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On Weyl multipliers of the rearranged trigonometric system

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Abstract. We prove that the condition $\sum_{n=1}^{\infty} 1/(nw(n)) < \infty$ is necessary for an increasing sequence of numbers $w(n)$ to be an almost everywhere unconditional convergence Weyl multiplier for the trigonometric system. This property was known long ago for Haar, Walsh, Franklin and some other classical orthogonal systems. The proof of this result is based on a new sharp logarithmic lower bound on L^2 for the majorant operator related to the rearranged trigonometric system.

Bibliography: 32 titles.

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§ 1. Introduction

Let $\Phi = \{\varphi_n : n = 1, 2, \dots\} \subset L^2(0,1)$ be an orthonormal system. Recall that a sequence of positive numbers $w(n) \nearrow \infty$ is said to be an *a.e. convergence Weyl multiplier* (a C-*multiplier* for short) if every series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{1.1}$$

with coefficients satisfying the condition

$$\sum_{n=1}^{\infty} a_n^2 w(n) < \infty, \tag{1.2}$$

is a.e. convergent (see [8] or [7]). The Menshov-Rademacher classical theorem (see [10] and [19]) states that the sequence $\log^2 n$ is a C-multiplier for any orthonormal system. The sharpness of $\log^2 n$ in this statement was established by Menshov in the same paper [10], proving that any sequence $w(n) = o(\log^2 n)$ fails to be a C-multiplier for some orthonormal system.

The following definitions are well known in the theory of orthogonal series.

Definition 1.1. A sequence of positive numbers $w(n) \nearrow \infty$ is said to be an *a.e. convergence Weyl multiplier for the rearrangements* (an RC-*multiplier*) *of an orthonormal system* Φ if it is a C-multiplier for any rearrangement of Φ.

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Definition 1.2. A sequence of positive numbers $w(n) \nearrow \infty$ is said to be an *a.e. unconditional convergence Weyl multiplier* (a UC-*multiplier*) *for an orthonormal system* Φ if under condition (1.2) the series (1.1) converges almost everywhere after any rearrangement of its terms.

For a given orthonormal system Φ, we denote by $\mathrm{RC}(\Phi)$ and $\mathrm{UC}(\Phi)$ the families of RC- and UC-multipliers, respectively. Observe that according to the Menshov-Rademacher theorem we have $\log^2 n \in \mathrm{RC}(\Phi)$ for any orthonormal system Φ, and a counterexample of Menshov tells us that $\log^2 n$ is optimal in this statement. The following two theorems provide a necessary and sufficient condition for a sequence to be a UC-multiplier for all orthonormal systems.

Theorem A (Orlicz; see [17]). *If an increasing sequence of positive numbers* $\lambda(n)$ *satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{n\lambda(n)\log n} < \infty, \tag{1.3}$$

then $w(n) = \lambda(n)\log^2 n$ *is a* UC-*multiplier for any orthonormal system.*

Theorem B (Tandori; see [21]). *If an increasing sequence of positive numbers* $\lambda(n)$ *does not satisfy* (1.3), *then there exists an orthonormal system for which the sequence* $w(n) = \lambda(n)\log^2 n$ *fails to be a* UC-*multiplier.*

In particular, these results imply that the sequence $\log^2 n(\log\log n)^{1+\varepsilon}$, $\varepsilon > 0$, is a UC-multiplier for any orthonormal system, while $\log^2 n \log\log n$ is not a UC-multiplier for some orthonormal systems.

The study of RC- and UC-multipliers of classical orthonormal systems is an old issue in the theory of orthogonal series. It is well known that the sequence $w(n) \equiv 1$ is a C-multiplier for trigonometric, Walsh, Haar and Franklin systems, while it fails to be an RC-multiplier for these systems. Kolmogorov [9] was the first who observed that the sequence $w(n) \equiv 1$ is not an RC-multiplier for the trigonometric system. However, he never published the proof of this fact. A proof of this assertion was later given by Zahorski [31]. Afterwards developing Zahorski's argument, Ul'yanov (see [23] and [24]) established such a property for the Haar and Walsh systems. Using the Haar functions technique, Olevskii [16] succeeded in proving that such a phenomenon may also occur for arbitrary complete orthonormal systems.

Later on Ul'yanov (see [25], [27]) found the optimal growth of the RC- and UC-multipliers of the Haar system. Moreover, his technique of the proof became a key argument in the study of the analogous problems for other classical systems.

Theorem C (Ul'yanov; see [26]). *The sequence* $\log n$ *is an* RC-*multiplier for the Haar system and any sequence* $w(n) = o(\log n)$ *is not.*

Theorem D (Ul'yanov; see [26]). *The sequence* $w(n)$ *is a* UC-*multiplier for the Haar system if and only if the bound*

$$\sum_{n=1}^{\infty} \frac{1}{nw(n)} < \infty$$

holds.

In his famous overview [28] of 1964 Ul'yanov raised two problems (see [28], § 11), and these have been further recalled several times in different papers of that author (see [27], [29], [30]):

1) find an optimal sequence $w(n)$ to be an RC-multiplier for the trigonometric (Walsh) system;
2) characterize the UC-multipliers of the trigonometric (Walsh) system.

The following result somehow clarifies the relationship between these two problems in terms of the Orlicz 'extra factor' $\lambda(n)$ (see (1.3)). It also tells us that the Orlicz theorem can be deduced from the Menshov-Rademacher theorem.

Theorem E (Ul'yanov and Poleshchuk; see [26], [18]). *If $w(n)$ is an RC-multiplier for an orthonormal system $\Phi = \{\varphi_n(x)\}$ and $\lambda(n)$ is an increasing sequence of positive numbers satisfying (1.3), then the sequence $\lambda(n)w(n)$ is a UC-multiplier for Φ.*

Relating to problem 1), we first note that the Menshov-Rademacher theorem implies that $\log^2 n$ is an RC-multiplier for the trigonometric and Walsh systems, and second, no RC-multiplier $w(n) = o(\log^2 n)$ is known for these systems. Similarly, the only known UC-multipliers of trigonometric and Walsh systems are the sequences $\lambda(n)\log^2 n$ coming from the result of Orlicz for the general orthonormal systems.

The lower estimates for RC- and UC-multipliers of the Walsh system were studied in [1], [14], [15] and [22]. The best result at this moment, proved independently by Bochkarev (see [2] and [1]) and Nakata (see [14]), says that if an increasing sequence $w(n)$ satisfies

$$\sum_{n=1}^{\infty} \frac{1}{nw(n)} = \infty, \tag{1.4}$$

then it is not a UC-multiplier for the Walsh system.

For the trigonometric system analogous bounds were studied in [11]–[13], [21], [5]. The most general result is due to Galstyan [5] (1992), who proved that under the condition

$$\sum_{n=1}^{\infty} \frac{1}{n \log\log n \, w(n)} = \infty \tag{1.5}$$

the sequence $w(n)$ fails to be a UC-multiplier for the trigonometric system. In contrast to the Haar and Walsh systems, in the trigonometric case we see an extra $\log\log n$ factor in (1.5). Corollary 1.3 stated below tells us that the factor $\log\log n$ can be removed also in the case of the trigonometric system.

Note that the following inequality is the key part of the proof of the Menshov-Rademacher theorem.

Theorem F (Menshov and Rademacher; see [10], [19] and also [8]). *For any orthonormal system $\{\varphi_k\colon k = 1, 2, \dots, n\} \subset L^2(0,1)$ and any coefficients a_k*

$$\left\| \max_{1\leqslant m\leqslant n} \left| \sum_{k=1}^{m} a_k\varphi_k \right| \right\|_2 \leqslant c\log n \left\| \sum_{k=1}^{n} a_k\varphi_k \right\|_2, \tag{1.6}$$

where $c > 0$ is an absolute constant.

Similarly, the counterexample of Menshov is based on the following result.

Theorem G (Menshov; see [10]). *For any natural number* $n \in \mathbb{N}$ *there exists an orthogonal system* φ_k, $k = 1, 2, \ldots, n$, *such that*

$$\left\| \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} \varphi_k \right| \right\|_2 \geqslant c \log n \left\| \sum_{k=1}^{n} \varphi_k \right\|_2,$$

for an absolute constant $c > 0$.

