4) we have

$$\mu(B)\le a\le 2\mu(G_1)\le 2\mu(B')\le 2a\le 4\mu(B).$$

That means $B\subset B'^{[1]}=B'$ and therefore $B^{[2]}\subset B'^{[2]}=B'$. Hence we obtain (5.1). Using Lemma 4.2 and (5.6), we get

$$\Delta(B^{[2]},B')\lesssim\left(\frac{\mu(B')}{\mu(B^{[2]})}\right)^{1/r}(\mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}})\lesssim \mathcal{L}_1+\|T\|_{L^r\to L^{r,\infty}},$$

that gives (5.2). Lemma is proved. □

Lemma 5.2. *Let $T\in \mathrm{BO}_{\mathfrak{B}}$ satisfy weak-L^r estimate and for a ball $B\in\mathfrak{B}$ we have $B^{[1]}=B$. Then there exists a ball B' satisfying (5.1) and*

$$\Delta(B^{[2]},B')\le \mathcal{L}_2, \tag{5.7}$$

$$\mu(B')>2\mu(B). \tag{5.8}$$

Proof. Applying T2)-condition, we find a ball $B'\supsetneq B$ such that

$$\Delta(B,B')\le\mathcal{L}_2. \tag{5.9}$$

From $B^{[1]}=B$ we get $B^{[2]}=B\subset B'$ that is (5.1). Then, applying (5.9), we get (5.7). Observe that (5.8) holds. Indeed, otherwise by two balls relation we will have $B'\subset B^{[1]}=B$, which contradicts the condition $B'\supsetneq B$. □

Lemma 5.3. *If $T\in\mathrm{BO}_{\mathfrak{B}}$ satisfies weak-L^r estimate, then for any $B\in\mathfrak{B}$ there exists a sequence of balls $B=B_0,B_1,B_2,\dots$ such that*

$$\bigcup_k B_k=X, \tag{5.10}$$

$$B_{k-1}^{[2]}\subset B_k,\quad k\ge 1, \tag{5.11}$$

$$\sup_{k\ge1}\Delta(B_{k-1}^{[2]},B_k)\lesssim \mathcal{L}_1+\mathcal{L}_2+\|T\|_{L^r\to L^{r,\infty}},\quad k\ge 1. \tag{5.12}$$

Proof. We construct a sequence of balls $\{B_k\}$ satisfying (5.11), (5.12) and the relation

$$\mu(B_k) > 2\mu(B_{k-2}), \quad k \ge 2. \tag{5.13}$$

We do it by induction. Take $B_0 = B$ and suppose we have already defined B_k for $k = 0, 1, \dots l$ satisfying the conditions (5.11), (5.12) and (5.13) for $k \le l$. If $B_l^{[1]} = B_l$, then applying Lemma 5.2, we get a ball $B_{l+1} = B'$ such that

$$\begin{aligned} &B_l^{[2]} \subset B_{l+1},\\ &\Delta(B_l^{[2]}, B_{l+1}) \le \mathcal{L}_2,\\ &\mu(B_{l+1}) \ge 2\mu(B_l). \end{aligned} \tag{5.14}$$

In the case of $B_l^{[1]} \ne B_l$ we use Lemma 5.1. At this time the ball $B_{l+1} = B'$ will satisfy (5.14) and the conditions

$$\begin{aligned} &\Delta(B_l^{[2]}, B_{l+1}) \lesssim \mathcal{L}_1 + \|T\|_{L^r \to L^{r,\infty}},\\ &\mu(B_{l+1}) \ge 2\mu(B_l). \end{aligned} \tag{5.15}$$

Applying this process, we get a sequence satisfying the conditions (5.11) and (5.12). Besides, one can easily observe that at least for one of any two consecutive integers k we will have $\mu(B_{k+1}) \ge 2\mu(B_k)$. So the condition (5.13) will also be satisfied. □

Lemma 5.4. *Let T be a $\mathrm{BO}_{\mathfrak{B}}$ operator satisfying weak-L^r inequality. If*

$$\lambda \ge 3\mathcal{K}^4, \tag{5.16}$$

then for every measurable set $F \subset X$ and a ball $A \in \mathfrak{B}$ with

$$F \cap A \ne \varnothing, \quad \mu(F) \le \mu(A)/\lambda, \tag{5.17}$$

there exists a family of balls $\mathcal{G} \subset \mathfrak{B}$ satisfying the conditions

$$F \cap A^{[1]} \cap G \ne \varnothing \text{ if } G \in \mathcal{G}, \tag{5.18}$$

$$F \cap A^{[1]} \subset \bigcup_{G \in \mathcal{G}} G \text{ a.s.} \tag{5.19}$$

$$\mu\left(\bigcup_{G \in \mathcal{G}} G^{[1]}\right) \lesssim \frac{\mu(A)}{\lambda}. \tag{5.20}$$

Besides, for each $G\in \mathcal{G}$ there is a ball $\tilde{G}$ (not necessarily in $\mathcal{G}$) such that

$$\tilde{G}\not\subset F. \tag{5.21}$$
$$G^{[2]}\subset \tilde{G}\subset A^{[1]}, \tag{5.22}$$
$$\Delta(G^{[2]},\tilde{G})\lesssim \mathcal{L}_1+\mathcal{L}_2+\|T\|_{L^r\to L^{r,\infty}}. \tag{5.23}$$

Proof. Applying Lemma 3.5 for the set $E=F\cap A^{[1]}$ we find a family of balls $\mathcal{P}\subset \mathfrak{B}$ such that

$$F\cap A^{[1]}\cap B\neq\varnothing,\quad B\in\mathcal{P}, \tag{5.24}$$
$$F\cap A^{[1]}\subset \bigcup_{B\in\mathcal{P}} B \text{ a.s.} \tag{5.25}$$
$$\sum_{B\in\mathcal{P}}\mu(B)<2\mathcal{K}\mu(F\cap A^{[1]}). \tag{5.26}$$

Take an arbitrary element $B\in\mathcal{P}$. Applying Lemma 5.3, we find a sequence of balls $B_k\in\mathfrak{B}$, $k=0,1,2,\ldots$, $B=B_0$, with conditions (5.10)-(5.12). To $B\in\mathcal{P}$ we can attach a ball $G=B_m$, where $m\geq 0$ is the least index satisfying the relation

$$B^{[1]}_{m+1}\not\subset F. \tag{5.27}$$

The collection of all such G defines the family $\mathcal{G}$. If $G\in\mathcal{G}$ is generated by $B\in\mathcal{P}$, then combining the relation

$$B\subset B_0^{[2]}\subset B_m=G \tag{5.28}$$

with (5.24) we obtain (5.18). From (5.28) we also get

$$\bigcup_{G\in\mathcal{G}} G\supset \bigcup_{B\in\mathcal{P}} B,$$

which together with (5.25) implies (5.19). Then, according to the definition of the integer m (see (5.27)) we have either $G^{[1]}\subset F$ or $G\in\mathcal{P}$ (the second relation holds only if $m=0$). This remark together with

(5.26) implies

$$\mu\left(\bigcup_{G\in\mathcal{G}} G^{[1]}\right) \le \mu(F) + \mu\left(\bigcup_{B\in\mathcal{P}} B^{[1]}\right) \tag{5.29}$$
$$\le \mu(F) + \sum_{B\in\mathcal{P}} \mu\left(B^{[1]}\right)$$
$$\le \mu(F) + \mathcal{K}\sum_{B\in\mathcal{P}} \mu(B),$$
$$\le \frac{(2\mathcal{K}^2+1)\mu(A^{[1]})}{\lambda},$$
$$\le \frac{3\mathcal{K}^2\mu(A)}{\lambda},$$

and we get (5.20). Now define

$$\tilde{G} = \begin{cases} A^{[1]} & \text{if} \quad \mu(B^{[1]}_{m+1}) > \mu(A), \\ B^{[1]}_{m+1} & \text{if} \quad \mu(B^{[1]}_{m+1}) \le \mu(A). \end{cases} \tag{5.30}$$

According to (5.17), we have $\mu(A^{[1]}) > \mu(F)$ and so $A^{[1]} \not\subset F$. This together with (5.27) and (5.30) implies (5.21). To check condition (5.22) notice that form (5.16) and (5.29) it follows that

$$\mu(G^{[2]}) \le \mathcal{K}^2 \cdot \mu(G) \le \mathcal{K}^2 \cdot \frac{3\mathcal{K}^2\mu(A)}{\lambda} \le \mu(A).$$

Thus, since $G \cap A \ne \varnothing$, we conclude

$$G^{[2]} \subset A^{[1]}. \tag{5.31}$$

If $\mu(B^{[1]}_{m+1}) > \mu(A)$, then by (5.30) we have $\tilde{G} = A^{[1]}$ and using (5.31) we get (5.22). If $\mu(B^{[1]}_{m+1}) \le \mu(A)$, then since $B^{[1]}_{m+1} \cap A \ne \varnothing$, we have $B^{[1]}_{m+1} \subset A^{[1]}$. Hence from (5.11) and (5.30) we obtain

$$G^{[2]} = B^{[2]}_m \subset B^{[1]}_{m+1} = \tilde{G} \subset A^{[1]}$$

and so (5.22). To prove (5.23) first we suppose that $\mu(B^{[1]}_{m+1}) \le \mu(A)$ and so $\tilde{G} = B^{[1]}_{m+1}$. Applying Lemma 4.4 and (5.12), we get

$$\begin{aligned} \Delta(G^{[2]}, \tilde{G}) &= \Delta(B^{[2]}_m, B^{[1]}_{m+1}) \\ &\lesssim \left(\frac{\mu(B^{[1]}_{m+1})}{\mu(B_{m+1})}\right)^{1/r} (\mathcal{L}_1 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(B^{[2]}_m, B_{m+1})) \\ &\lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}}. \end{aligned}$$

In the case $\mu(B^{[2]}_{m+1}) > \mu(A^{[1]})$ we have $\tilde{G} = A^{[1]} \subset B^{[3]}_{m+1}$ and therefore $\tilde{G}^{[1]} \subset B^{[4]}_{m+1}$. Once again applying Lemma 4.4, we obtain

$$\begin{aligned}\Delta(G^{[2]}, \tilde{G}) &\le \Delta(B^{[2]}_m, B^{[3]}_{m+1})\\ &\lesssim \left(\frac{\mu(B^{[2]}_{m+1})}{\mu(B_{m+1})}\right)^{1/r} (\mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}} + \Delta(B^{[2]}_m, B_{m+1}))\\ &\lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}},\end{aligned}$$

which completes the proof of lemma. □

Definition 5.1. *We say a set of balls $\mathfrak{A}$ is a family-tree if*

F1) there is an element $A_0 \in \mathfrak{A}$ called grandparent of $\mathfrak{A}$,
F2) to each $A \in \mathfrak{A}$ except the grandparent A_0 a unique parent $\mathrm{pr}(A) \in \mathfrak{A}$ is attached,
F3) for each $A \in \mathfrak{A}$, $A \neq A_0$ we have $A_0 = \mathrm{pr}^n(A) = \mathrm{pr}(\mathrm{pr}(\ldots\mathrm{pr}(A)\ldots))$ for some $n \in \mathbb{N}$.

Given ball $A \in \mathfrak{A}$ we denote

$$\begin{aligned}\mathfrak{Ch}_n(A) &= \{B \in \mathfrak{A} : \mathrm{pr}^n(B) = A\}, \quad n = 1, 2, \ldots\\ \mathfrak{Gen}(A) &= \bigcup_{n=1}^{\infty} \mathfrak{Ch}_n(A),\end{aligned}$$

where M is the maximal operator (4.1). The family $\mathfrak{Ch}(A) = \mathfrak{Ch}_1(A)$ is said to be the children of A and $\mathfrak{Gen}(A)$ is the generation of A.

The notation $n \ll m$ ($n \gg m$) for two integers n, m denotes $n < m - 1$ ($n > m + 1$) and $n \asymp m$ stands for the condition $|m - n| \le 1$.