To state the results of the present paper let us introduce some notation. For two positive quantities a and b the notation $a \lesssim b$ will stand for the inequality $a < cb$, where $c > 0$ is an absolute constant, and we write $a \sim b$ whenever $a \lesssim b \lesssim a$. Let Σ_N denote the family of one-to-one mappings (permutations) on $\{1, 2, \ldots, N\}$. We will consider the trigonometric system on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For a given integer $N \geqslant 1$ and $\sigma \in \Sigma_N$, we consider the operator $T_{\sigma,N} \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by

$$T_{\sigma,N} f(x) = \max_{1 \leqslant m \leqslant N} \left| \sum_{k=1}^{m} c_{\sigma(k)} e^{2\pi i \sigma(k) x} \right|, \quad \text{where } c_k = \int_{\mathbb{T}} f(x) e^{-2\pi i k x}\, dx.$$

Our main result is the following.

Theorem 1.1. *For any integer* $N > 1$ *there exists a permutation* $\sigma \in \Sigma_N$ *such that*

$$\|T_{\sigma,N}\|_{L^2 \to L^2} \sim \log N. \tag{1.7}$$

We note that the upper bound in (1.7) follows from the Menshov-Rademacher inequality (1.6). Recall the weak L^2-norm of an operator $T \colon L^2 \to L^2$, defined by

$$\|T\|_{L^2 \to L^{2,\infty}} = \sup_{\|f\|_2 \leqslant 1,\, \lambda > 0} \lambda (|\{|Tf(x)| > \lambda\}|)^{1/2}.$$

From the lower bound of (1.7), applying Lemma 8.1, we easily deduce also a lower estimate for the weak L^2-norm of the operator $T_{\sigma,N}$. Namely, the following holds.

Corollary 1.1. *For any integer* $N > 1$ *one can find a permutation* $\sigma \in \Sigma_N$ *such that*

$$\|T_{\sigma,N}\|_{L^2 \to L^{2,\infty}} \gtrsim \sqrt{\log N}. \tag{1.8}$$

Applying (1.8) we prove the following results.

Corollary 1.2. *Any increasing sequence of positive numbers* $w(n)$, *satisfying*

$$w(n) = o(\log n), \tag{1.9}$$

fails to be an RC-*multiplier for the trigonometric system. Moreover, there are coefficients* a_n *satisfying* (1.2) *such that the series*

$$\sum_{n=1}^{\infty} a_n e^{2\pi i \sigma(n) x}$$

is almost everywhere divergent for some permutation σ.

Corollary 1.3. *If an increasing sequence of positive numbers $w(n)$ satisfies* (1.4), *then it is not a* UC*-multiplier for the trigonometric system. Namely, there are coefficients a_n satisfying* (1.2) *such that the series*

$$\sum_{n=1}^{\infty} a_n e^{2\pi i n x} \tag{1.10}$$

can be rearranged into an almost everywhere divergent series.

Remark 1.1. Corollaries 1.2 and 1.3 can be stated in terms of real trigonometric series, by considering

$$\sum_{n=1}^{\infty} a_n \cos(nx + \rho_n)$$

instead of the series (1.10). In fact, using an elementary argument, one can deduce the real trigonometric versions of Corollaries 1.2 and 1.3 from their complex analogues.

Remark 1.2. We do not know whether the reverse inequality to (1.8) holds for every permutation σ, that is,

$$\max_{\sigma \in \Sigma_N} \|T_{\sigma,N}\|_{L^2 \to L^{2,\infty}} \lesssim \sqrt{\log N}.$$

Remark 1.3. An estimate like (1.8) is not known for the Walsh system. Note that our proof of (1.8) is based on a specific argument which works only for the trigonometric system and it is not applicable in the case of the Walsh system. Namely, we use a logarithmic lower bound due to Demeter [3] for the directional Hilbert transform on the plane.

Remark 1.4. Recall the following problem posed by Kashin [20], which has become more interesting after the result of Theorem 1.1: is there a sequence of positive numbers $\gamma(n) = o(\log n)$ such that for any orthonormal system φ_n on $(0,1)$ the inequality

$$\left(\int_0^1 \int_0^1 \max_{1 \leqslant m \leqslant n} \left|\sum_{k=1}^{m} \varphi_k(x)\varphi_k(y)\right|^2 dx\, dy\right)^{1/2} \leqslant \gamma(n)\sqrt{n} \tag{1.11}$$

holds?

Remark 1.5. Finally, we note that the result analogous to Theorem D for the Franklin system was proved by Gevorkyan [4]. In a recent paper of this author [6] the analogues of Theorems C and D were proved for the orthonormal systems of nonoverlapping martingale-difference (in particular, Haar) polynomials.

§ 2. Directional Hilbert transform and Demeter's example

The starting point for our construction is an example given by Demeter [3] for the directional Hilbert transform. To state it we need the notation

$$B(a,b) = \{\mathbf{x} \in \mathbb{R}^2 \colon a \leqslant \|\mathbf{x}\| < b\}, \qquad 0 \leqslant a < b \leqslant \infty$$

and

$$\Gamma_\theta = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \colon x_1 \cos\theta + x_2 \sin\theta \geqslant 0\}.$$

For a rapidly decreasing function f and a unit vector $(\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$, we define

$$H_\theta f(\mathbf{x}) = \text{p.v.}\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(\mathbf{x} - t(\cos\theta, \sin\theta))}{t}\, dt, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2,$$

which is the one-dimensional Hilbert transform corresponding to the direction θ. It is well known that this operator can be extended to a bounded operator on $L^2(\mathbb{R}^2)$. For the collection of uniformly distributed unit vectors

$$\Theta = \left\{\theta_k = \frac{\pi k}{N},\ k = 1, 2, \ldots, N\right\},$$

consider the operator

$$H_\Theta^* f(\mathbf{x}) = \sup_{\theta \in \Theta} |H_\theta f(\mathbf{x})|.$$

The result of [3] is the lower bound $\|H_\Theta^*\|_{2\to 2} \gtrsim \log N$. We find it appropriate to give a detailed proof of this result.

Lemma 2.1 (Demeter; see [3]). *For any integer $N > N_0$, where N_0 is an absolute constant, the function*

$$f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|} \mathbf{1}_{B(10N^{-9}, N^{-8})}(\mathbf{x}) \tag{2.1}$$

satisfies the inequality

$$\|H_\Theta^*(f)\|_2 \gtrsim \log N \|f\|_2. \tag{2.2}$$

Proof. A change of variable allows us to prove (2.2) for the function

$$h(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|} \mathbf{1}_{B(10,N)}(\mathbf{x})$$

instead of f. Fix a point $\mathbf{x}$ in the upper half-plane $\mathbb{R}^2_+$ satisfying

$$10^5 \leqslant \|\mathbf{x}\| \leqslant \frac{N}{3} \tag{2.3}$$

and consider the unit vector $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\| = (\cos\theta, \sin\theta)$. Clearly, there is a unit vector $\mathbf{u}_k = (\cos\theta_k, \sin\theta_k)$ such that

$$\|\mathbf{u}_k - \mathbf{u}\| \leqslant \frac{\pi}{N}. \tag{2.4}$$

A geometric argument shows that the line $\mathbf{x} - t\mathbf{u}_k$, $t \in \mathbb{R}$, has four points of intersection with the boundary of $B(10, N)$. Moreover, we have

$$E = \{t \in \mathbb{R} \colon \mathbf{x} - t\mathbf{u}_k \in B(10, N)\} = (A, a] \cup [b, B), \tag{2.5}$$

where the numbers $A < a < b < B$ satisfy

$$\|\mathbf{x} - A\mathbf{u}_k\| = \|\mathbf{x} - B\mathbf{u}_k\| = N, \qquad \|\mathbf{x} - a\mathbf{u}_k\| = \|\mathbf{x} - b\mathbf{u}_k\| = 10, \tag{2.6}$$

$$t \in (a, b) \iff \|\mathbf{x} - t\mathbf{u}_k\| < 10, \tag{2.7}$$

$$t \notin [A, B] \iff \|\mathbf{x} - t\mathbf{u}_k\| > N.$$

For any $t \in E$ we have

$$|t| \leqslant \|\mathbf{x}\| + \|\mathbf{x} - t\mathbf{u}_k\| \leqslant \frac{N}{3} + N = \frac{4N}{3}. \tag{2.8}$$