Proof of Theorem 1.1. Let an operator $T \in \mathrm{BO}_{\mathfrak{B}}$ satisfy weak-L^r inequality. Define

$$\Gamma f(x) = \max\{|Tf(x)|, T^*f(x), \mathcal{L} \cdot M_r f(x)\},$$

where

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}}.$$

Applying Theorem 4.1 and Theorem 4.2, we conclude that the operator Γ satisfy weak-L^r estimate and besides

$$\|\Gamma\|_{L^r\to L^{r,\infty}} \lesssim \mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^r\to L^{r,\infty}}. \tag{5.32}$$

Denote

$$T^{**}f(x) = \sup_{B\in\mathfrak{B},\, x\in B} |T(f \cdot \mathbb{I}_{B^{[1]}})(x)|. \tag{5.33}$$

Subadditivity of T implies

$$T^{**}f(x) \le T^*f(x) + |Tf(x)| \le 2\Gamma f(x), \quad x \in X. \tag{5.34}$$

Let $f \in L^r(X)$ and B be the ball from the statement of theorem. Clearly we can chose a ball A_0 such that

$$B^{[1]} \subset A_0, \quad \int_{A_0} |f| > \frac{\|f\|_1}{2}. \tag{5.35}$$

Let $\lambda > 0$ satisfy (5.16). We shall construct a family-tree $\mathfrak{A} \subset \mathfrak{B}$ with the grandparent A_0 such that

1) If $G \in \mathfrak{Ch}(A)$, then $A^{[1]} \cap G \neq \varnothing$.
2) We have

$$\mu\left(\bigcup_{G\in\mathfrak{Ch}(A)} G^{[1]}\right) \lesssim \frac{\mu(A)}{\lambda}. \tag{5.36}$$

3) If $G \in \mathfrak{Ch}(A)$, then there exist a ball $\tilde{G}$ and a point $\xi \in \tilde{G}$ such that

$$G^{[2]} \subset \tilde{G} \subset A^{[1]}, \tag{5.37}$$

$$\Gamma f_A(\xi) \lesssim \mathcal{L}\lambda \cdot \langle f\rangle_{A^{[3]},r}, \tag{5.38}$$

$$|T(f \cdot \mathbb{I}_{\tilde{G}^{[1]}\setminus G^{[3]}})(x)| \lesssim \mathcal{L}\lambda \cdot \langle f\rangle_{A^{[3]},r},\ x \in G^{[2]}, \tag{5.39}$$

where

$$f_A = \begin{cases} f \cdot \mathbb{I}_{A^{[3]}} \text{ if } & A \neq A_0, \\ f \text{ if } & A = A_0. \end{cases} \tag{5.40}$$

4) We have

$$|\Gamma f_A(x)| \lesssim \mathcal{L}\lambda \cdot \langle f\rangle_{A^{[3]},r}, \tag{5.41}$$

$$\text{for almost all } x \in A^{[1]} \setminus \bigcup_{G\in\mathfrak{Ch}(A)} G,\ \text{if } A \neq A_0.$$

The elements of $\mathfrak{A}$ will be determined inductively by an increasing order of generations levels. The first element of $\mathfrak{A}$ is A_0. Then we suppose inductively that we have already defined all the members of

$$\bigcup_{1\le n\le l} \mathfrak{Ch}_n(A)$$

such that any member

$$A \in \bigcup_{1\le n\le l-1} \mathfrak{Ch}_n(A)$$

satisfies the conditions 1)-4). To define the members of $\mathfrak{Ch}_{l+1}(A_0)$ we take an arbitrary $A \in \mathfrak{Ch}_l(A_0)$ and define the children of A as follows.

Take a measurable set $F = F_A$ such that

$$F \supset \{x \in X : \Gamma f_A(x) > \beta \cdot \langle f \rangle_{A^{[3]},r}\}, \tag{5.42}$$

$$\mu(F) = \mu^*\{x \in X : \Gamma f_A(x) > \beta \cdot \langle f \rangle_{A^{[3]},r}\}. \tag{5.43}$$

Suppose that $F \neq \varnothing$. Using (5.32), for any A (include the case $A = A_0$) we will have

$$\begin{aligned}\mu(F) &\le \frac{2\mu(A^{[3]})}{\beta^r}\|\Gamma\|^r_{L^r\to L^{r,\infty}}\\ &\le \frac{2\mathcal{K}^3\cdot\mu(A)}{\beta^r}\|\Gamma\|^r_{L^r\to L^{r,\infty}}\\ &\le \frac{\mu(A)}{\lambda},\end{aligned}$$

for a suitable constant

$$\beta \sim \mathcal{L}\lambda$$

since $r \ge 1$ and we have (5.32). Not that in the case $A = A_0$ one needs additionally use the inequality (5.35). Thus, applying Lemma 5.4 for A and $F = F_A$, we get a family $\mathcal{G}$ satisfying the conditions of the lemma. The family $\mathcal{G}$ will form the children collection of A, that is

$$\mathfrak{Ch}(A) = \mathcal{G}\,.$$

The relations 1) and 2) are immediate consequences of (5.18) and (5.20) respectively, while (5.37) follows from condition (5.22) of Lemma 5.4. From (5.21) it follows the existence of $\xi \in \tilde{G} \setminus F$ and by the definition (5.42) we get (5.38) ((5.57) if $A = A_0$). Since $\xi \in \tilde{G}$, using (5.22),(5.23) and (5.38), for $f \in L^r(X)$ we get the inequality

$$\begin{aligned}\sup_{x\in G^{[2]}}|T(f\cdot\mathbb{I}_{\tilde{G}^{[1]}\setminus G^{[3]}})(x)| &= \sup_{x\in G^{[2]}}|T(f\cdot\mathbb{I}_{A^{[3]}}\cdot\mathbb{I}_{\tilde{G}^{[1]}\setminus G^{[3]}})(x)|\\ &\le \Delta(G^{[2]},\tilde{G})\cdot\langle f\cdot\mathbb{I}_{A^{[3]}}\rangle_{\tilde{G}^{[1]},r}\\ &\lesssim \mathcal{L}\langle f\cdot\mathbb{I}_{A^{[3]}}\rangle^*_{\tilde{G},r}\\ &\le \mathcal{L}M_r(f\cdot\mathbb{I}_{A^{[3]}})(\xi)\\ &\le \Gamma(f\cdot\mathbb{I}_{A^{[3]}})(\xi) = \Gamma f_A(\xi)\\ &\le \beta\cdot\langle f\rangle_{A^{[3]},r}\\ &\lesssim \mathcal{L}\lambda\cdot\langle f\rangle_{A^{[3]},r},\end{aligned}$$

which implies (5.39). From (5.19) we get

$$\mu\left(F\bigcap\left(A^{[1]}\setminus\bigcup_{G\in\mathfrak{Ch}(A)}G\right)\right) = 0,$$

therefore by the definition of F we have (5.41). Hence, the properties of the family $\mathfrak{A}$ are satisfied.

Now we construct a sparse subfamily $\mathcal{S} \subset \mathfrak{A}$ consisting of countable collection of balls, which will satisfy the conditions of the theorem. We will do that by removing some elements of $\mathfrak{A}$. As we will see below, removing an element $A \in \mathfrak{A}$, we also remove all the elements of its generation $\mathfrak{Gen}(A)$. Thus one can easily check that the properties 1)-3) will hold during the whole process.

To start the description of the process we let $R = \mathcal{K}^2$, where $\mathcal{K} > 1$ is the constant from (1.6). For $B \in \mathfrak{B}$ denote

$$\mathrm{r}(B) = [\log_R \mu(B)]$$

Observe that the collections of balls

$$\begin{aligned}\mathfrak{A}_k &= \{B \in \mathfrak{A} : \mathrm{r}(B) = k\} \\ &= \left\{B \in \mathfrak{A} : R^k \le \mu(B) < R^{k+1}\right\}, \quad k \le k_0,\end{aligned}$$

gives a partition of $\mathfrak{A}$, i.e. we have $\mathfrak{A} = \cup_{k \le k_0} \mathfrak{A}_k$, where $k_0 = \mathrm{r}(A_0)$ and $A_{k_0} = \{A_0\}$. The reduction of the elements of $\mathfrak{A}$ will be realized in different stages. The content of $\mathfrak{A}_{k_0}$ will not be changed. In the n-th stage only the contents of the families $\mathfrak{A}_k$ with $k \le k_0 - n$ can be changed. Besides, at the end of the n-th stage $\mathfrak{A}_{k_0-n}$ will be fixed and remain the same till the end of the process. Suppose by induction the l-th stage of reduction has been already finished and so the families $\mathfrak{A}_k$, $k = k_0, k_0-1, k_0-2, \ldots, k_0-l$ have already fixed. In the next $(l+1)$-th stage we will apply the following two procedures consecutively:

Procedure 1. *Remove any element $G \in \mathfrak{A}_{k_0-l-1}$ together with all the elements of his generation $\mathfrak{Gen}(G)$ if there exists a $B \in \mathfrak{A}$ satisfying the conditions*

$$G^{[2]} \cap B \ne \varnothing, \tag{5.44}$$

$$\mathrm{r}(\mathrm{pr}^k(G)) \lll \mathrm{r}(B) \lll \mathrm{r}(\mathrm{pr}^{k+1}(G)), \tag{5.45}$$

for some integer $k \ge 0$.

Remark 5.1. *Observe that if an element G is removed because of a ball B satisfying the conditions (5.44) and (5.45) of Procedure 1, then we should have*

$$\mathrm{r}(G) \lll \mathrm{r}(B) \lll \mathrm{r}(\mathrm{pr}(G))$$

that means (5.45) can hold only with $k = 0$. Indeed, the left inequality immediately follows from (5.45). To prove the right one, suppose to *the contrary in (5.45) we have* $k \geq 1$. *Denote*

$$G' = \mathrm{pr}^k(G) \in \bigcup_{j=0}^{l} \mathfrak{A}_{k_0-j}.$$

Since $G'^{[2]} \supset G^{[2]}$ *(see (5.37)), we have* $G'^{[2]} \cap B \neq \varnothing$. *On the other hand (5.45) can be written by* $\mathrm{r}(G') \ll \mathrm{r}(B) \ll \mathrm{r}(\mathrm{pr}(G'))$. *We thus conclude that* G' *satisfies the conditions of the Procedure 1, so* G' *together with his generation* $\mathfrak{Gen}(G')$ *(include* G*) had to be removed in one of the previous stages of the process, when* B *was already fixed. This is a contradiction and so* $k = 0$.

Procedure 2. *Apply Lemma 3.1 to the rest of the elements* $\mathfrak{A}_{k_0-l-1}$ *having after Procedure 1. The application of lemma removes some elements of* $\mathfrak{A}_{k_0-l-1}$. *If an element* A *is removed, then the generation* $\mathfrak{Gen}(A)$ *will also be removed.*

Remark 5.2. *After the Procedure 2 the elements of* $\mathfrak{A}_{k_0-l-1}$ *become pairwise disjoint. Besides, we will have*

$$\bigcup_{G\in\mathfrak{A}_{k_0-l-1}(\text{before Procedure 2})} G \subset \bigcup_{G\in\mathfrak{A}_{k_0-l-1}(\text{after Procedure 2})} G^{[1]}. \tag{5.46}$$

After these two procedures the family $\mathfrak{A}_{k_0-l-1}$ will be fixed. Hence, finishing the induction process, we get the final state of $\mathfrak{A}$ which will be denoted by $\mathcal{D}$. Since after Procedure 2 in n-th stage $\mathfrak{A}_{k_0-n}$ gets countable number of balls so the family $\mathcal{D}$ will also be countable at the end of whole process.

Now we shall prove that for an admissible constant $\lambda > 0$ the family $\mathcal{D}$ is a union of two 1/2-sparse collections of balls. For $A \in \mathcal{D}$ define

$$E(A) = A \setminus \bigcup_{G\in\mathcal{D}:\, \mathrm{r}(G)\ll\mathrm{r}(A)} G = A \setminus \bigcup_{G\in\mathcal{D}:\, G\cap A\neq\varnothing,\, \mathrm{r}(G)\ll\mathrm{r}(A)} G. \tag{5.47}$$

Observe that

$$E(A) \cap E(B) = \varnothing, \text{ if } \mathrm{r}(A) \not\ll \mathrm{r}(B) \text{ or } \mathrm{r}(A) = \mathrm{r}(B). \tag{5.48}$$

Indeed, take arbitrary $A, B \in \mathcal{D}$. If $\mathrm{r}(A) = \mathrm{r}(B)$, then the balls A, B are result of the application of Procedure 2 (Lemma 3.1). That means we have $A \cap B = \varnothing$ and therefore according to (5.47) it follows that $E(A) \cap E(B) = \varnothing$. If $\mathrm{r}(A) \gg \mathrm{r}(B)$, then $E(A) \cap B = \varnothing$ immediately follows from the definition (5.47) and so we will get again $E(A) \cap E(B) = \varnothing$. To prove

$$\mu(E(A)) \geq \mu(A)/2 \tag{5.49}$$

take an arbitrary $A \in \mathcal{D}$ and denote

$$\mathcal{P} = \{P \in \mathcal{D} : \mathrm{r}(P) \asymp \mathrm{r}(A),\, P^{[2]} \cap A \neq \varnothing\}.$$

We have

$$R^{-3} \cdot \mu(A) \le \mu(P) \le R^3 \cdot \mu(A), \quad P \in \mathcal{P}. \tag{5.50}$$

On the other hand

$$\mathcal{P} \subset \mathfrak{A}_{l-1} \cup \mathfrak{A}_l \cup \mathfrak{A}_{l+1},$$

where $l = \mathrm{r}(A)$. Hence $\mathcal{P}$ consists of three families of pairwise disjoint balls. Thus, applying the remark after the Lemma 3.6, we get

$$\#\mathcal{P} \lesssim 1. \tag{5.51}$$

Suppose that $G \in \mathcal{D}$ satisfies

$$\mathrm{r}(G) \ll \mathrm{r}(A), \quad G \cap A \neq \varnothing,$$

and so $G^{[2]} \cap A \neq \varnothing$. Since G was not removed by Procedure 1, we have $\mathrm{r}(\mathrm{pr}^k(G)) \asymp \mathrm{r}(A)$ for some integer $k \ge 1$. Denote $P = \mathrm{pr}^k(G)$. We have $\mathrm{r}(P) \asymp \mathrm{r}(A)$ and $P^{[2]} \supset G^{[2]}$ by (5.37) and so $P^{[2]} \cap A \neq \varnothing$. This implies that $P \in \mathcal{P}$ and $G \in \mathfrak{Gen}(P)$. Hence, from (5.47), (5.36), (5.51) and (5.50) it follows that

$$\begin{aligned}\mu(A \setminus E_A) &\le \mu\left(\bigcup_{G\in\mathcal{D}:\, G\cap A\neq\varnothing,\, \mathrm{r}(G)\ll \mathrm{r}(A)} G\right)\\ &\le \mu\left(\bigcup_{P\in\mathcal{P}} \bigcup_{G\in\mathfrak{Gen}(P)} G\right) \le \sum_{P\in\mathcal{P}}\sum_{k=1}^{\infty} \mu\left(\bigcup_{G:\, \mathrm{pr}^k(G)=P} G\right)\\ &\lesssim \sum_{P\in\mathcal{P}}\sum_{k=1}^{\infty} \frac{\mu(P)}{\lambda^k} \lesssim \frac{\mu(A)}{\lambda}.\end{aligned}$$

Thus for an admissible constant λ we get (5.49). From (5.48) and (5.49) we conclude that two families

$$\mathcal{D}_1 = \{A \in \mathcal{D} : \mathrm{r}(A) \text{ is odd }\}, \quad \mathcal{D}_2 = \mathcal{D} \setminus \mathcal{D}_1,$$

are sparse. Further for an $A \in \mathcal{D}$ we will need the bound

$$\begin{aligned}&|\Gamma f_A(x)| \lesssim \mathcal{L}\lambda\langle f\rangle_{A^{[3]},r},\\ &\qquad \text{for a.a. } x \in A^{[1]} \setminus \bigcup_{G\in\mathcal{D}:\, \mathrm{r}(G)<\mathrm{r}(A)} G^{[1]},\end{aligned} \tag{5.52}$$

which is based on the inequality (5.41). It is enough to prove that

$$A^{[1]} \setminus \bigcup_{B\in\mathcal{D}:\, \mathrm{r}(B)<\mathrm{r}(A)} B^{[1]} \subset A^{[1]} \setminus \bigcup_{G\in\mathfrak{A}:\, G\in\mathfrak{Ch}(A)} G$$

or equivalently

$$D=\bigcup_{B\in\mathcal{D}\colon \mathrm{r}(B)<\mathrm{r}(A)} B^{[1]}\supset \bigcup_{G\in\mathfrak{A}\colon G\in\mathfrak{Ch}(A)} G. \tag{5.53}$$

Take $A\in\mathfrak{A}$ and arbitrary $G\in\mathfrak{Ch}(A)$. We have $\mathrm{r}(G)<\mathrm{r}(A)$. In the case $G\in\mathcal{D}$, that is G has not been removed during the Procedures 1 and 2, G is an element of the left union of (5.53) and so $G\subset D$. If $G\notin\mathcal{D}$, then G has been removed during the removal process. If G was removed by an application of Procedure 1, then there exists a ball $B\in\mathcal{D}$ such that $G^{[2]}\cap B\neq\varnothing$ and $\mathrm{r}(G)\ll\mathrm{r}(B)\ll\mathrm{r}(\mathrm{pr}(G))=\mathrm{r}(A)$ (see remark after the Procedure 1). From the inequality $R\geq\mathcal{K}^3$ it follows that

$$\mu(G^{[2]})\leq\mathcal{K}^2\mu(G)\leq\mathcal{K}^2\cdot\frac{\mu(B)}{\mathcal{K}^2}=\mu(B).$$

Thus we get $G\subset G^{[2]}\subset B^{[1]}$, which means $G\subset D$. If G was removed by an application of Procedure 2, then according to (5.46) we have $G\subset\cup_k B_k^{[1]}$ for a family of balls B_k satisfying $\mathrm{r}(B_k)=\mathrm{r}(G)<\mathrm{r}(A)$ and so $B_k^{[1]}\subset D$. This again implies $G\subset D$ and so we get (5.52). To prove the theorem we need to prove

$$\begin{aligned}|Tf(x)|&\lesssim\mathcal{L}\lambda\cdot\mathcal{A}_{\mathcal{S},r}f(x)\\&=\mathcal{L}\lambda\cdot\sum_{S\in\mathcal{S}}\langle f\rangle_{S,r}\cdot\mathbb{I}_S(x)\text{ a.e. } x\in X,\end{aligned}\tag{5.54}$$

where $\mathcal{S}=\{S^{[3]}:S\in\mathcal{D}\}$ clearly is a union of two sparse collections. Observe that the set

$$E=\bigcap_{k\leq k_0}\bigcup_{G\in\mathcal{D}\colon \mathrm{r}(G)\leq k}A^{[1]}$$

has zero measure, since $\mathcal{D}$ consists of countable number of balls with bounded sum of their measures. Besides we can fix a set F of zero measure such that for each $A\in\mathcal{D}$ inequality (5.52) holds for any

$$x\in\left(A^{[1]}\setminus\bigcup_{G\in\mathcal{D}\colon \mathrm{r}(G)<\mathrm{r}(A)}G^{[1]}\right)\setminus F.$$

Hence, it is enough to prove the bound (5.54) for arbitrary $x\in A_0\setminus(E\cup F)$. Observe that for such x there exists a ball $A\in\mathcal{D}$ such that

$$x\in\left(A^{[1]}\setminus\bigcup_{G\in\mathcal{D}\colon \mathrm{r}(G)<\mathrm{r}(A)}G^{[1]}\right),$$

and so from (5.52) we conclude

$$|Tf_A(x)| \le |\Gamma f_A(x)| \lesssim \mathcal{L}\lambda\langle f\rangle_{A^{[3]},r}. \tag{5.55}$$

According to the property 1) one can find a unique sequence of balls $A_0, A_1, A_2, \dots A_k = A$ in $\mathcal{D}$ such that $A_j = \mathrm{pr}(A_{j+1})$. According to the properties (5.37)-(5.39) there exist balls $\tilde{A}_j$ and points ξ_j, $j = 0, 1, \dots, k-1$, such that

$$A_{j+1}^{[3]} \subset \tilde{A}_{j+1}^{[1]} \subset A_j^{[2]}, \tag{5.56}$$

$$\Gamma f_A(\xi_j) \lesssim \mathcal{L}\lambda\langle f\rangle_{A_j^{[3]},r}, \quad \xi_j \in \tilde{A}_{j+1}, \tag{5.57}$$

$$|T(f\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}\setminus A_{j+1}^{[3]}})(t)| \lesssim \mathcal{L}\lambda\langle f\rangle_{A_j^{[3]},r}, \quad t \in A_{j+1}^{[2]}. \tag{5.58}$$

Since

$$x \in A^{[1]} = A_k^{[1]} \subset A_{j+1}^{[2]},$$

the condition (5.58) holds for $t = x$. Thus, we get

$$T(f\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}\setminus A_{j+1}^{[3]}})(x)| \lesssim \mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]}}. \tag{5.59}$$

We claim that

$$|Tf_{A_j}(x)| \le C\mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r} + |Tf_{A_{j+1}}(x)|, \tag{5.60}$$

where $C > 1$ is an admissible constant. Indeed, by T1)-condition and (5.57), we will have

$$\begin{aligned}\left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus\tilde{A}_{j+1}^{[1]}})(x) - T(f_{A_j}\cdot \mathbb{I}_{X\setminus\tilde{A}_{j+1}^{[1]}})(\xi_j)\right| &\lesssim \mathcal{L}\langle f_{A_j}\rangle^*_{\tilde{A}_{j+1},r} \\ &\le \mathcal{L}M_r f_{A_j}(\xi_j) \le \Gamma f_{A_j}(\xi_j) \lesssim \mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r}.\end{aligned} \tag{5.61}$$

Besides, from (5.59) we get

$$\begin{aligned}\left|T(f_{A_j}\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}\setminus A_{j+1}^{[3]}})(x)\right| &= \left|T(f\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}\setminus A_{j+1}^{[3]}})(x)\right| \\ &\lesssim \mathcal{L}\lambda\cdot\langle f\rangle_{A_j^{[3]},r}.\end{aligned} \tag{5.62}$$

From the definition of f_{A_j} (see (5.40)) and (5.56) it follows that

$$f_{A_j}\cdot \mathbb{I}_{X\setminus\tilde{A}_{j+1}^{[1]}} = f\cdot \mathbb{I}_{A_j^{[3]}\setminus\tilde{A}_{j+1}^{[1]}} = f\cdot \mathbb{I}_{A_j^{[3]}} - f\cdot \mathbb{I}_{\tilde{A}_{j+1}^{[1]}}. \tag{5.63}$$

Thus, applying (5.34), (5.61), (5.57), (5.62) and (5.63), we conclude

$$\begin{aligned}
\left|Tf_{A_j}(x)\right| &= \left|Tf_{A_j}(x)\right| \\
&\le \left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}^{[1]}_{j+1}})(x)\right| + \left|T(f_{A_j}\cdot \mathbb{I}_{\tilde{A}^{[1]}_{j+1}})(x)\right| \\
&\le \left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}^{[1]}_{j+1}})(x) - T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}^{[1]}_{j+1}})(\xi_j)\right| + \left|T(f_{A_j}\cdot \mathbb{I}_{X\setminus \tilde{A}^{[1]}_{j+1}})(\xi_j)\right| \\
&\quad + \left|T(f_{A_j}\cdot \mathbb{I}_{\tilde{A}^{[1]}_{j+1}\setminus A^{[3]}_{j+1}})(x)\right| + \left|T(f_{A_j}\cdot \mathbb{I}_{A^{[3]}_{j+1}})(x)\right| \\
&\le C\mathcal{L}\lambda\cdot \langle f\rangle_{A^{[3]}_j,r} + \left|Tf_{A_j}(\xi_j)\right| \\
&\quad + \left|T(f\cdot \mathbb{I}_{\tilde{A}^{[1]}_{j+1}})(\xi_j)\right| + \left|Tf_{A_{j+1}}(x)\right| \\
&\le C\mathcal{L}\lambda\cdot \langle f\rangle_{A^{[3]}_j,r} \\
&\quad + 2T^{**}f_{A_j}(\xi_j) + \left|Tf_{A_{j+1}}(x)\right| \\
&\le C\mathcal{L}\lambda\cdot \langle f\rangle_{A^{[3]}_j,r} \\
&\quad + 4\Gamma f_{A_j}(\xi_j) + \left|Tf_{A_{j+1}}(x)\right| \\
&\le C\mathcal{L}\lambda\cdot \langle f\rangle_{A^{[3]}_j,r} + \left|Tf_{A_{j+1}}(x)\right|,
\end{aligned}$$

where $C>0$ is an admissible constant that can vary in the above inequalities. Thus we get (5.60). Applying (5.60) for each $j=0,1,2,\ldots,k-1$, (5.40) and (5.55), we get

$$\begin{aligned}
|Tf(x)| = |Tf_{A_0}(x)| &\le C\mathcal{L}\lambda\cdot\sum_{j=0}^{k-1}\langle f\rangle_{A^{[3]}_j,r} + |T(f\cdot\mathbb{I}_{A^{[3]}})(x)| \\
&\lesssim \mathcal{L}\cdot\sum_{j=0}^{k}\langle f\rangle_{A^{[3]}_j,r} \\
&\lesssim \mathcal{L}\cdot\mathcal{A}_{\mathcal{S},r}f(x).
\end{aligned}$$

completing the proof of Theorem 1.1. □

Proof of Theorem 1.2. Theorem 1.2 immediately follows from Theorem 1.1, Theorem 4.2 and Theorem 4.3, □

6. Weighted estimates in abstract measure spaces

6.1. **The general case.** Let w satisfy A_p-condition. We denote $q=\frac{p}{p-1}$ and let $\sigma=w^{-\frac{1}{p-1}}$ be the dual weight of w. Note that if $w\in A_p$, then $\sigma=w^{1/(1-p)}\in A_q$ and

$$[\sigma]_{A_q}=[w]_{A_p}^{1/(p-1)}. \tag{6.1}$$

Besides we have

$$[w]_{A_p} = \sup_{B\in\mathfrak{B}} \frac{w(B)}{\mu(B)} \left(\frac{\sigma(B)}{\mu(B)}\right)^{p-1}.$$

The notation dw in the integrals will stand for $wd\mu$, where μ is the basic measure. Any weight w defines a measure on the basic measurable space $(X, \mathfrak{M})$ and the w-measure of a set E is defined as $w(E) = \int_E dw$. Everywhere below we denote by c_p different constants depending only on $1 < p < \infty$. In this section we shall consider maximal functions with respect to different measures. So we denote the maximal function associated to a measure β by

$$M_\beta f(x) = \sup_{B\in\mathfrak{B}:\, x\in B} \frac{1}{\mu(B)} \int_B |f(t)| d\beta(t).$$

Recall that $\mathcal{A}_{\mathcal{S}}$ denotes the sparse operator corresponding to the case of $r = 1$.