Based on (2.4), (2.6) and (2.8), we claim that

$$\|\mathbf{x}\| - 15 \leqslant a \leqslant \|\mathbf{x}\| - 5, \tag{2.9}$$

$$\|\mathbf{x}\| + 5 \leqslant b \leqslant \|\mathbf{x}\| + 15, \tag{2.10}$$

$$||A| - N| \leqslant \frac{N}{3}, \tag{2.11}$$

$$||B| - N| \leqslant \frac{N}{3}. \tag{2.12}$$

Indeed, observe first that $t = \|\mathbf{x}\| \notin E$, since

$$\big\|\mathbf{x} - \|\mathbf{x}\|\mathbf{u}_k\big\| \leqslant \big\|\mathbf{x} - \|\mathbf{x}\|\mathbf{u}\big\| + \|\mathbf{x}\|\,\|\mathbf{u} - \mathbf{u}_k\| = \|\mathbf{x}\|\,\|\mathbf{u} - \mathbf{u}_k\| \leqslant 2.$$

So from (2.7) we conclude that $a < \|\mathbf{x}\| < b$. Thus, using also the inequality

$$\big|10 - |\|\mathbf{x}\| - a|\big| = \big|\|\mathbf{x} - a\mathbf{u}_k\| - \|\mathbf{x} - a\mathbf{u}\|\big| \leqslant |a|\,\|\mathbf{u} - \mathbf{u}_k\| \leqslant 5,$$

we easily get (2.9). Similarly, we have (2.10). From

$$\big||A| - N\big| = \big|\|A\mathbf{u}_k\| - \|\mathbf{x} - A\mathbf{u}_k\|\big| \leqslant \|\mathbf{x}\| \leqslant \frac{N}{3}$$

and the same bound for B we obtain (2.11) and (2.12), respectively. If $t \in E$, then by (2.4) and (2.8) we have

$$\big|\|\mathbf{x}\| - t\big| = \|\mathbf{x} - t\mathbf{u}\| \geqslant \|\mathbf{x} - t\mathbf{u}_k\| - |t|\|\mathbf{u} - \mathbf{u}_k\| \geqslant 10 - 5 = 5,$$

and therefore

$$\|\mathbf{x} - t\mathbf{u}_k\| \geqslant \|\mathbf{x} - t\mathbf{u}\| - |t|\|\mathbf{u} - \mathbf{u}_k\| \geqslant \big|\|\mathbf{x}\| - t\big| - 4.8 \geqslant \frac{\big|\|\mathbf{x}\| - t\big|}{25}.$$

Thus we get

$$\left|\frac{1}{\|\mathbf{x} - t\mathbf{u}\|} - \frac{1}{\|\mathbf{x} - t\mathbf{u}_k\|}\right| \leqslant \frac{|t|\|\mathbf{u} - \mathbf{u}_k\|}{\|\mathbf{x} - t\mathbf{u}\|\|\mathbf{x} - t\mathbf{u}_k\|} \leqslant \frac{25\pi|t|}{N\big|t - \|\mathbf{x}\|\big|^2},$$

and hence, using also (2.5), (2.9) and (2.10),

$$\left|\pi H_{\theta_k} h(\mathbf{x}) - \text{p.v.} \int_E \frac{1}{t\|\mathbf{x} - t\mathbf{u}\|}\, dt\right| \leqslant \frac{25\pi}{N} \int_E \frac{1}{|t - \|\mathbf{x}\||^2}\, dt \leqslant \frac{50\pi}{N} \int_5^\infty \frac{1}{t^2}\, dt = \frac{10\pi}{N}. \tag{2.13}$$

On the other hand,

$$\begin{aligned}
\text{p.v.}\int_E \frac{1}{t\|\mathbf{x}-t\mathbf{u}\|}\,dt &= \text{p.v.}\int_A^a \frac{dt}{t\|\mathbf{x}-t\mathbf{u}\|} + \text{p.v.}\int_b^B \frac{dt}{t\|\mathbf{x}-t\mathbf{u}\|}\\
&= \text{p.v.}\int_A^a \frac{dt}{t(\|\mathbf{x}\|-t)} + \text{p.v.}\int_b^B \frac{dt}{t(t-\|\mathbf{x}\|)}\\
&= \text{p.v.}\frac{1}{\|\mathbf{x}\|}\int_A^a \left(\frac{1}{\|\mathbf{x}\|-t}+\frac{1}{t}\right)dt + \frac{1}{\|\mathbf{x}\|}\int_b^B \left(\frac{1}{t-\|\mathbf{x}\|}-\frac{1}{t}\right)dt\\
&= \frac{1}{\|\mathbf{x}\|}\big(\log\big|\|\mathbf{x}\|-A\big| - \log\big|\|\mathbf{x}\|-a\big| + \log|a| - \log|A|\big)\\
&\quad + \frac{1}{\|\mathbf{x}\|}\big(\log\big|B-\|\mathbf{x}\|\big| - \log\big|b-\|\mathbf{x}\|\big| + \log|b| - \log|B|\big).
\end{aligned}$$

Using (2.3), (2.11) and (2.12) we can say that

$$\log\big|\|\mathbf{x}\|-A\big|,\qquad \log\big|B-\|\mathbf{x}\|\big|,\qquad \log|A| \quad\text{and}\quad \log|B|$$

are equal to $\log N + c$ for different constants $c\in[\log(1/3),\log(5/3)]$. From (2.9) and (2.10) we get $\log\big|\|\mathbf{x}\|-a\big|, \log\big|b-\|\mathbf{x}\|\big| \in [\log 5, \log 15]$. On the other hand, for $\log|a|$ and $\log|b|$ we have a lower bound by $\log(\|\mathbf{x}\|/2)$ in view of (2.3). All these imply that

$$\text{p.v.}\int_E \frac{1}{t\|\mathbf{x}-t\mathbf{u}\|}\,dt \geqslant \frac{2\log(10^{-5}\|\mathbf{x}\|)}{\|\mathbf{x}\|}. \tag{2.14}$$

Combining (2.13) and (2.14), we obtain

$$\pi H^*_\Theta h(\mathbf{x}) \geqslant \pi|H_{\theta_k} h(\mathbf{x})| \geqslant \frac{\log(10^{-5}\|\mathbf{x}\|)}{\|\mathbf{x}\|} - \frac{5\pi}{N}$$

for all $\mathbf{x}\in\mathbb{R}^2_+$ satisfying (2.3). Thus, simple integration shows that

$$\begin{aligned}
\|H^*_\Theta(h)\|_2^2 &\gtrsim \int_{B(10^5,N/3)\cap\mathbb{R}^2_+} |H_{\theta_k}(h)|^2\\
&\gtrsim \int_{10^5}^{N/3}\left(\frac{\log^2(10^{-5}r)}{r} - \frac{10\pi\log(10^{-5}r)}{N} + \frac{25\pi^2}{N^2}r\right)dr \gtrsim \log^3 N
\end{aligned}$$

and $\|h\|_2 \lesssim \sqrt{\log N}$ for $N > N_0$. This implies (2.2).

Lemma 2.1 is proved.

§ 3. Smooth modification of the function f

Since the one-dimensional Hilbert transform is the multiplier operator of $i\,\mathrm{sign}\,x$, for any direction $\theta = (\cos\theta,\sin\theta)$ we have

$$\widehat{H_\theta f}(\mathbf{x}) = i\,\mathrm{sign}(x_1\cos\theta + x_2\sin\theta)\widehat{f}(\mathbf{x}).$$

Recall the multiplier operator T_D corresponding to a region $D\subset\mathbb{R}^2$ and defined by

$$\widehat{T_D(f)} = \mathbf{1}_D\widehat{f}.$$

One can check that

$$T_{\Gamma_\theta}(f) = \frac{f - iH_\theta f}{2}. \tag{3.1}$$

We denote

$$T^* f = \sup_{\theta\in\Theta} |T_{\Gamma_\theta} f|.$$

So the bound (2.2) is equivalent to the inequality

$$\|T^*(f)\|_2 \gtrsim \log N \|f\|_2, \tag{3.2}$$

which will be used in the next sections. In this section we examine some properties of the function (2.1).

Lemma 3.1. *The function* (2.1) *satisfies the relations*

$$\|f\|_1 \sim N^{-8}, \qquad \|f\|_2 \sim \sqrt{\log N} \tag{3.3}$$

and

$$\omega_2(\delta, f) = \sup_{\|\mathbf{h}\|<\delta} \left(\int_{\mathbb{R}^2} |f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|^2 \, d\mathbf{x}\right)^{1/2} \lesssim N^5\sqrt{\delta} \tag{3.4}$$

for any $0 < \delta < N^{-10}$.

Proof. Equations (3.3) are results of a simple integration. Fix a vector $\mathbf{h}$, $\|\mathbf{h}\| < \delta$. Observe that

$$\mathbf{x} \in B(10N^{-9} + \delta, N^{-8} - \delta) \tag{3.5}$$

implies that $\mathbf{x}+\mathbf{h} \in B(10N^{-9}, N^{-8})$ and so

$$|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})| = \left|\frac{1}{\|\mathbf{x}+\mathbf{h}\|} - \frac{1}{\|\mathbf{x}\|}\right| \leqslant \frac{\|\mathbf{h}\|}{\|\mathbf{x}+\mathbf{h}\|\,\|\mathbf{x}\|} \leqslant N^{18}\delta.$$

Using this we get

$$\int_{B(10N^{-9}+\delta, N^{-8}-\delta)} |f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|^2 \, d\mathbf{x} \lesssim |B(10N^{-9}+\delta, N^{-8}-\delta)| N^{36}\delta^2 \lesssim N^{20}\delta^2. \tag{3.6}$$

If

$$\mathbf{x} \in B(10N^{-9} - \delta, 10N^{-9} + \delta) \cup B(N^{-8} - \delta, N^{-8} + \delta), \tag{3.7}$$

then $|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})| \leqslant 2\|f\|_\infty \lesssim N^9$ and so

$$\int_{B(10N^{-9}+\delta, 10N^{-9}-\delta)\cup B(N^{-8}-\delta, N^{-8}+\delta)} |f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|^2 \, dx \lesssim N^{10}\delta. \tag{3.8}$$

If $\mathbf{x}$ is outside the regions that we have in (3.5) and (3.7), then $f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) = 0$. So combining (3.6) and (3.8) we obtain (3.4).

The lemma is proved.

It is well known that there exists a spherical function $K \in L^\infty(\mathbb{R}^2)$ satisfying the relations

$$\int_{\mathbb{R}^2} K(\mathbf{t})\, d\mathbf{t} = 1, \tag{3.9}$$

$$\operatorname{supp} \widehat{K} \subset B(0,1) \tag{3.10}$$

and

$$0 < K(\mathbf{x}) \leqslant \frac{c}{|\mathbf{x}|^{50}}, \tag{3.11}$$

where $c > 0$ is a constant. Indeed, choose a spherical function $\varphi \in C^\infty(\mathbb{R}^2)$ with $\operatorname{supp} \varphi \subset B(0, 1/2)$ and define $K(x)$ by $\widehat{K} = c_1(\varphi * \varphi)$. Clearly, we have (3.10), as well as (3.11) for any power instead of 50 in the denominator. The relation (3.9) will be satisfied after a suitable choice of the constant c_1. We are going to replace the function (2.1) by the function

$$g(\mathbf{x}) = \int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{t}) \mathscr{K}(\mathbf{t})\, d\mathbf{t}, \tag{3.12}$$

where

$$\mathscr{K}(\mathbf{x}) = N^{30} K(N^{15}\mathbf{x}).$$

Lemma 3.2. *For N large enough, the function (3.12) is spherical and satisfies the relations*

$$\operatorname{supp} \widehat{g} \subset B(0, N^{15}), \tag{3.13}$$

$$\|g - f\|_2 \lesssim \frac{1}{N^2} \tag{3.14}$$

and

$$\|T^*(g)\|_2 \gtrsim \log N \|g\|_2. \tag{3.15}$$

Proof. The function g is spherical since f and $\mathscr{K}$ are spherical. Applying the Fourier transform to the convolution (3.12) we get

$$\widehat{g}(\mathbf{x}) = \widehat{f}(\mathbf{x}) \widehat{\mathscr{K}}(\mathbf{x}) = \widehat{f}(\mathbf{x}) \widehat{K}\left(\frac{\mathbf{x}}{N^{15}}\right), \tag{3.16}$$

so (3.10) immediately implies (3.13). Write g in the form

$$\begin{aligned} g(\mathbf{x}) &= \int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{t}) \mathscr{K}(\mathbf{t})\, d\mathbf{t} \mathbf{1}_{B(0,1)}(\mathbf{x}) + \int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{t}) \mathscr{K}(\mathbf{t})\, d\mathbf{t} \mathbf{1}_{B(1,\infty)}(\mathbf{x}) \\ &= I_1(\mathbf{x}) + I_2(\mathbf{x}). \end{aligned}$$

Applying (3.3) and (3.11), we can roughly estimate

$$|I_2(\mathbf{x})| \lesssim \frac{\mathbf{1}_{B(1,\infty)}(\mathbf{x})}{N^2 \|\mathbf{x}\|},$$

then after a simple integration we get

$$\|I_2\|_2 \lesssim \frac{1}{N^2}. \tag{3.17}$$

Choosing $\delta = N^{-14}$, for every $\mathbf{x} \in B(0,1)$ we can write

$$
\begin{aligned}
|I_1(\mathbf{x}) - f(\mathbf{x})| &\leqslant \int_{\mathbb{R}^2} |f(\mathbf{x}-\mathbf{t}) - f(\mathbf{x})| \mathscr{K}(\mathbf{t})\, d\mathbf{t} \\
&= \int_{B(0,\delta)} |f(\mathbf{x}-\mathbf{t}) - f(\mathbf{x})| \mathscr{K}(\mathbf{t})\, d\mathbf{t} + \int_{B(\delta,\infty)} |f(\mathbf{x}-\mathbf{t}) - f(\mathbf{x})| \mathscr{K}(\mathbf{t})\, d\mathbf{t} \\
&= I_{11}(\mathbf{x}) + I_{12}(\mathbf{x}).
\end{aligned}
$$

From (3.4) and (3.9) it follows that

$$
\begin{aligned}
\|I_{11}\|_2 &\leqslant \left(\int_{B(0,\delta)} \mathscr{K}(\mathbf{t}) \int_{\mathbb{R}^2} |f(\mathbf{x}-\mathbf{t}) - f(\mathbf{x})|^2 \, d\mathbf{x}\, d\mathbf{t} \right)^{1/2} \\
&\leqslant \omega_2(\delta, f) \lesssim N^5 \sqrt{\delta} \leqslant \frac{1}{N^2}. \qquad (3.18)
\end{aligned}
$$

Applying (3.11) and the bound $\|f\|_\infty < N^9$, the second integral can again be roughly estimated as follows:

$$
\begin{aligned}
|I_{12}(\mathbf{x})| &\leqslant 2N^{30} \|f\|_\infty \int_{B(\delta,\infty)} K(N^{15}\mathbf{t})\, d\mathbf{t} \\
&\lesssim N^{39} \int_{B(\delta,\infty)} \frac{1}{(N^{15}|\mathbf{t}|)^{50}}\, d\mathbf{t} \lesssim \frac{1}{N^2},
\end{aligned}
$$

and so

$$
\|I_{12} \mathbf{1}_{B(0,1)}\|_2 \lesssim \frac{1}{N^2}. \qquad (3.19)
$$

From (3.17)–(3.19) we obtain (3.14). Finally, having (3.14), (3.2) and (4.1), we get

$$
\begin{aligned}
\|T^*(g)\|_2 &\geqslant \|T^*(f)\|_2 - \|T^*(g-f)\|_2 \\
&\geqslant \|T^*(f)\|_2 - \sum_{k=1}^{N} \big\|T_{\Gamma_{\theta_k}}(g-f)\big\|_2 \geqslant \|T^*(f)\|_2 - c \gtrsim \log N \|g\|_2.
\end{aligned}
$$

This completes the proof of the lemma.