Lemma 6.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$. If $\mathcal{S} \subset \mathfrak{B}$ is a γ-sparse collection $(0 < \gamma < 1)$ and the weight w satisfies the A_p condition for $1 < p \le 2$, then*

(6.2)
$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} \le \gamma^{-1}[w]_{A_p}^{1/(p-1)} \|M_w\|^{1/(p-1)}_{L^p(w)\to L^p(w)} \cdot \|M_\sigma\|_{L^p(\sigma)\to L^p(\sigma)}.$$

Proof. Applying the Hölder inequality, for a measurable $E \in \mathfrak{M}$ we have

$$\begin{aligned} \mu(E) &= \int_E w^{1/p} \cdot w^{-1/p} d\mu \\ &\le \left(\int_E w d\mu\right)^{1/p} \cdot \left(\int_E w^{-\frac{1}{p-1}} d\mu\right)^{1/q} \\ &= (w(E))^{1/p} (\sigma(E))^{1/q}. \end{aligned} \tag{6.3}$$

Now suppose $\mathcal{S}$ is a sparse collection of balls and $f \in L^p(w)$ is positive. Using the inequality $(\sum a_k)^{p-1} \le \sum a_k^{p-1}$ for $1 < p < 2$, we obtain

$$\begin{aligned}
\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w)}^p &= \int_X \left((\mathcal{A}_{\mathcal{S}} f)^{p-1}\right)^{\frac{p}{p-1}} w d\mu \\
&= \int_X \left(\left(\sum_{B\in\mathcal{S}} \langle f\rangle_B \cdot \mathbb{I}_B\right)^{p-1}\right)^q w d\mu \\
&\le \int_X \left(\sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \cdot \mathbb{I}_B\right)^q w d\mu \\
&= \left\|\sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \cdot \mathbb{I}_B\right\|_{L^q(w)}^q.
\end{aligned} \tag{6.4}$$

There is a function $g \in L^p(w)$ with $\|g\|_{L^p(w)} = 1$ such that

$$\begin{aligned}
\left\|\sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \cdot \mathbb{I}_B\right\|_{L^q(w)} &= \int_X \sum_{B\in\mathcal{S}} \langle f\rangle_B^{p-1} \cdot \mathbb{I}_B g w d\mu \\
&= \sum_{B\in\mathcal{S}} \left(\frac{1}{\mu(B)} \int_B f d\mu\right)^{p-1} \int_B g dw \\
&= \sum_{B\in\mathcal{S}} \left(\frac{1}{\sigma(B)} \int_B f d\mu\right)^{p-1} \times \\
&\times \frac{1}{w(B)} \int_B |g| dw \cdot \frac{w(B)}{\mu(B)} \left(\frac{\sigma(B)}{\mu(B)}\right)^{p-1} \cdot \mu(B).
\end{aligned}$$

Thus, applying (6.3), A_p-condition and Hölder's inequality, we get

(6.5)

$$\begin{aligned}
&\left\|\sum_{B\in\mathcal{S}}\langle f\rangle_B^{p-1}\cdot\mathbb{I}_B\right\|_{L^q(w)}\\
&\leq \gamma^{-1}[w]_{A_p}\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B f d\mu\right)^{p-1}\times\frac{1}{w(B)}\int_B |g|dw\cdot\mu(E_B)\\
&\leq \gamma^{-1}[w]_{A_p}\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B f d\mu\right)^{p-1}(\sigma(E_B))^{1/q}\times\\
&\qquad\times\frac{1}{w(B)}\int_B |g|dw\cdot(w(E_B))^{1/p}\\
&\leq \gamma^{-1}[w]_{A_p}\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B f\sigma^{-1}d\sigma\right)^p\sigma(E_B)\right)^{1/q}\times\\
&\qquad\times\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{w(B)}\int_B |g|dw\right)^p\cdot w(E_B)\right)^{1/p}.
\end{aligned}$$

The last two factors can be estimated by the maximal functions M_σ and M_w respectively. Namely, for the second one we have

(6.6)

$$\begin{aligned}
\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{w(B)}\int_B |g|dw\right)^p\cdot w(E_B)\right)^{1/p} &\leq \|M_w g\|_{L^p(w)}\\
&\leq \|M_w\|_{L^p(w)\to L^p(w)}\cdot\|g\|_{L^p(w)}\\
&= \|M_w\|_{L^p(w)\to L^p(w)}.
\end{aligned}$$

Similarly, the first factor is estimated by

$$\begin{aligned}
&\left(\sum_{B\in\mathcal{S}}\left(\frac{1}{\sigma(B)}\int_B f\sigma^{-1}d\sigma\right)^p\sigma(E_B)\right)^{1/q}\\
&\leq \|M_\sigma\|_{L^p(\sigma)\to L^p(\sigma)}^{p/q}\cdot\|f\sigma^{-1}\|_{L^p(\sigma)}^{p/q}\\
&= \|M_\sigma\|_{L^p(\sigma)\to L^p(\sigma)}^{p/q}\cdot\left(\int_X f^p\sigma^{-p}\sigma d\mu\right)^{1/q}\\
&= \|M_\sigma\|_{L^p(\sigma)\to L^p(\sigma)}^{p/q}\cdot\left(\int_X f^p dw\right)^{1/q}\\
&= \|M_\sigma\|_{L^p(\sigma)\to L^p(\sigma)}^{p/q}\cdot\|f\|_{L^p(w)}^{p/q}.
\end{aligned}\tag{6.7}$$

From (6.4), (6.5), (6.6) and (6.7) we immediately get (6.2). □

Lemma 6.2. *Let $(X,\mathfrak{M},\mu)$ be measure space with a ball-basis $\mathfrak{B}$, $1<p,q<\infty$ and $p^{-1}+q^{-1}=1$. If $\mathcal{S}$ is a sparse collection and the weight w satisfies the A_p condition, then*

$$\|\mathcal{A}_\mathcal{S}\|_{L^p(w)\to L^p(w)}=\|\mathcal{A}_\mathcal{S}\|_{L^q(\sigma)\to L^q(\sigma)},\tag{6.8}$$

where σ is the dual weight of w.

Proof. We have

$$\|\mathcal{A}_\mathcal{S}\|_{L^p(w)\to L^p(w)}=\sup_{f\in L^p(w),\, g\in L^q(w)}\int_X \mathcal{A}_\mathcal{S}f\cdot gdw.$$

By the duality argument for $f\in L^p(w)$ and $g\in L^q(w)$ we get the estimate

$$\begin{aligned}\int_X \mathcal{A}_\mathcal{S}f\cdot gdw &= \int_X \mathcal{A}_\mathcal{S}f\cdot gwd\mu=\sum_{B\in\mathcal{S}}\frac{1}{\mu(B)}\int_B fd\mu\int_B gwd\mu\\
&=\int_X \mathcal{A}_\mathcal{S}(gw)\cdot fd\mu=\int_X\frac{\mathcal{A}_\mathcal{S}(gw)}{w}\cdot fwd\mu=\int_X\frac{\mathcal{A}_\mathcal{S}(gw)}{w}\cdot fdw\\
&\le\left\|\frac{\mathcal{A}_\mathcal{S}(gw)}{w}\right\|_{L^q(w)}\|f\|_{L^p(w)}=\left(\int_X(\mathcal{A}_\mathcal{S}(gw))^q w^{-q}dw\right)^{1/q}\cdot\|f\|_{L^p(w)}\\
&=\left(\int_X(\mathcal{A}_\mathcal{S}(gw))^q\sigma d\mu\right)^{1/q}\cdot\|f\|_{L^p(w)}\\
&\le\|\mathcal{A}_\mathcal{S}\|_{L^q(\sigma)\to L^q(\sigma)}\|gw\|_{L^q(\sigma)}\|f\|_{L^p(w)}\\
&=\|\mathcal{A}_\mathcal{S}\|_{L^q(\sigma)\to L^q(\sigma)}\left(\int_X(gw)^q\sigma d\mu\right)^{1/q}\|f\|_{L^p(w)}\\
&=\|\mathcal{A}_\mathcal{S}\|_{L^q(\sigma)\to L^q(\sigma)}\|g\|_{L^q(w)}\|f\|_{L^p(w)},\end{aligned}$$

which implies $\|\mathcal{A}_\mathcal{S}\|_{L^p(w)\to L^p(w)}\le\|\mathcal{A}_\mathcal{S}\|_{L^q(\sigma)\to L^q(\sigma)}$. Similarly we have the reverse inequality and so (6.8). □

Lemma 6.3. *Let $(X,\mathfrak{M},\mu)$ be a measure space with a ball-basis $\mathfrak{B}$ and w be a weight satisfying the A_p-condition, $1<p<\infty$, with respect to the measure μ. Then for any balls A,B with $A\subset B$ we have*

$$\frac{w(B)}{w(A)}\le 2^p\cdot[w]_{A_p}\cdot\left(\frac{\mu(B)}{\mu(A)}\right)^p.\tag{6.9}$$

Proof. Denote

$$a=\frac{1}{\mu(A)}\int_A wd\mu=\frac{w(A)}{\mu(A)}.\tag{6.10}$$

By Chebishev's inequality we find

$$\mu\{t \in A :\ w \le 2a\} > \frac{\mu(A)}{2}.$$

Thus we get

$$\left(\int_B w^{1/(1-p)}\right)^{p-1} \ge \left(\frac{\mu(A)}{2}(2a)^{1/(1-p)}\right)^{p-1} = \frac{(\mu(A))^p}{2^p w(A)} \tag{6.11}$$

and then by (1.1) and (6.11) we obtain

$$\begin{aligned}
[w]_{A_p} &\ge \frac{1}{\mu(B)}\int_B w \cdot \left(\frac{1}{\mu(B)}\int_B w^{1/(1-p)}\right)^{p-1}\\
&\ge \frac{w(B)}{(\mu(B))^p} \cdot \left(\int_B w^{1/(1-p)}\right)^{p-1}\\
&\ge 2^{-p} \cdot \frac{w(B)}{w(A)} \cdot \frac{(w(A))^p}{(\mu(B))^p}
\end{aligned}$$

and so (6.10). □

Lemma 6.4. *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$ and w be a weight, satisfying the A_p-condition, $1 < p < \infty$. Then the maximal function associated to the measure w satisfies the inequalities*

$$\|M_w\|_{L^1(w)\to L^{1,\infty}(w)} \le (2\mathcal{K})^p \cdot [w]_{A_p}, \tag{6.12}$$

$$\|M_w\|_{L^{p'}(w)\to L^{p'}(w)} \le c(p,p') \cdot \mathcal{K}^{p/p'}[w]_{A_p}^{1/p'}, \quad 1 < p' < \infty. \tag{6.13}$$

where the constant $c(p,p')$ depends on p and p'.

Proof. Denote

$$E = \{x \in B :\ M_w f(x) > \lambda\}.$$

For any $x \in E$ there exists a ball $B(x) \subset X$ such that

$$x \in B(x), \quad \frac{1}{w(B(x))}\int_{B(x)} |f| dw > \lambda.$$

Given ball $B \in \mathfrak{B}$ consider the collection of balls $\{B(x) :\ x \in E \cap B\}$. Apply Lemma 3.1, we find a sequence of pairwise disjoint balls $\{B_k\}$ taken from this collection satisfying

$$E \cap B \subset \bigcup_k B_k^{[1]}.$$

Note that $B_k^{[1]}$ as usual is defined with respect to the measure μ. From Lemma 6.3 we easily get $w(B_k^{[1]})\le (2\mathcal{K})^p\cdot[w]_{A_p}w(B_k)$ and hence

$$\begin{aligned} w\left(\cup_k B_k^{[1]}\right) &\le \sum_k w(B_k^{[1]})\\ &\le (2\mathcal{K})^p\cdot[w]_{A_p}\sum_k w(B_k)\\ &\le (2\mathcal{K})^p\cdot[w]_{A_p}\frac{1}{\lambda}\sum_k\int_{B_k}|f|dw\\ &\le \frac{(2\mathcal{K})^p\cdot[w]_{A_p}}{\lambda}\int_X|f|dw. \end{aligned}$$

Similarly as in the proof of Theorem 4.1, thus we conclude

$$w^*(E)\le\frac{(2\mathcal{K})^p\cdot[w]_{A_p}}{\lambda}\int_X|f|dw$$

and so (6.12). Applying Theorem 2.1 (Marcinkiewicz interpolation theorem), we then get (6.13). □

Theorem 6.1. *If $\mathcal{S}$ is a γ-sparse collection and the weight w satisfies the A_p condition for some $1<p<\infty$, then the corresponding sparse operator satisfies the bound*

$$\|\mathcal{A}_{\mathcal{S}}f\|_{L^p(w)\to L^p(w)}\le c(p,\mathcal{K})\cdot\gamma^{-1}[w]_{A_p}^{\max\left\{\frac{p+2}{p(p-1)},\frac{3p-2}{p}\right\}}. \tag{6.14}$$

Proof. First we suppose that $1<p\le 2$. Applying Lemma 6.1, Lemma 6.4 and (6.1), we obtain

$$\begin{aligned} \|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} &\le c(p,\mathcal{K})\gamma^{-1}[w]_{A_p}^{1/(p-1)}\cdot[w]_{A_p}^{1/p(p-1)}\cdot[\sigma]_{A_q}^{1/p}\\ &= c(p,\mathcal{K})\gamma^{-1}[w]_{A_p}^{\frac{p+2}{p(p-1)}} \end{aligned}$$

and so (6.14). If $2<p<\infty$, then by Lemma 6.2 and (6.1) we obtain

$$\begin{aligned} \|\mathcal{A}_{\mathcal{S}}f\|_{L^p(w)\to L^p(w)} &= \|\mathcal{A}_{\mathcal{S}}f\|_{L^q(\sigma)\to L^q(\sigma)}\\ &\le c(q,\mathcal{K})\gamma^{-1}[\sigma]_{A_q}^{\frac{q+2}{q(q-1)}} = c(p,\mathcal{K})\gamma^{-1}[w]_{A_p}^{\frac{3p-2}{p}}. \end{aligned}$$

Theorem is proved. □

Combining Theorem 1.1 with Theorem 6.1, we obtain

Theorem 6.2. *If $(X,\mathfrak{M},\mu)$ is a measure space with a ball-basis $\mathfrak{B}$ and the operator $T\in\mathrm{BO}_{\mathfrak{B}}(X)$ satisfies weak-L^1 inequality, then*

$$\|Tf\|_{L^p(w)\to L^p(w)}\le c(p,\mathcal{K})(\mathcal{L}_1+\mathcal{L}_2+\|T\|_{L^1\to L^{1,\infty}})[w]_{A_p}^{\max\left\{\frac{p+2}{p(p-1)},\frac{3p-2}{p}\right\}}.$$

6.2. **The case of Besicovitch condition.**

Definition 6.1. *Let $\mathfrak{B}$ be a family of sets of an arbitrary set X. We say $\mathfrak{B}$ satisfies the Besicovitch D-condition with a constant $D \in \mathbb{N}$, if for any collection $\mathcal{A} \subset \mathfrak{B}$ one can find a subscollection $\mathcal{A}' \subset \mathcal{A}$ such that*

$$\bigcup_{A\in\mathcal{A}} A = \bigcup_{A\in\mathcal{A}'} A,$$
$$\sum_{A\in\mathcal{A}'} \mathbb{I}_A(x) \le D.$$

We say $\mathfrak{B}$ is martingale system if $D = 1$.