§ 4. A basic sequence of orthogonal functions

In the sequel we always suppose N to be a large enough integer. For the functions f and g introduced in the previous sections we will often use the relation

$$
\|g\|_2 \sim \|f\|_2 \sim \sqrt{\log N}, \qquad (4.1)
$$

which easily follows from (3.3) and (3.14).

Lemma 4.1. *Let $f \in L^2(\mathbb{R})$ and $\operatorname{supp} f \subset B(0,\delta)$. Then for any direction θ and number $A \geqslant 2\delta$ the inequality*

$$
\|T_{\Gamma_\theta}(f) \mathbf{1}_{B(A,\infty)}\|_2 \lesssim \sqrt{\frac{\delta}{A}} \|f\|_2 \qquad (4.2)
$$

holds.

Proof. In light of (3.1) and the conditions of the lemma we have

$$\|T_{\Gamma_\theta}(f)\mathbf{1}_{B(A,\infty)}\|_2 = \frac{1}{2}\|H_\theta(f)\mathbf{1}_{B(A,\infty)}\|_2,$$

so it is enough to prove (4.2) for H_θ instead of the operator T_{Γ_θ}. Without loss of generality we can suppose that $\theta = 0$. So we have

$$H_\theta f(\mathbf{x}) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x_1 - t, x_2)}{t}\,dt = \frac{1}{\pi}\int_{-\delta}^{\delta} \frac{f(t,x_2)}{t - x_1}\,dt.$$

Observe that

$$H_\theta f(\mathbf{x}) = 0, \qquad |x_2| > \delta,$$

and

$$|H_\theta f(\mathbf{x})| \lesssim \frac{1}{|x_1|}\int_{-\delta}^{\delta} |f(t,x_2)|\,dt, \qquad |x_2| \leqslant \delta, \quad |x_1| > 1.6\delta.$$

Thus, using $A \geqslant 2\delta$ and a simple geometric argument, we obtain

$$\begin{aligned}\|H_\theta(f)\mathbf{1}_{B(A,\infty)}\|_2^2 &\lesssim 2\int_{-\delta}^{\delta}\int_{0,8A}^{\infty} \frac{1}{|x_1|^2}\left(\int_{-\delta}^{\delta} |f(t,x_2)|\,dt\right)^2 dx_1\,dx_2 \\ &\lesssim \frac{1}{A}\int_{-\delta}^{\delta}\left(\int_{-\delta}^{\delta} |f(t,x_2)|\,dt\right)^2 dx_2 \lesssim \frac{\delta}{A}\|f\|_2^2,\end{aligned}$$

and so (4.2) holds.

Lemma 4.1 is proved.

Denote

$$S(\alpha,\beta) = \Gamma_\beta \setminus \Gamma_\alpha = \{\mathbf{x} \in \mathbb{R}^2 \colon x_1\cos\beta + x_2\sin\beta \geqslant 0,\ x_1\cos\alpha + x_2\sin\alpha < 0\},$$

which is a sectorial region.

Lemma 4.2. *Let* $0 < \delta < 1/16$, $f \in L^2(\mathbb{R})$ *and* $\operatorname{supp} f \subset B(0,\delta)$. *Then for any directions* α *and* β *the inequality*

$$\|T_{S(\alpha,\beta)}(f)\mathbf{1}_{B(1/2,\infty)}\|_2 \lesssim \sqrt[4]{\delta}\|f\|_2$$

holds.

Proof. Observe that

$$T_{S(\alpha,\beta)} = T_{\Gamma_\beta} \circ T_{\Gamma_{\pi+\alpha}}.$$

Consider the functions

$$f_1 = T_{\Gamma_{\pi+\alpha}}(f)\mathbf{1}_{B(0,\sqrt{\delta})} \quad \text{and} \quad f_2 = T_{\Gamma_{\pi+\alpha}}(f)\mathbf{1}_{B(\sqrt{\delta},\infty)}.$$

Applying Lemma 4.1 for $A = \sqrt{\delta}$, we obtain

$$\|f_2\|_2 = \|T_{\Gamma_{\pi+\alpha}}(f)\mathbf{1}_{B(\sqrt{\delta},\infty)}\|_2 \lesssim \sqrt{\frac{\delta}{\sqrt{\delta}}}\|f\|_2 = \sqrt[4]{\delta}\|f\|_2, \tag{4.3}$$

and so

$$\|T_{\Gamma_\beta}(f_2)\|_2 \leqslant \|f_2\|_2 \lesssim \sqrt[4]{\delta}\|f\|_2.$$

Once again applying Lemma 4.1 for $A = 1/2$ we get

$$\|T_{\Gamma_\beta}(f_1)\mathbf{1}_{B(1/2,\infty)})\|_2 \lesssim \sqrt[4]{\delta}\|f_1\|_2 \leqslant \sqrt[4]{\delta}\|f\|_2. \tag{4.4}$$

Finally, combining (4.3) and (4.4) we obtain

$$\begin{aligned}\|T_{S(\alpha,\beta)}(f)\mathbf{1}_{B(1/2,\infty)}\|_2 &= \|T_{\Gamma_\beta}(T_{\Gamma(\pi+\alpha)}(f))\mathbf{1}_{B(1/2,\infty)}\|_2\\ &\leqslant \|T_{\Gamma_\beta}(f_1)\mathbf{1}_{B(1/2,\infty)}\|_2 + \|T_{\Gamma_\beta}(f_2)\|_2 \lesssim \sqrt[4]{\delta}\|f\|_2.\end{aligned}$$

The lemma is proved.

Denote

$$S_k^+ = S(\theta_k, \theta_{k-1}), \qquad S_k^- = S(\theta_{k-1}, \theta_k) \quad \text{and} \quad S_k = S_k^+ \cup S_k^-,$$

and consider the functions

$$g_k = T_{S_k^+}(g) - T_{S_k^-}(g), \qquad k = 1, 2, \dots, N, \tag{4.5}$$

where g is (3.12).

Lemma 4.3. *The sequence of functions (4.5) satisfies the bound*

$$\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m g_k\right|\right\|_2 \gtrsim \log N\left\|\sum_{k=1}^N g_k\right\|_2 = \log N\|g\|_2. \tag{4.6}$$

Proof. One can check that

$$T_{\Gamma_m}(g) = T_{\Gamma_0}(g) + \sum_{k=1}^m g_k, \qquad \left\|\sum_{k=1}^N g_k\right\|_2 = \|g\|_2.$$

So from (3.15) we obtain

$$\begin{aligned}\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m g_k\right|\right\|_2 &\geqslant \left\|\max_{1\leqslant m\leqslant N}\left|T_{\Gamma_0}(g) + \sum_{k=1}^m g_k\right|\right\|_2 - \|T_{\Gamma_0}(g)\|_2\\ &= \left\|\max_{1\leqslant m\leqslant N}|T_{\Gamma_m}(g)|\right\|_2 - \|T_{\Gamma_0}(g)\|_2 \geqslant \|T^*(g)\|_2 - \|g\|_2 \gtrsim \log N\|g\|_2.\end{aligned}$$

Lemma 4.3 is proved.

Now denote

$$\begin{aligned}D_k^+ &= B(5N^4, N^{15}) \cap S\left(\theta_k - \frac{1}{N^4}, \theta_{k-1} + \frac{1}{N^4}\right),\\ D_k^- &= B(5N^4, N^{15}) \cap S\left(\theta_{k-1} + \frac{1}{N^4}, \theta_k - \frac{1}{N^4}\right)\end{aligned}$$

and

$$D_k = D_k^+ \cup D_k^-$$

and consider the functions

$$f_k = T_{D_k^+}(g) - T_{D_k^-}(g), \qquad k = 1, 2, \dots, N. \tag{4.7}$$

Lemma 4.4. *We have the inequality*

$$\|f_k - g_k\|_2 \lesssim \frac{\|g\|_2}{N^2}, \qquad k = 1, 2, \ldots, N. \tag{4.8}$$

Proof. First observe that, since g and so $\widehat{g}$ are spherical functions, we have

$$\|T_{S(\alpha,\beta)}(g)\|_2^2 = \|\widehat{g}\,\mathbf{1}_{S(\alpha,\beta)}\|_2^2 = \frac{|\alpha-\beta|}{2\pi}\|\widehat{g}\|_2^2 = \frac{|\alpha-\beta|}{2\pi}\|g\|_2^2. \tag{4.9}$$

In view of (3.13), (4.5) and (4.7) it follows that

$$\operatorname{supp}(\widehat{f}_k - \widehat{g}_k) \subset B(0, 5N^4) \cup \left(\bigcup_{j=1}^{4} U_j\right),$$

where

$$U_1 = S(\theta_j - N^{-4}, \theta_j), \qquad U_2 = S(\theta_{j-1} + N^{-4}, \theta_{j-1}),$$
$$U_3 = S(\theta_j, \theta_j - N^{-4}), \qquad U_4 = S(\theta_{j-1}, \theta_{j-1} + N^{-4}).$$

In addition, according to (3.16) we have

$$\|\widehat{g}\|_\infty \leqslant \|\widehat{f}\|_\infty \|\widehat{K}\|_\infty \lesssim \|\widehat{f}\|_\infty \leqslant \|f\|_1 \lesssim N^{-8}. \tag{4.10}$$

Thus, using (4.9) and (4.1) we get

$$\begin{aligned}\|f_k - g_k\|_2 = \|\widehat{f}_k - \widehat{g}_k\|_2 &\leqslant \|\widehat{g}\,\mathbf{1}_{B(0,5N^4)}\|_2 + \sum_{k=1}^{4} \|\widehat{g}\,\mathbf{1}_{U_k}(g)\|_2 \\ &\lesssim \|\widehat{g}\|_\infty N^4 + N^{-2}\|g\|_2 \lesssim N^{-4} + N^{-2}\|g\|_2 \lesssim N^{-2}\|g\|_2.\end{aligned}$$

The lemma is proved.

Lemma 4.5. *The inequality*

$$\|f_k \mathbf{1}_{B(1/2,\infty)}\|_2 \lesssim \frac{\|g\|_2}{N^2} \tag{4.11}$$

holds.

Proof. Letting

$$g_1 = T_{B(5N^4,\infty)}(g) \quad \text{and} \quad g_2 = T_{B(0,5N^4)}(g),$$

we write

$$g = g_2 + g_1 = g_2 + g_1\mathbf{1}_{B(0,N^{-8})} + g_1\mathbf{1}_{B(N^{-8},\infty)} = g_2 + U + V.$$

Using (3.13) and the definitions of the domains D_k^+ and D_k^-, one can write

$$\begin{aligned}T_{D_k^+}(g) - T_{D_k^-}(g) &= T_{D_k^+}(g_1) - T_{D_k^-}(g_1) = T_{G_k^+}(g_1) - T_{G_k^-}(g_1) \\ &= T_{G_k^+}(U) - T_{G_k^-}(U) + T_{G_k^+}(V) - T_{G_k^-}(V),\end{aligned} \tag{4.12}$$

where

$$G_k^+ = S\left(\theta_k - \frac{1}{N^4}, \theta_{k-1} + \frac{1}{N^4}\right) \quad \text{and} \quad G_k^- = S\left(\theta_{k-1} + \frac{1}{N^4}, \theta_k - \frac{1}{N^4}\right).$$

By (4.10) we have

$$\|g_2\|_2 = \|\widehat{g}\,\mathbf{1}_{B(0,5N^4)}\|_2 \lesssim \frac{1}{N^8}\|\mathbf{1}_{B(0,5N^4)}\|_2 \lesssim \frac{1}{N^4}.$$

Combining $\operatorname{supp} f \subset B(0, N^{-8})$ with inequality (3.14) implies that

$$\begin{aligned}\|T_{G_k^+}(V) - T_{G_k^-}(V)\|_2 &\leqslant \|V\|_2 = \|(f - g_1)\mathbf{1}_{B(N^{-8},\infty)}\|_2 \\ &\leqslant \|f - g_1\|_2 \lesssim \|f - g\|_2 + \|g_2\|_2 \lesssim N^{-2}. \end{aligned} \tag{4.13}$$

Then, applying Lemma 4.2 with $\delta = N^{-8}$ and taking into account that $\operatorname{supp} U \subset B(0, N^{-8})$ we obtain

$$\|(T_{G_k^+}(U) - T_{G_k^-}(U))\mathbf{1}_{B(1/2,\infty)}\|_2 \lesssim \frac{\|U\|_2}{N^2} \leqslant \frac{\|g\|_2}{N^2}. \tag{4.14}$$

From (4.7), (4.12), (4.13), (4.14) and (4.1) we obtain

$$\begin{aligned}&\|f_k\mathbf{1}_{B(1/2,\infty)}\|_2 = \|(T_{D_k^+}(g) - T_{D_k^-}(g))\mathbf{1}_{B(1/2,\infty)}\|_2 \\ &\leqslant \|(T_{G_k^+}(V) - T_{G_k^-}(V))\mathbf{1}_{B(1/2,\infty)}\|_2 + \|(T_{G_k^+}(U) - T_{G_k^-}(U))\mathbf{1}_{B(1/2,\infty)}\|_2 \lesssim \frac{\|g\|_2}{N^2},\end{aligned}$$

and so (4.11) follows.

The lemma is proved.

Lemma 4.6. *There exists a sequence of functions $r_k \in L^2(\mathbb{R}^2)$, $k = 1, 2, \ldots, N$, such that*

$$\operatorname{supp}(r_k) \subset \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{4.15}$$

$$\|\widehat{r}_k\mathbf{1}_{\mathbb{R}^2\setminus D_k}\|_2 \lesssim \frac{\|g\|_2}{N^2} \tag{4.16}$$

and

$$\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m r_k\right|\right\|_2 \gtrsim \log N\left\|\sum_{k=1}^N r_k\right\|_2 \sim \log N\|g\|_2. \tag{4.17}$$

Proof. Set

$$r_k(\mathbf{x}) = f_k(\mathbf{x})\mathbf{1}_{B(0,1/2)}(\mathbf{x}).$$

We immediately have (4.15). From (4.11) it follows that

$$\|r_k - f_k\|_2 \lesssim \frac{\|g\|_2}{N^2},$$

and in light of (4.8) we get $\|r_k - g_k\|_2 \lesssim \|g\|_2/N^2$. Thus, taking (4.6) into account we get (4.17). Since

$$r_k = f_k - f_k\mathbf{1}_{B(1/2,\infty)},$$

and supp $\widehat{f}_k \subset D_k$, by (4.11) and (4.1) we get

$$\|\widehat{r}_k \mathbf{1}_{\mathbb{R}^2\setminus D_k}\|_2 = \|\widehat{f_k \mathbf{1}}_{B(1/2,\infty)} \mathbf{1}_{\mathbb{R}^2\setminus D_k}\|_2 \leqslant \|\widehat{f_k \mathbf{1}}_{B(1/2,\infty)}\|_2$$
$$= \|f_k \mathbf{1}_{B(1/2,\infty)}\|_2 \lesssim \frac{\|f\|_2}{N^2}$$

and so (4.16) holds.

The lemma is proved.

§ 5. Double trigonometric polynomials

The following lemma is a version of Lemma 4.6 for double trigonometric sums.