Theorem 6.3. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and the collection of measurable sets $\mathfrak{B} \subset \mathfrak{M}$ satisfy the Besicovitch D-condition. Then the maximal operator M_μ satisfies the bounds*

$$\begin{aligned} &\|M_\mu\|_{L^1(\mu)\to L^{1,\infty}(\mu)} \le D,\\ &\|M_\mu\|_{L^p(\mu)\to L^p(\mu)} \le c_p \cdot D^{1/p}, \quad 1 < p < \infty. \end{aligned} \tag{6.15}$$

Proof. Define

$$E = \{x \in X : M_\mu f(x) > \lambda\}.$$

For any $x \in E$ there exists a set $B(x) \subset \mathfrak{B}$ such that

$$\frac{1}{w(B(x))}\int_{B(x)} |f|dw > \lambda, \quad x \in B(x). \tag{6.16}$$

According to the Besicovitch condition there is a subcollection $\mathfrak{A} \subset \{B(x) : x \in E\}$ such that

$$\bigcup_{A\in\mathfrak{A}} A = \bigcup_{x\in E} B(x),$$
$$\sum_{A\in\mathfrak{A}} \mathbb{I}_A(x) \le D.$$

Thus we get

$$\mu^*(E) \le \sum_{A\in\mathfrak{A}} \mu(A) \le \frac{1}{\lambda}\sum_{A\in\mathfrak{A}}\int_A |f| \le \frac{D}{\lambda}\int_X |f|d\mu.$$

The second inequality immediately follows from (6.15), according to Marcinkiewicz interpolation theorem (Theorem 2.1). □

The following theorem gives a sharp weighted estimate in general measure spaces with a ball basis satisfying the Besicovitch condition.

Theorem 6.4. *Let $\mathfrak{B}$ be a family of sets in a measure space $(X, \mathfrak{M}, \mu)$ satisfying the Besicovitch D-condition. If $\mathcal{S}$ is a γ-sparse collection and the weight w satisfies the A_p condition for $1 < p < \infty$, then*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w)\to L^p(w)} \lesssim c_p \gamma^{-1} D^{\max\{1/(p-1),p-1\}} \cdot [w]_{A_p}^{\max\{1,1/(p-1)\}}. \tag{6.17}$$

Proof. First suppose that $1 < p \le 2$. Applying Lemma 6.1 and Theorem 6.3 we obtain

$$\begin{aligned}\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} &\le c_p \gamma^{-1} [w]_{A_p}^{1/(p-1)} \cdot D^{1/p(p-1)} \cdot D^{1/p} \\ &= c_p \gamma^{-1} [w]_{A_p}^{1/(p-1)} \cdot D^{1/(p-1)}\end{aligned}$$

and so (6.17). In the case $2 < p < \infty$ we use the same argument as in the proof of Theorem 6.1. □

Applying Theorem 1.1 and Theorem 6.4 we immediately get the following

Theorem 6.5. *Let a family of measurable sets $\mathfrak{B}$ in a measure space $(X, \mathfrak{M}, \mu)$ satisfy the Besicovitch D-condition and w be a A_p weight with $1 < p < \infty$. Then if an operator $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfy weak-L^1 inequality, then*

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim C(\mathcal{L}_1 + \mathcal{L}_2 + \|T\|_{L^1 \to L^{1,\infty}}) \cdot [w]_{A_p}^{\max\{1,1/(p-1)\}},$$

where C is a constant depending on p and the Besicovitch constant.

7. Bounded oscillation operators on spaces of homogeneous type

Definition 7.1. *A quasimetric on a set X is a function $\rho : X \times X \to [0, \infty)$ satisfying the conditions*

1) $\rho(x, y) \ge 0$ for every $(x, y) \in X$ and $\rho(x, y) = 0$ if and only if $x = y$,

2) $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$,

3) $\rho(x, y) \le \mathcal{D}(\rho(x, z) + \rho(z, y))$ for every $x, y, z \in X$, where $\mathcal{D} > 1$ is a fixed constant.

Define the ball of a center x and a radius r by

$$B(x, r) = \{y \in X : \rho(x, y) < r\}, \quad x \in X, \quad 0 < r < \infty$$

and denote by $\mathfrak{U}(\rho)$ the family of all such balls, calling them ρ-balls. Quasimetric defines a topology, for which the ρ-balls form a base. In general, the balls need not to be open sets in this topology. For $B \in \mathfrak{U}(\rho)$ denote by $c(B)$ and $r(B)$ respectively the center and the radius of B. For any $t > 0$ we set $tB = B(c(B), tr(B))$. We define also an enlarged family of balls $\mathfrak{U}'(\rho)$ as follows: if $\mu(X) = \infty$, then

$\mathfrak{U}'(\rho)$ coincides with $\mathfrak{U}(\rho)$, in the case $\mu(X) < \infty$ we include in $\mathfrak{U}'(\rho)$ additionally the set X.

Definition 7.2. *Let ρ be a quasimetric on X and μ be a positive measure defined on a σ-algebra $\mathfrak{M}$ of subsets of X, containing the ρ-open sets and the ρ-balls. The collection $(X, \rho, \mathfrak{M}, \mu)$ is said to be a space of homogeneous type if*

$$\mu(2B) \le \mathcal{H} \cdot \mu(B) \tag{7.1}$$

for any ball $B \in \mathfrak{U}(\rho)$.

Note that (7.1) implies a more general inequality. Namely,

$$\mu(a \cdot B) \le \mathcal{H}(a) \cdot \mu(B), \quad a > 0, \tag{7.2}$$

where $\mathcal{H}(a)$ is a constant depending on a and $\mathcal{H}$. From property 3) of quasimetric it easily follows that for any $B \in \mathfrak{U}(\rho)$ it holds the inequality

$$\operatorname{diam} B = \sup_{x,y \in B} \rho(x, y) \le 2\mathcal{D} \cdot r(B).$$

In this section the notation $a \lesssim b$ will stand for $a \le c \cdot b$, where $c > 0$ is a constant depending on the constants $\mathcal{H}$ and $\mathcal{D}$ of the space homogeneous type .

Theorem 7.1. *Let $(X, \rho, \mathfrak{M}, \mu)$ be a space of homogeneous type such that $\mathfrak{U}(\rho)$ satisfies the density condition. Then the enlarged family of balls $\mathfrak{U}'(\rho)$ forms a ball-basis for the measure space $(X, \mathfrak{M}, \mu)$ and satisfies the doubling condition. Besides, the hull ball of any $B = B(x_0, r) \in \mathfrak{U}(\rho)$ has the form $B^* = B(x_0, R)$, $2r \le R \le \infty$.*

The proof of the theorem is based on the following lemmas.

Lemma 7.1. *If $(X, \rho, \mathfrak{M}, \mu)$ is a space of homogeneous type, then for any point $x_0 \in X$ and ball $G \in \mathfrak{U}(\rho)$ we have*

$$G \subset B(x_0, 2\mathcal{D}^2(\operatorname{dist}(x_0, G) + r(G))). \tag{7.3}$$

Proof. Fix a point $y \in G$ with $\rho(x_0, y) < 2\operatorname{dist}(x_0, G)$. For arbitrary $x \in G$ we have

$$\begin{aligned} \rho(x, x_0) &\le \mathcal{D}(\rho(x, y) + \rho(y, x_0)) \\ &\le \mathcal{D}(2\mathcal{D}r(G) + 2\operatorname{dist}(x_0, G)) \le 2\mathcal{D}^2(\operatorname{dist}(x_0, G) + r(G)) \end{aligned}$$

that means x belongs to the right side of (7.3). □

Lemma 7.2. *If* $(X, \rho, \mathfrak{M}, \mu)$ *is a space of homogeneous type and the balls* $B \in \mathfrak{U}(\rho)$*,* $G_k \in \mathfrak{U}(\rho)$*,* $k = 1, 2, \ldots$*, satisfy the relations*

$$B \cap G_k \neq \varnothing, \tag{7.4}$$
$$r(G_k) \to \infty \text{ as } k \to \infty,$$

then

$$\mu(X) \lesssim \limsup_{k\to\infty} \mu(G_k).$$

Proof. Without loss of generality we can suppose that

$$r(G_k) > r(B).$$

From (7.4) it follows that $\operatorname{dist}(c(G_k), B) < r(G_k)$. Thus, applying Lemma 7.1, we get

$$\begin{aligned} B(c(B), r(G_k)) &\subset B(c(G_k), 2\mathfrak{D}^2(\operatorname{dist}(c(G_k), B) + r(G_k))) \\ &\subset B(c(G_k), 4\mathfrak{D}^2 r(G_k)), \end{aligned}$$

and therefore by (7.2) we obtain

$$\mu(B(c(B), r(G_k))) \le \mathcal{H}(4\mathfrak{D}^2) \cdot \mu(G_k).$$

On the other hand, since $r(G_k) \to \infty$, we have $X = \cup_k B(c(B), r(G_k))$. Therefore we get

$$\mu(X) = \lim_{k\to\infty} \mu(B(c(B), r(G_k)) \lesssim \limsup_{k\to\infty} \mu(G_k).$$

□

Lemma 7.3. *Let* $(X, \rho, \mathfrak{M}, \mu)$ *be a space of homogeneous type. Then for any* $B = B(x_0, r) \in \mathfrak{U}(\rho)$ *there exists a ball* $B^* = B(x_0, R)$ *with* $2r \le R \le \infty$ *such that*

$$\mu(B^*) \lesssim \mu(B), \tag{7.5}$$

$$\bigcup_{A \in \mathfrak{U}(\rho):\, \mu(A) \le 2\mu(B),\, A\cap B \neq \varnothing} A \subset B^*. \tag{7.6}$$

Proof. For a given $B \in \mathfrak{U}(\rho)$ let $\mathfrak{A}$ be the family of balls $A \in \mathfrak{U}(\rho)$ satisfying

$$A \cap B \neq \varnothing, \quad \mu(A) \le 2\mu(B).$$

First suppose that

$$\gamma = \sup_{A \in \mathfrak{A}} r(A) < \infty.$$

Applying Lemma 7.1, for an arbitrary $A \in \mathfrak{A}$ we get

$$\begin{aligned} A = B(c(A), r(A)) &\subset B(c(B), 2\mathfrak{D}^2(r(B) + r(A))) \\ &\subset B(c(B), 4\mathfrak{D}^2\gamma). \end{aligned}$$

It is clear that $B^* = B(c(B), 4\mathfrak{D}^2\gamma)$ satisfies (7.6). Take a ball $G \in \mathfrak{A}$ such that $r(G) > \gamma/2$. Again applying Lemma 7.1, we get

$$\begin{aligned} B^* &= B(c(B), 4\mathfrak{D}^2\gamma) \subset B(c(G), 2\mathfrak{D}(r(G) + 4\mathfrak{D}^2\gamma)) \\ &\subset B(c(G), 10\mathfrak{D}^3\gamma)) \subset B(c(G), 20\mathfrak{D}^3 r(G)) \\ &= 20\mathfrak{D}^3 \cdot G. \end{aligned}$$

Thus we conclude

$$\mu(B^*) \le \mu(20\mathfrak{D}^3 \cdot G) \le \mathcal{H}(20\mathfrak{D}^3)\mu(G) \lesssim \mu(B)$$

that is just (7.5). Now consider the case $\gamma = \infty$. There is a sequence of balls $G_k \in \mathfrak{A}$ such that $r(G_k) \to \infty$. Applying Lemma 7.2, we get

$$\mu(X) \lesssim \limsup_{k\to\infty} \mu(G_k) \le 2\mu(B).$$

Obviously $B^* = B(x_0, \infty) = X$ satisfies (7.5) and (7.6). $\square$

Proof of Theorem 7.1. We need to check conditions B1)-B4) of the definition of ball-basis. The conditions B1) and B2) immediately follows from the axioms of quasi-metric space and B4) follows from Lemma 7.3 and moreover for $B = B(x_0) \in \mathfrak{U}(\rho)$ the hull ball B^* has the form $B(x_0, R)$. The B3)-condition follows from the density property, since by Lemma 3.4 those are equivalent. In order to prove the doubling condition, take a ball $A = B(x_0, r)$ such that $A^* = B(x, R) \neq X$. Denote

$$R' = \sup_{r' \ge R:\, B(x_0') = B(x_0, R)} r'.$$

Since $B(x_0, R) \neq X$, one can check that $R' < \infty$ and

$$A^* = B(x_0, R) = B(x_0, R') \subsetneq B(x_0, 2R').$$

Thus defining $B = B(x_0, 2R')$, we get $A \subsetneq B$ and

$$\mu(B) = \mu(B(x_0, 2R')) \lesssim \mu(B(x_0, R')) = \mu(A^*) \lesssim \mu(A),$$

that proves the doubling condition. $\square$

Theorem 7.2. *Let $(X, \rho, \mathfrak{M}, \mu)$ be a space homogeneous type satisfying the density condition. If $\mathcal{S} \subset \mathfrak{U}(\rho)$ is a sparse collection of balls and the weight w satisfies the A_p-condition for $1 < p < \infty$ (with respect to the family $\mathfrak{U}(\rho)$), then the corresponding sparse operator satisfies the bound*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w) \to L^p(w)} \lesssim c_p [w]_{A_p}^{\max\{1, 1/(p-1)\}}. \tag{7.7}$$

The proof of this theorem is based on the Hytönen-Kairema [10] dyadic decomposition theorem, which reduces Theorem 7.2 to its martingale version (the case of $D = 1$ in Theorem 6.4).