Proposition 5.1. *There exist two-dimensional nonoverlapping trigonometric polynomials*

$$p_k(\mathbf{x}) = \sum_{\mathbf{n}\in G_k} a_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}}, \qquad k = 1, 2, \dots, N, \tag{5.1}$$

such that

$$G_k \subset B(0.4N^{15}) \cap \mathbb{Z}^2_+$$

and

$$\left\| \max_{1\leqslant m\leqslant N} \left| \sum_{k=1}^{m} p_k \right| \right\|_2 \gtrsim \log N \left\| \sum_{k=1}^{N} p_k \right\|_2. \tag{5.2}$$

Proof. Let $\mathbf{u}$ be a fixed vector. In light of (4.15) the function

$$r_k(\mathbf{u}, \mathbf{x}) = e^{-2\pi i \mathbf{u}\cdot\mathbf{x}} r_k(\mathbf{x}) \tag{5.3}$$

as a function of $\mathbf{x}$ can be continued periodically and considered as a function in $L^2(\mathbb{T}^2)$ with the Fourier representation

$$r_k(\mathbf{u}, \mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \widehat{r}_k(\mathbf{n} + \mathbf{u}) e^{2\pi i \mathbf{n}\cdot\mathbf{x}}.$$

For any $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ we denote $\Delta_{\mathbf{n}} = [n_1, n_1 + 1) \times [n_2, n_2 + 1)$ and let

$$U_k = \{\mathbf{n} \in \mathbb{Z}^2 \colon \Delta_{\mathbf{n}} \cap D_k \neq \varnothing\} \subset B(0, 2N^{15}).$$

From the definition of D_k it follows that

$$D_k \subset \bigcup_{\mathbf{n}\in U_k} \Delta_{\mathbf{n}} \subset B(0, 2N^{15}). \tag{5.4}$$

A simple geometric argument shows that $\operatorname{dist}(D_k, \mathbb{R}^2 \setminus S_k) > 2$, which implies that

$$\bigcup_{\mathbf{n}\in U_k} \Delta_{\mathbf{n}} \subset S_k = S_k^+ \cup S_k^-,$$

so the U_k are pairwise disjoint. Consider the functions

$$p_k(\mathbf{u}, \mathbf{x}) = \sum_{\mathbf{n}\in U_k} \widehat{r}_k(\mathbf{n} + \mathbf{u}) e^{2\pi i \mathbf{n}\cdot\mathbf{x}}$$

and

$$q_k(\mathbf{u},\mathbf{x})=\sum_{\mathbf{n}\in\mathbb{Z}^2\setminus U_k}\widehat{r}_k(\mathbf{n}+\mathbf{u})e^{2\pi i\mathbf{n}\cdot\mathbf{x}}.$$

For fixed $\mathbf{u}$ the polynomials $p_k(\mathbf{u},\mathbf{x})$ are nonoverlapping since $U_k\subset S_k$. Clearly,

$$r_k(\mathbf{u},\mathbf{x})=p_k(\mathbf{u},\mathbf{x})+q_k(\mathbf{u},\mathbf{x}), \tag{5.5}$$

and by (4.16) and (5.4) we obtain

$$\begin{aligned}\int_{\mathbb{T}^2}\|q_k(\mathbf{u},\cdot)\|_2^2\,d\mathbf{u}&=\int_{\mathbb{T}^2}\sum_{n\in\mathbb{Z}^2\setminus U_k}|\widehat{r}_k(\mathbf{n}+\mathbf{u})|^2\,d\mathbf{u}=\sum_{\mathbf{n}\in\mathbb{Z}^2\setminus U_k}\int_{\Delta_\mathbf{n}}|\widehat{r}_k(t)|^2\,dt\\&\leqslant\|\widehat{r}_k\mathbf{1}_{\mathbb{R}^2\setminus D_k}\|_2^2\lesssim\frac{\|g\|_2^2}{N^4}\end{aligned}$$

and so

$$\sum_{k=1}^N\int_{\mathbb{T}^2}\|q_k(\mathbf{u},\cdot)\|_2^2\,d\mathbf{u}\lesssim\frac{\|g\|_2^2}{N^3}.$$

This inequality produces $\mathbf{u}=\mathbf{u}_0$ such that

$$\sum_{k=1}^N\|q_k(\mathbf{u}_0,\cdot)\|_2^2\lesssim\frac{\|g\|_2^2}{N^3},$$

and by Hölder's inequality we get

$$\sum_{k=1}^N\|q_k(\mathbf{u}_0,\cdot)\|_2\leqslant\sqrt{N}\left(\sum_{k=1}^N\|q_k(\mathbf{u}_0,\cdot)\|_2^2\right)^{1/2}\lesssim\frac{\|g\|_2}{N}. \tag{5.6}$$

Finally, we can define the polynomials

$$p_k(\mathbf{x})=e^{2\pi i2(N^{15}x_1+N^{15}x_2)}p_k(\mathbf{u}_0,\mathbf{x}),\qquad \mathbf{x}=(x_1,x_2),$$

with the nonoverlapping spectra

$$G_k=U_k+(2N^{15},2N^{15})\subset B(0.4N^{15})\cap\mathbb{Z}^2_+.$$

Combining (4.17), (5.3), (5.5) and (5.6) we get

$$\begin{aligned}\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m p_k\right|\right\|_2&=\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m p_k(\mathbf{u}_0,\cdot)\right|\right\|_2\\&\geqslant\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m r_k(\mathbf{u}_0,\cdot)\right|\right\|_2-\sum_{k=1}^N\|q_k(\mathbf{u}_0,\cdot)\|_2\\&\geqslant\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m r_k\right|\right\|_2-c_2\frac{\|g\|_2}{N}\geqslant c_1\log N\left\|\sum_{k=1}^N r_k\right\|_2-c_2\frac{\|g\|_2}{N}.\end{aligned} \tag{5.7}$$

Likewise, using (4.1) one can show that

$$\left|\left\|\sum_{k=1}^{N} r_k\right\|_2 - \left\|\sum_{k=1}^{N} p_k\right\|_2\right| \lesssim \frac{\|g\|_2}{N}. \tag{5.8}$$

From (4.17), (5.7) and (5.8) one can easily get (5.2).

Proposition 5.1 is proved.

§ 6. Equivalence of discrete trigonometric systems

Let $\{f_k : k \in A\}$ and $\{g_k : k \in B\}$ be families of measurable complex-valued functions defined on measure spaces (X, μ) and (Y, ν), respectively. We say that these sequences are equivalent if there is a one-to-one mapping $\sigma : A \to B$ such that the equality

$$\mu\{f_{\alpha_j} \in B_j,\, j = 1, 2, \dots, m\} = \nu\{g_{\sigma(\alpha_j)} \in B_j,\, j = 1, 2, \dots, m\}$$

holds for any choice of indices $\alpha_j \in A$ and open balls $B_j \subset \mathbb{R}^2$, $j = 1, 2, \dots, m$. For an integer $l \geqslant 1$ we denote $\mathbb{N}_l = \{1, 2, \dots, l\}$. The discrete trigonometric system of order l on $[0, 1)$ is defined by

$$T^{(l)} = \left\{ t_n^{(l)}(x) = \sum_{k=1}^{l} \exp\left(2\pi i \frac{nk}{l}\right) \mathbf{1}_{\delta_k^{(l)}}(x),\, n \in \mathbb{N}_l \right\},$$

where $\delta_k^{(l)} = [(k-1)/l, k/l)$. The tensor product of the two one-dimensional systems of orders p and q is the collection of functions

$$T^{(p)} \times T^{(q)} = \left\{ (t_{n_1}^{(p)} \times t_{n_2}^{(q)})(\mathbf{x}) = t_{n_1}^{(p)}(x_1) t_{n_2}^{(q)}(x_2),\, \mathbf{n} \in \mathbb{N}_p \times \mathbb{N}_q \right\}.$$

Notice that

$$(t_{n_1}^{(p)} \times t_{n_2}^{(q)})(\mathbf{x}) = \sum_{u_1=1}^{p} \sum_{u_2=1}^{q} \exp\left(2\pi i \left(\frac{n_1 u_1}{p} + \frac{n_2 u_2}{q}\right)\right) \mathbf{1}_{\delta_{u_1}^{(p)} \times \delta_{u_2}^{(q)}}(\mathbf{x}). \tag{6.1}$$

We prove the following.