Definition 7.3. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space. For two families of measurable sets* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *we write* $\mathfrak{B} \prec \mathfrak{B}'$ *if for any* $B \in \mathfrak{B}$ *there exists* $B' \in \mathfrak{B}'$ *such that*

$$B \subset B', \quad \mu(B') \le \gamma \mu(B),$$

where $\gamma > 0$ *is a constant. The minimum value of such constants* γ *will be denoted by* $\gamma(\mathfrak{B} \prec \mathfrak{B}')$*. If the relations* $\mathfrak{B} \prec \mathfrak{B}'$ *and* $\mathfrak{B}' \prec \mathfrak{B}$ *hold simultaneously, then we write* $\mathfrak{B} \sim \mathfrak{B}'$ *and denote*

$$\gamma(\mathfrak{B} \sim \mathfrak{B}') = \max\{\gamma(\mathfrak{B} \prec \mathfrak{B}'), \gamma(\mathfrak{B}' \prec \mathfrak{B})\}.$$

Remark 7.1. *One can verify that if for two families of measurable sets in* $(X, \mathfrak{M}, \mu)$ *we have* $\mathfrak{B} \sim \mathfrak{B}'$*, then the* A_p *characteristics with respect these families are equivalent. That is*

$$0 < c_1 < \frac{\sup_{B \in \mathfrak{B}} \left(\frac{1}{|B|} \int_B w\right) \left(\frac{1}{|B|} \int_B w^{-1/(p-1)}\right)^{p-1}}{\sup_{B \in \mathfrak{B}'} \left(\frac{1}{|B|} \int_B w\right) \left(\frac{1}{|B|} \int_B w^{-1/(p-1)}\right)^{p-1}} < c_2,$$

for some constants c_1 *and* c_2 *depending on* $\gamma(\mathfrak{B} \sim \mathfrak{B}')$*.*

Theorem 7.3 (Hytönen-Kairema [10])**.** *If* $(X, \rho, \mathfrak{M}, \mu)$ *is a space homogeneous type, then there exist martingale systems* $\mathfrak{B}_k \subset \mathfrak{M}$*,* $k = 1, 2, \dots, l$*, such that*

$$\mathfrak{U}(\rho) \sim \mathfrak{B} = \bigcup_{j=1}^{l} \mathfrak{B}_j.$$

where l *and* $\gamma(\mathfrak{U}(\rho) \sim \mathfrak{B})$ *are constants depending on* $\mathcal{H}$ *and* $\mathcal{D}$*.*

Proof of Theorem 7.2. Apply Theorem 7.3. For every $B \in \mathfrak{U}(\rho)$ there exists a set $Q(B) \in \mathfrak{B}$ such that

$$B \subset Q(B), \quad \mu(Q(B)) \lesssim \mu(B). \tag{7.8}$$

We shall consider the sparse operators

$$\mathcal{A}_k f(x) = \sum_{B \in \mathcal{S} \colon Q(B) \in \mathfrak{B}_k} \langle f \rangle_{Q(B)} \mathbb{I}_{Q(B)}(x), \quad k = 1, 2, \dots, l.$$

From (7.8) it follows that

$$\begin{aligned}\mathcal{A}_{\mathcal{S}}f(x) &= \sum_{B\in\mathcal{S}}\langle f\rangle_B\mathbb{I}_B(x) \lesssim \sum_{B\in\mathcal{S}}\langle f\rangle_{Q(B)}\mathbb{I}_{Q(B)}(x)\\ &\le \sum_{k=1}^{l}\sum_{B\in\mathcal{S}:\,Q(B)\in\mathfrak{B}_k}\langle f\rangle_{Q(B)}\mathbb{I}_{Q(B)}(x)\\ &= \sum_{k=1}^{l}\mathcal{A}_k f(x).\end{aligned} \tag{7.9}$$

Since each $\mathfrak{B}_k$ is martingale system, by Theorem 6.4 (for $D = 1$) we conclude that

$$\|\mathcal{A}_k f\|_{L^p(\omega)} \lesssim c_p[\omega]_{A_p}^{\max\{1,1/(p-1)\}}.$$

Combining this and (7.9), we get (7.7). □

Let $(X,\rho,\mathfrak{M},\mu)$ be a space of homogeneous type and $K(x,y) : X\times X\to\mathbb{R}$ be a measurable function. Given ball $\mathfrak{B}\in\mathfrak{U}(\rho)$ define the function

$$\phi_B(t) = \sup_{x,x'\in B,\, y\in X\setminus B(c(B),t)}|K(x,y)-K(x',y)|, \text{ if } t\ge 2r(B), \tag{7.10}$$

$$\phi_B(t) = \phi_B(2r(B)) \text{ if } 0\le t<2r(B), \tag{7.11}$$

which is clearly decreasing on $[0,\infty)$. Denote

$$R = \sup_{B\in\mathfrak{B}}\int_X \phi_B\left(\rho(y,c(B))\right)d\mu(y), \tag{7.12}$$

$$d_B = \sup_{x\in B,\, y\in X\setminus 2B}|K(x,y)|.$$

(7.13)

Definition 7.4. *An operator $T : L^1(X)\to L^0(X)$ is said to be of Calderón-Zygmund type if for any $B\in\mathfrak{U}(\rho)$ it admits the representation*

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y) \text{ whenever } x\in B,\ \operatorname{supp} f\subset X\setminus 2B, \tag{7.14}$$

where the kernel $K(x,y)$ satisfies the conditions

$$R<\infty,\quad d_B<\infty \text{ for any } B\in\mathfrak{U}(\rho). \tag{7.15}$$

Theorem 7.4. *If $(X,\rho,\mathfrak{M},\mu)$ is a space of homogeneous type such that $\mathfrak{U}(\rho)$ satisfies the density condition, then any Calderón-Zygmund type operator (7.14) is* BO *operator with respect to the ball-basis $\mathfrak{U}'(\rho)$. Moreover we have $\mathcal{L}_1(T)\le R$, where R is (7.12).*

Proof. First note that Theorem 7.1 implies that $\mathfrak{U}'(\rho)$ is a ball basis having the doubling property and for $B = B(x_0, r) \in \mathfrak{U}(\rho)$ the hull ball B^* has the form $B(x_0, R)$, $2r \le R < \infty$. Since $\mathfrak{U}'(\rho)$ satisfies the doubling condition, according to Theorem 4.4, we need to verify only T1)-condition. Since $\phi_B(t)$ is decreasing, we can prove

$$\int_X \phi_B(\rho(y, c(B)))|f(y)|d\mu(y) \le R \cdot \langle f\rangle^*_B. \tag{7.16}$$

Indeed, one can easily find a step function $\psi(t)$ on $[0, \infty)$ such that

$$\phi_B(t) \le \psi(t) = \sum_{k=1}^{\infty} a_k \mathbb{I}_{[0,r_k]}(t),\ a_k > 0,\ 2r(B) = r_1 < r_2 < \dots,$$

$$\int_X \psi\left(\rho(y, c(B))\right) d\mu(y) < \int_X \phi_B\left(\rho(y, c(B))\right) d\mu(y) + \delta \le R + \delta,$$

where $\delta > 0$ can be enough small. We have

$$\begin{aligned}\int_X \phi_B(\rho(y, c(B)))|f(y)|d\mu(y) &\le \int_X \psi(\rho(y, c(B)))|f(y)|d\mu(y)\\ &= \sum_{k=1}^{\infty} a_k \int_{B(c(B),r_k)} |f(y)|d\mu(y)\\ &\le \langle f\rangle^*_B \sum_{k=1}^{\infty} a_k \mu(B(c(B), r_k))\\ &= \langle f\rangle^*_B \int_X \psi\left(\rho(y, c(B))\right) d\mu(y)\\ &\le \langle f\rangle^*_B (R + \delta).\end{aligned}$$

Since $\delta > 0$ is small enough, we get (7.16). Now take $B \in \mathfrak{U}(\rho)$, $f \in L^1(X)$ and suppose that $x, x' \in B$. From (7.10) it follows that

$$|K(x, y) - K(x', y)| \le \phi_B(\rho(y, c(B)))| \text{ whenever } y \in X \setminus 2B.$$

Thus, using (7.16) and the relation $B^* \supset 2B$, we get the bound

$$\begin{aligned}&|T(f \cdot \mathbb{I}_{X\setminus B^*})(x) - T(f \cdot \mathbb{I}_{X\setminus B^*})(x')|\\ &\qquad = \left|\int_{X\setminus B^*} (K(x, y) - K(x', y))f(y)d\mu(y)\right|\\ &\qquad \le \int_{X\setminus 2B} |K(x, y) - K(x', y)||f(y)|d\mu(y)\\ &\qquad \le \int_X \phi_B(\rho(y, c(B)))|f(y)|d\mu(y)\\ &\qquad \le R \cdot \langle f\rangle^*_B,\end{aligned}$$

which gives T1)-condition. □

Let $(X,\rho,\mathfrak{M},\mu)$ be a space of homogeneous type and $\omega:[0,\infty)\to[0,\infty)$ be an increasing function satisfying $\omega(t+s)\le\omega(t)+\omega(s)$, $\omega(0)=0$, and the Dini condition

$$C_1=\int_0^1\frac{\omega(t)}{t}dt<\infty. \tag{7.17}$$

An operator $T:L^1(X)\to L^0(X)$ is said to be ω-Calderón-Zygmund operator if it has the representation (7.14) and for any ball $B\in\mathfrak{U}(\rho)$ we have

$$\sup_{x\in B,\, y\in X\setminus B(c(B),t)}|K(x,y)|\le\frac{C_2}{\mu(B(c(B),t))}, \tag{7.18}$$

$$\sup_{x,x'\in B,\, y\in X\setminus B(c(B),t)}|K(x,y)-K(x',y)|\le\frac{\omega\left(\frac{r(B)}{t}\right)}{\mu(B(c(B),t))}, \tag{7.19}$$

$$\sup_{y,y'\in B,\, x\in X\setminus B(c(B),t)}|K(x,y)-K(x,y')|\le\frac{\omega\left(\frac{r(B)}{t}\right)}{\mu(B(c(B),t))}, \tag{7.20}$$

where all these inequalities hold for any $t>2r(B)$.

Theorem 7.5. *Let $(X,\rho,\mathfrak{M},\mu)$ be a space of homogeneous type such that $\mathfrak{U}(\rho)$ satisfies the density condition. If T is a ω-Calderón-Zygmund operator, then it is a* BO *operator with respect to the ball-basis $\mathfrak{U}'(\rho)$. Moreover, we have the estimates*

$$\mathcal{L}_1(T)\lesssim C_1,\quad \mathcal{L}_2(T)\lesssim C_2. \tag{7.21}$$

Proof. Taking into account (7.19) and the definition of function ϕ_B in (7.10), (7.11), for every $B=B(x_0,r)\in\mathfrak{U}(\rho)$ we have

$$\phi_B(t)\le\omega\left(\frac{r(B)}{t}\right)\frac{1}{\mu(B(c(B),t))}\text{ if }t\ge 2r(B),$$
$$\phi_B(t)\le\omega\left(\frac{1}{2}\right)\frac{1}{\mu(2B)}\text{ if }0\le t<2r(B).$$

Thus, applying doubling property (7.1), (7.17) and the subadditivity of ω, we get

$$\begin{aligned}\int_X \phi_B\left(\rho(y,c(B))\right)d\mu(y) &= \int_{2B} \phi_B\left(\rho(y,c(B))\right)d\mu(y)\\ &\quad+\sum_{k=1}^{\infty}\int_{(2^{k+1}B)\setminus(2^kB)} \phi_B\left(\rho(y,c(B))\right)d\mu(y)\\ &\le \omega\left(\frac{1}{2}\right)+\sum_{k=1}^{\infty}\omega\left(2^{-k}\right)\frac{\mu(2^{k+1}B)-\mu(2^kB)}{\mu(2^kB)}\\ &\lesssim \sum_{k=1}^{\infty}\omega(2^{-k})\lesssim C_1.\end{aligned}$$

Since this inequality holds for any ball B, applying Theorem 7.4, we get the first estimate in (7.21). To estimate $\mathcal{L}_2(T)$ take a ball $A = B(x_0,r)\in\mathfrak{U}(\rho)$ with $A^*\neq X$. According to Theorem 7.1, $A^* = B(x_0,R)$, $R\ge 2r$. Denote

$$L=\sup_{r'\ge R:\, B(x_0,r')=B(x_0,R)} r',$$

and let $B = B(x_0,2L)$. Since $B(x_0,R)\neq X$, we have $L<\infty$ and

$$A^*=B(x_0,R)=B(x_0,L)\subsetneq B(x_0,2L)=B.$$

Thus we get $A\subsetneq B$ and

$$\mu(B^*)\lesssim \mu(B(x_0,2L))\le \mathcal{H}\mu(B(x_0,L))=\mathcal{H}\mu(A^*). \tag{7.22}$$

Since $r(A^*)\ge 2r(A)$, from (7.18) we obtain

$$\sup_{x\in A,\, y\in X\setminus A^*}|K(x,y)|\le \frac{C_2}{\mu(A^*)}.$$

Thus, using also (7.22), for any point $x\in A$ we get

$$\begin{aligned}|T(f\cdot \mathbb{I}_{B^*\setminus A^*})(x)| &\le \int_{B^*\setminus A^*}|K(x,y)||f(y)|dy\\ &\le \frac{C_2}{\mu(A^*)}\int_{B^*}|f(y)|dy\\ &\lesssim C_2\langle f\rangle_{B^*}.\end{aligned}$$

Hence we obtain $\mathcal{L}_2(T)\lesssim C_2$ completing the proof of theorem. □

Combining Theorem 1.1, Theorem 7.2 and Theorem 7.5, we immediately get

Theorem 7.6. *Let $(X,\rho,\mathfrak{M},\mu)$ be a space of homogeneous type such that $\mathfrak{U}(\rho)$ satisfies the density condition. If T is a ω-Calderón-Zygmund operator and the weight w satisfies A_p condition with respect to the ball-basis $\mathfrak{U}(\rho)$, $1<p<\infty$, then we have*

$$\|T\|_{L^p(w)\to L^p(w)} \le c_p(C_1+C_2+\|T\|_{L^1\to L^{1,\infty}})[w]_{A_p}^{\max\{1,1/(p-1)\}}. \tag{7.23}$$

It is well known that any ω-Calderón-Zygmund operator, which is bounded on $L^2(X)$ satisfies the bound

$$\|T\|_{L^1\to L^{1,\infty}} \lesssim \|T\|_{L^2\to L^2}.$$

So in (7.23) $\|T\|_{L^1\to L^{1,\infty}}$ can be replaced by $\|T\|_{L^2\to L^2}$. Note that the Hytönen-Roncal-Tapiola [17] inequality is the case of (7.23) for the ω-Calderón-Zygmund operators on $\mathbb{R}^n$. Besides, (7.23) is a stronger version of the Anderson-Vagharshakyan [1] inequality, where the case of $\omega(t)=t^\delta$ was considered.

8. Other examples of BO operators

Theorem 8.1. *If $(X,\mathfrak{M},\mu)$ is a measure space with a ball-basis $\mathfrak{B}$, then the maximal operator M corresponding to $r=1$ in (4.1) is* BO *operator with respect to $\mathfrak{B}$.*

Proof. In order to establish T1) condition we let B be an arbitrary ball. Take two points $x,x'\in B$ and a nonzero function $f\in L^1(X)$ with

$$\operatorname{supp} f \in X\setminus B^{[1]}. \tag{8.1}$$

Suppose that

$$Mf(x)\ge Mf(x'). \tag{8.2}$$

We have $\langle f\rangle_B^*>0$. Thus, by the definition of maximal operator we get

$$Mf(x)\le \frac{1}{\mu(A)}\int_A |f| + \langle f\rangle_B^* \tag{8.3}$$

for some ball $A\ni x$. If $\mu(A)\le\mu(B)$, then by two balls relation we have $A\subset B^{[1]}$ and so by (9.4) we get

$$\frac{1}{\mu(A)}\int_A |f| = 0.$$

Therefore according to (9.5) and (9.6) we have

$$|Mf(x)-Mf(x')| = Mf(x)-Mf(x')\le \langle f\rangle_B^*.$$

If $\mu(A)>\mu(B)$, then we get $B\subset A^{[1]}$ and so

$$\begin{aligned} Mf(x)-Mf(x')\le \frac{1}{\mu(A)}\int_A |f|+\langle f\rangle_B^*\\ \lesssim \frac{1}{\mu(A^{[1]})}\int_{A^{[1]}} |f|+\langle f\rangle_B^*\lesssim \langle f\rangle_B^*. \end{aligned}$$

This gives T1)-condition. To prove T2)-condition fix a ball B and set

$$\begin{aligned} &\mathcal{A}=\{A\in \mathfrak{B}:\, A\cap B\neq\varnothing,\, \mu(A)>\mu(B)\},\\ &\gamma=\inf_{A\in\mathcal{A}}\mu(A). \end{aligned}$$

There exist a ball $A\in\mathcal{A}$ such that

$$\gamma\le\mu(A)<2\gamma.$$

Define $B'=A^{[1]}$. One can check that

$$B\subsetneq A^{[1]}=B'. \tag{8.4}$$

On the other hand for any function $f\in L^1(X)$ and any point $x\in B$ we have

$$M(f\cdot \mathbb{I}_{B'^{[1]}\setminus B^{[1]}})(x)=\frac{1}{\mu(C)}\int_C |f|\cdot \mathbb{I}_{B'^{[1]}\setminus B^{[1]}}+\delta, \tag{8.5}$$

for some ball $C\ni x$ and a number $\delta>0$ that can be taken arbitrarily small. We can suppose that $\mu(C)>\mu(B)$, since otherwise we should have $C\subset B^{[1]}$, which will imply

$$\frac{1}{\mu(C)}\int_C |f|\cdot \mathbb{I}_{B'^{[1]}\setminus B^{[1]}}=0.$$

Hence, since we also have $C\cap B\neq\varnothing$, we get $C\in\mathcal{A}$. Thus we obtain $\mu(C)\ge\gamma$ and therefore

$$\mu(C)>\frac{\mu(A)}{2}\ge\frac{\mu(A^{[2]})}{2\mathcal{K}^2}=\frac{\mu(B'^{[1]})}{2\mathcal{K}^2}.$$

Hence we have

$$\frac{1}{\mu(C)}\int_C |f|\cdot \mathbb{I}_{B'^{[1]}\setminus B^{[1]}}\lesssim \frac{1}{\mu(B'^{[1]})}\int_{B'^{[1]}}|f|=\langle f\rangle_{B'^{[1]}}. \tag{8.6}$$

Combining (9.7) and (8.6), we get

$$M(f\cdot \mathbb{I}_{B'^{[1]}\setminus B^{[1]}})(x)\lesssim \langle f\rangle_{B^{[1]}}$$

and so T2)-condition is proved. □

Thus, applying Theorem 1.1, we get

Theorem 8.2. *Let $(X,\mathfrak{M},\mu)$ be a measure space with a ball-basis $\mathfrak{B}$ and w be a A_p weight with $1<p<\infty$. Then the maximal function (4.1) satisfies the bound*

$$\|M\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\left\{\frac{p+2}{p(p-1)},\frac{3p-2}{p}\right\}}.$$

If in addition $\mathfrak{B}$ satisfies the Besicovitch condition, then

$$\|M\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\{1,1/(p-1)\}}.$$

Remark 8.1. *Theorem 8.2 does not give the full weighted estimate like (1.2), which is known to be optimal for the maximal function in Euclidean spaces ([3]). In the general case the optimality only occurs when $1<p\le 2$. The Buckley [3] argument can be applied to get full bound (1.2) in the case of Besicovitch condition.*

Another example of BO operator is the martingale transform. Let $(X,\mathfrak{M},\mu)$ be a measure space, and let $\{\mathfrak{B}_n : n\in\mathbb{Z}\}$ be a collections of measurable sets such that

(1) Each $\mathfrak{B}_n$ is a finite or countable partition of X.
(2) For each n and $A\in\mathfrak{B}_n$ the set A is the union of sets from $\mathfrak{B}_{n+1}$.
(3) The collection $\mathfrak{B}=\cup_{n\in\mathbb{Z}}\mathfrak{B}_n$ generates the σ-algebra $\mathfrak{M}$.
(4) For any points $x,y\in X$ there is a set $A\in X$ such that $x,y\in A$.

For a given $A\in\mathfrak{B}$ let $\operatorname{pr}(A)$ (parent of A) be the minimal element of $\mathfrak{B}$ satisfying $A\subsetneq \operatorname{pr}(A)$. One can easily check that $\mathfrak{B}$ satisfies the ball-basis conditions B1)-B4). Moreover, for $A\in\mathfrak{B}$ we can define

$$A^*=\begin{cases} A \text{ if} & \mu(\operatorname{pr}(A))>2\mu(A),\\ \operatorname{pr}^n(A) \text{ if} & \mu(\operatorname{pr}^n(A))\le 2\mu(A)<\mu(\operatorname{pr}^{n+1}(A)),\end{cases} \tag{8.7}$$

and take $\mathcal{K}=2$. Consider a function $f\in L^1(X)$. The martingale difference associated with $A\in\mathfrak{B}$ is

$$\Delta_A f(x)=\sum_{B:\,\operatorname{pr}(B)=A}\left(\frac{1}{\mu(B)}\int_B f-\frac{1}{\mu(A)}\int_A f\right)\mathbb{I}_B(x).$$

The martingale transform operator is defined by

$$Tf(x)=\sum_{A\in\mathfrak{B}}\varepsilon_A\Delta_A f(x),$$

where $\varepsilon_A=\pm 1$ are fixed.

Lemma 8.1. *Any martingale transform T satisfies T1)-condition and moreover, $\mathcal{L}_1(T)=0$.*

Proof. Take a function $f\in L^1(X)$ with

$$\operatorname{supp} f\in X\setminus A^*$$

and two points $x,x'\in A$. Observe that

$$\begin{aligned}&\Delta_B f(x)=\Delta_B f(x'),\ \text{ if } B\supset A^*,\\&\Delta_B f(x)=\Delta_B f(x')=0,\ \text{ if } B\subseteq A^* \text{ or } B\cap A^*=\varnothing.\end{aligned}$$

Thus we have $\Delta_B f(x)=\Delta_B f(x')$ for any ball $B\in\mathfrak{B}$ and so $Tf(x)=Tf(x')$. This implies $\mathcal{L}_1(T)=0$. □

Lemma 8.2. *If T is a martingale transform, then for any ball $A\in\mathfrak{B}$ we have*

$$\sup_{x\in A,\, f\in L^1(X)}\frac{|T(f\cdot\mathbb{I}_{\mathrm{pr}(A)\setminus A})(x)|}{\langle f\rangle_{\mathrm{pr}(A)}}\le 2. \tag{8.8}$$

Proof. Take a function $f\in L^1(X)$ with

$$\operatorname{supp} f\subset \mathrm{pr}(A)\setminus A.$$

Consider the sequence A_k, $k=1,2,\ldots$, defined by $A_0=A$, $A_{k+1}=\mathrm{pr}(A_k)$. For every point $x\in A$ we have

$$\begin{aligned}&\Delta_{A_k} f(x)=\left(\frac{1}{\mu(A_{k-1})}-\frac{1}{\mu(A_k)}\right)\int_{\mathrm{pr}(A)} f \text{ if } k>1,\\&\Delta_B f(x)=0,\ \text{ if } B\subseteq A,\\&\Delta_{A_1} f(x)=-\frac{1}{\mu(A_1)}\int_{\mathrm{pr}(A)} f.\end{aligned}$$

Hence we get

$$\begin{aligned}|Tf(x)|&\le\sum_{k\ge 1}|\Delta_k f(x)|\le\frac{1}{\mu(A_1)}\int_{\mathrm{pr}(A)}|f|+\sum_{k>1}\left(\frac{1}{\mu(A_{k-1})}-\frac{1}{\mu(A_k)}\right)\int_{\mathrm{pr}(A)}|f|\\&\le\frac{2}{\mu(\mathrm{pr}(A))}\int_{\mathrm{pr}(A)}|f|\\&=2\langle f\rangle_{\mathrm{pr}(A)}\end{aligned}$$

Lemma is proved. □

Lemma 8.3. *If T is a martingale transform, then T satisfies T2)-condition and $\mathcal{L}_2(T)$ is bounded by an absolute constant.*

Proof. We need to prove the inequality

$$\sup_{x\in A,\, f\in L^1(X)}\frac{|T(f\cdot\mathbb{I}_{\mathrm{pr}(A)^*\setminus A^*})(x)|}{\langle f\rangle_{\mathrm{pr}(A)^*}}\le c, \tag{8.9}$$

where $c > 0$ is an absolute constant (see the definition of T2)-condition). If $\mu(\mathrm{pr}(A)) \le 2\mu(A)$, then applying Lemma 4.2 and Lemma 8.1, the left hand side of (8.9) can be estimated by $c \cdot \|T\|_{L^1 \to L^{1,\infty}}$. Since $\mathcal{K} = 2$, we can say that here $c > 0$ is an absolute constant. It is well-known that $\|T\|_{L^1 \to L^{1,\infty}}$ is estimated by an absolute constant too. This implies (8.9). In the case $\mu(\mathrm{pr}(A)) > 2\mu(A)$, applying $\mathcal{K} = 2$, we obtain

$$\begin{aligned} &\sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus A^*})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}} \\ &\qquad \le \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A) \setminus A^*})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}} \\ &\qquad\qquad + \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}} \\ &\qquad \le 2 \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A) \setminus A})(x)|}{\langle f \rangle_{\mathrm{pr}(A)}} \\ &\qquad\qquad + \sup_{x \in A,\, f \in L^1(X)} \frac{|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)|}{\langle f \rangle_{\mathrm{pr}(A)^*}}. \end{aligned}$$

The first terms in the last sum is estimated by (8.8). Now let us estimate the second term. Applying weak-L^1 inequality, for a $\lambda > 0$ we can write

$$\begin{aligned} \mu\{x \in \mathrm{pr}(A) : &|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)| > \lambda \langle f \rangle_{\mathrm{pr}(A)^*}\} \\ &\le \frac{\|T\|_{L^1 \to L^{1,\infty}}}{\lambda \cdot \langle f \rangle_{\mathrm{pr}(A)^*}} \int_{\mathrm{pr}(A)^*} |f| \\ &= \frac{\|T\|_{L^1 \to L^{1,\infty}} \cdot \mu(\mathrm{pr}(A)^*)}{\lambda} \le \frac{\mu(\mathrm{pr}(A))}{2}, \end{aligned}$$

where the last inequality is obtained with a suitable absolute constant $\lambda > 0$. This implies that the inequality

$$|T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)| \le \lambda \langle f \rangle_{\mathrm{pr}(A)^*} \tag{8.10}$$

holds for some point $x \in \mathrm{pr}(A)$. Observe that the function $T(f \cdot \mathbb{I}_{\mathrm{pr}(A)^* \setminus \mathrm{pr}(A)})(x)$ is constant on $\mathrm{pr}(A)$ and it can be shown likewise the proof of Lemma 8.1. Hence we will have (8.10) for any $x \in \mathrm{pr}(A)$. Thus we will give a bound of the second term by an absolute constant. □

Lemma 8.1 and Lemma 8.3 immediately imply

Theorem 8.3. *The martingale transform is a* BO *operator with respect to the ball-basis* $\mathfrak{B}$*. Moreover,* $\mathcal{L}_1(T) = 0$ *and* $\mathcal{L}_2(T)$ *is bounded by an absolute constant.*

Applying Theorem 1.1, Theorem 6.4 and Theorem 8.3 we deduce the following results:

Theorem 8.4 (Lacey [21]). *Let T be a martingale transform. If $B\in\mathfrak{B}$ and $f\in L^1(X)$, then there is a sparse operator $\mathcal{A}$ such that*

$$|Tf(x)|\le C\cdot\mathcal{A}f(x),\quad x\in B,$$

where C is an absolute constant.

Theorem 8.5 (Thiele, Treil and Volberg [34]). *If T is a martingale transform and the weight w satisfies A_p condition with respect to the ball-basis $\mathfrak{B}$, $1<p<\infty$, then we have*

$$\|T\|_{L^p(w)\to L^p(w)}\le c_p[w]_{A_p}^{\max\{1,1/(p-1)\}}.$$

9. Bounded oscillation of Carleson operators

Let $\{T_\alpha\}$ be a family of BO operators. In this section we prove that if the characteristic constants of operators T_α are uniformly bounded, then the domination operator

$$Tf(x)=\sup_\alpha|T_\alpha f(x)| \tag{9.1}$$

is also BO operator. More precisely, we have

Theorem 9.1. *If $(X,\mathfrak{M},\mu)$ is a measure space with a ball-basis $\mathfrak{B}$. If a BO-family of operators $\{T_\alpha\}$ satisfies weak-L^r inequality, then the operator (9.1) satisfies the bounds*

$$\mathcal{L}_1(T)\lesssim\sup_\alpha\mathcal{L}_1(T_\alpha), \tag{9.2}$$

$$\mathcal{L}_2(T)\lesssim\sup_\alpha\mathcal{L}_1(T_\alpha)+\sup_\alpha\mathcal{L}_2(T_\alpha)+\sup_\alpha\|T_\alpha\|_{L^r\to L^{r,\infty}}. \tag{9.3}$$

Proof. Let $A\in\mathfrak{B}$ be an arbitrary ball. Take two points $x,x'\in A$ and a nonzero function $f\in L^r(X)$ with

$$\operatorname{supp}f\subset X\setminus A^{[1]}. \tag{9.4}$$

Suppose that

$$Tf(x)\ge Tf(x'). \tag{9.5}$$

According to the definition of T, for any $\delta>0$ there exists an index α such that

$$Tf(x)\le|T_\alpha f(x)|+\delta. \tag{9.6}$$

On the other hand for the same α we have $Tf(x') \geq |T_\alpha f(x')|$. Thus, applying (9.5), (9.6) and the localization property of T_α, we obtain

$$\begin{aligned}|Tf(x) - Tf(x')| &= Tf(x) - Tf(x') \\ &\leq |T_\alpha f(x)| + \delta - |T_\alpha f(x')| \\ &\leq |T_\alpha f(x) - T_\alpha f(x')| + \delta \\ &\leq \mathcal{L}_1(T_\alpha)\langle f\rangle^*_{A,r} + \delta.\end{aligned}$$

Since $\delta > 0$ can be taken enough small, we get (9.2).

To prove (9.3) fix a ball A ($A^* \neq X$) and consider the number

$$\gamma = \inf_{B\in\mathfrak{B}:\, B\supsetneq A} \mu(B).$$

Since $A \neq X$, from Lemma 3.2 it easily follows that the set of balls B satisfying $B \supsetneq A$ is nonempty. So there exists a ball $B \supsetneq A$ such that

$$\gamma \leq \mu(B) < 2\gamma.$$

Take $f \in L^r(X)$ and a point $x \in A$. We have

$$T(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x) = |T_\alpha(f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)| + \delta, \tag{9.7}$$

for an index α, where $\delta > 0$ can be arbitrarily small. Since T_α satisfies T2)-condition, there exists a ball $C \supsetneq A$ such that

$$T_\alpha(g\cdot \mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x) \lesssim \mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r} \tag{9.8}$$

holds for any $g \in L^r(X)$. Consider the function $g = f\cdot \mathbb{I}_{B^{[1]}\setminus A^{[1]}}$. If $\mu(C) > \mu(B)$, then we have $B \subset C^{[1]}$ and so $B^{[1]} \subset C^{[2]}$. Thus, applying (9.8) and Lemma 4.2, we obtain

(9.9)

$$\begin{aligned}|T_\alpha g(x)| &= |T_\alpha(g\cdot \mathbb{I}_{C^{[2]}\setminus A^{[1]}})(x) \\ &\leq |T_\alpha(g\cdot \mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x) + |T_\alpha(g\cdot \mathbb{I}_{C^{[2]}\setminus C^{[1]}})(x) \\ &\leq \mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r} + \sup_{x\in C,\, u\in L^r(X)} \frac{|T_\alpha(u\cdot \mathbb{I}_{C^{[2]}\setminus C^{[1]}})(x)|}{\langle u\rangle_{C^{[2]},r}}\cdot \langle g\rangle_{C^{[2]},r} \\ &\lesssim \mathcal{L}_2(T_\alpha)\left(\frac{1}{\mu(C^{[1]})}\int_{B^{[1]}}|f|^r\right)^{1/r} \\ &\quad + (\mathcal{L}_1(T_\alpha) + \|T_\alpha\|_{L^r\to L^{r,\infty}})\left(\frac{\mu(C^{[1]})}{\mu(C)}\right)^{1/r}\left(\frac{1}{\mu(C^{[2]})}\int_{B^{[1]}}|f|^r\right)^{1/r} \\ &\lesssim \sup_\alpha(\mathcal{L}_2(T_\alpha) + \mathcal{L}_1(T_\alpha) + \|T_\alpha\|_{L^r\to L^{r,\infty}})\cdot \langle f\rangle_{B^{[1]},r}.\end{aligned}$$

In the case $\mu(C)\le\mu(B)$ we have $C\subset B^{[1]}$, and since $C\supsetneq A$, we obtain

$$\mu(B)\gtrsim\mu(B^{[1]})\ge\mu(C)\ge\gamma\ge\frac{\mu(B)}{2}.$$

Thus, again applying Lemma 4.2 and (9.8), we conclude

(9.10)

$$\begin{aligned}|T_\alpha(g)(x)|&\le|T_\alpha(g\cdot\mathbb{I}_{B^{[2]}\setminus C^{[1]}})(x)|+|T_\alpha(g\cdot\mathbb{I}_{C^{[1]}\setminus A^{[1]}})(x)|\\&\le\sup_{x\in C,\,u\in L^r(X)}\frac{|T_\alpha(u\cdot\mathbb{I}_{B^{[2]}\setminus C^{[1]}})(x)|}{\langle u\rangle_{B^{[2]},r}}\cdot\langle g\rangle_{B^{[2]},r}+\mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r}\\&\lesssim(\mathcal{L}_1(T_\alpha))+\|T_\alpha\|_{L^r\to L^{r,\infty}})\langle g\rangle_{B^{[2]},r}+\mathcal{L}_2(T_\alpha)\langle g\rangle_{C^{[1]},r}\\&\lesssim\sup_\alpha(\mathcal{L}_1(T_\alpha))+\|T_\alpha\|_{L^r\to L^{r,\infty}}+\mathcal{L}_2(T_\alpha))\cdot\langle f\rangle_{B^{[1]},r}.\end{aligned}$$

Observe that the admissible constants used in (9.9) and (9.10) do not depend on f, point x the number δ from (9.7). Hence, since δ can be taken arbitrarily small, from (9.7), (9.9) and (9.10) we get the inequality

$$\begin{aligned}&\sup_{x\in A,\,f\in L^r(X)}\frac{T(f\cdot\mathbb{I}_{B^{[1]}\setminus A^{[1]}})(x)}{\langle f\rangle_{B^{[1]},r}}\\&\qquad\lesssim\mathcal{L}_1(T_\alpha))+\mathcal{L}_2(T_\alpha)+\|T_\alpha\|_{L^r\to L^{r,\infty}},\end{aligned}$$

which implies (9.3). □

Let T be a BO operator and $\mathcal{G}=\{g_\alpha\}\subset L^\infty(X)$ be a family of functions such that

$$\beta=\sup_\alpha\|g_\alpha\|_\infty<\infty,\quad\|T\|_{L^r\to L^{r,\infty}}<\infty.\tag{9.11}$$

One can easily check that the operators

$$T_\alpha f(x)=T(g_\alpha\cdot f)(x),\tag{9.12}$$

are BO operator. Moreover, we have

(9.13)

$$\mathcal{L}_1(T_\alpha)\le\beta\mathcal{L}_1(T),\ \mathcal{L}_2(T_\alpha)\le\beta\mathcal{L}_2(T),\ \|T_\alpha\|_{L^r\to L^{r,\infty}}\le\beta\|T\|_{L^r\to L^{r,\infty}}.$$

Define the maximal modulation of the operator$T$ by

$$T^{\mathcal{G}}f(x)=\sup_\alpha|T_\alpha f(x)|.\tag{9.14}$$

According to Theorem 9.1 and relations (9.13), we conclude that $T^{\mathcal{G}}$ is also BO operator. Hence, applying Theorem 1.1, we obtain

Theorem 9.2. *If $T \in \mathrm{BO}_{\mathfrak{B}}(X)$ satisfies (9.11), then for any function $f \in L^r(X)$ and a ball $B \in \mathfrak{B}$ there exists a family of balls $\mathcal{S}$, which is a union of two sparse collections and*

$$|T^{\mathcal{G}} f(x)| \lesssim \sup_{\alpha} \|g_\alpha\|_\infty (\mathcal{L}_1(T) + \mathcal{L}_2(T) + \|T^{\mathcal{G}}\|_{L^r \to L^{r,\infty}}) \cdot \mathcal{A}_{\mathcal{S},r} f(x),$$

for a.e. $x \in B$.

Weighted estimates of the maximal modulations of Calderón-Zygmund operators on $\mathbb{R}^n$ (in particular Carleson or Walsh-Carleson operators) were considered in the papers [6, 8]. Theorem 9.2 implies a pointwise sparse domination of such operators, which is the strongest version of the weighted norm domination of Carleson operators by sparse operators proved in [6].

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