Proposition 6.1. *If p and q are coprime integers, then the systems $T^{(pq)}$ and $T^{(p)} \times T^{(q)}$ are equivalent.*

Lemma 6.1. *For any coprime numbers p and q, there are two one-to-one mappings φ and ψ acting from $\mathbb{N}_p \times \mathbb{N}_q$ to $\mathbb{N}_{pq}$ such that*

$$\left\{ \frac{n_1 u_1}{p} + \frac{n_2 u_2}{q} \right\} = \left\{ \frac{\varphi(\mathbf{n}) \psi(\mathbf{u})}{pq} \right\}, \tag{6.2}$$

where $\{a\}$ denotes the fractional part of a real number a.

Proof. According to the Chinese remainder theorem, for every pair (n_1, n_2) of integers $n_1 \in \mathbb{N}_p$ and $n_2 \in \mathbb{N}_q$ one can find a unique $l \in \mathbb{N}_{pq}$ such that

$$l = n_1 \mod p \quad \text{and} \quad l = n_2 \mod q$$

and this defines a one-to-one mapping τ from $\mathbb{N}_p\times\mathbb{N}_q$ to $\mathbb{N}_{pq}$ such that $\tau(n_1,n_2)=l$. For our further convenience we extend τ over the whole of $\mathbb{Z}_+^2$ periodically by $\tau(n_1,n_2)=\tau(n_1+pk,n_2+qj)$ for any pair of positive integers k, j. Define

$$\varphi(\mathbf{n})=\tau(n_1,n_2)\quad\text{and}\quad\psi(\mathbf{u})=\tau(u_1,-u_2)(q-p)\ \operatorname{mod}^*pq,$$

where

$$m\ \operatorname{mod}^* n=\begin{cases}n, & m\ \operatorname{mod} n=0,\\ m\ \operatorname{mod} n, & m\ \operatorname{mod} n\neq 0.\end{cases}\tag{6.3}$$

Clearly, φ and ψ determine one-to-one mappings from $\mathbb{N}_p\times\mathbb{N}_q$ to $\mathbb{N}_{pq}$. Moreover,

$$\begin{aligned}\left\{\frac{\varphi(\mathbf{n})\psi(\mathbf{u})}{pq}\right\}&=\left\{\frac{\tau(n_1,n_2)\tau(u_1,-u_2)(q-p)}{pq}\right\}\\ &=\left\{\frac{\tau(n_1,n_2)\tau(u_1,-u_2)}{p}-\frac{\tau(n_1,n_2)\tau(u_1,-u_2)}{q}\right\}=\left\{\frac{n_1u_1}{p}+\frac{n_2u_2}{q}\right\}.\end{aligned}$$

Proof of Proposition 6.1. Let φ and ψ be mappings taken from Lemma 6.1. Then φ produces a one-to-one correspondence

$$t_{n_1}^{(p)}\times t_{n_2}^{(q)}\to t_{\varphi(\mathbf{n})}^{(pq)},$$

while ψ produces a one-to-one correspondence

$$\delta_{u_1}^{(p)}\times\delta_{u_2}^{(q)}\to\delta_{\psi(u_1,u_2)}^{(pq)}.$$

In light of (6.1) and (6.2) one can see that each function $t_{n_1}^{(p)}\times t_{n_2}^{(q)}$ takes the same value on $\delta_{u_1}^{(p)}\times\delta_{u_2}^{(q)}$ as $t_{\varphi(n_1,n_2)}^{(pq)}$ on $\delta_{\psi(u_1,u_2)}^{(pq)}$. This obviously implies the equivalence of the systems $T^{(pq)}$ and $T^{(p)}\times T^{(q)}$.

The proposition is proved.

§ 7. The main lemma

Lemma 7.1. *If functions $f,g\in L^2(\mathbb{T})$ satisfy the strong orthogonality condition*

$$\int_{\mathbb{T}}f(x)g(h-x)\,dx=0\quad\textit{for any } h\in\mathbb{R},\tag{7.1}$$

then they have nonoverlapping Fourier series.

Proof. Condition (7.1) implies that

$$\widehat{f}(n)\widehat{g}(n)=(\widehat{f\star g})(n)=0.$$

So for each $n\in\mathbb{Z}$ either $\widehat{f}(n)=0$ or $\widehat{g}(n)=0$. This completes the proof.

Now we are able to prove the main lemma.

Lemma 7.2. *There exists a sequence of one-dimensional trigonometric polynomials*

$$Q_k(x)=\sum_{n\in U_k}a_ne^{2\pi inx},\qquad k=1,2,\ldots,N,$$

with nonoverlapping spectra U_k such that

$$U_k\subset[1,N^{70}] \tag{7.2}$$

and

$$\left\|\max_{1\leqslant m\leqslant N}\left|\sum_{k=1}^m Q_k\right|\right\|_2\gtrsim\log N\left\|\sum_{k=1}^N Q_k\right\|_2. \tag{7.3}$$

Proof. For the coprime numbers $p=N^{31}$ and $q=N^{31}+1$ we consider the discrete double trigonometric system $T^{(p)}\times T^{(q)}$. It is easy to see that

$$\left|e^{2\pi i\mathbf{n}\cdot\mathbf{x}}-(t_{n_1}^{(p)}\times t_{n_2}^{(q)})(\mathbf{x})\right|\lesssim\frac{1}{N^{16}},\qquad \mathbf{x}\in\mathbb{T}^2,\quad \mathbf{n}=(n_1,n_2)\in B(0.4N^{15})\cap\mathbb{Z}_+. \tag{7.4}$$

Consider the nonoverlapping double discrete trigonometric polynomials

$$P_k(\mathbf{x})=\sum_{\mathbf{n}\in G_k}a_{\mathbf{n}}(t_{n_1}^{(p)}\times t_{n_2}^{(q)})(\mathbf{x}),\qquad k=1,2,\ldots,N,$$

with the same coefficients as in (5.1), where $G_k\subset\mathbb{N}_p\times\mathbb{N}_q$. According to Proposition 6.1 the systems $T^{(pq)}$ and $T^{(p)}\times T^{(q)}$ are equivalent. So the sequence of double polynomials P_k generates a one-dimensional sequence of nonoverlapping polynomials $R_k\in T^{(pq)}$, $k=1,2,\ldots,N$. Both sequences share the same logarithmic bound (5.2) for the p_k, since it follows from (7.4) that

$$\|P_k-p_k\|_2\lesssim\frac{1}{N}\left(\sum_{\mathbf{n}\in G_k}a_{\mathbf{n}}^2\right)^{1/2}\leqslant\frac{1}{N}\left\|\sum_{k=1}^N p_k\right\|_2.$$

The disjointness of the spectra of R_k as polynomials in $T^{(pq)}$ implies that

$$\int_{\mathbb{T}}R_k(x)R_m(h-x)\,dx=0\quad\text{for any}\quad h\in\mathbb{R},\quad k,m\in\mathbb{N}_{pq},\quad k\neq m.$$

Thus, according to Lemma 7.1, the functions $R_k\in L^2(\mathbb{T})$ have nonoverlapping spectra of Fourier series. Moreover, they are step functions with intervals of constancy having length $(pq)^{-1}\sim N^{-62}$. Let

$$Q_k(x)=e^{2\pi i(N^{66}+1)x}\sigma_{N^{66}}(x,R_k),$$

where $\sigma_n(x,f)$ denotes the nth-order $(C,1)$-mean of the function f. Clearly, we have (7.2). Recall the approximation property of the $(C,1)$-means:

$$\begin{aligned}\|\sigma_n(f)-f\|_2 &\leqslant \biggl(\int_{\mathbb{T}}\int_{\mathbb{T}} K_n(t)|f(x+t)-f(x)|^2\,dt\,dx\biggr)^{1/2}\\ &\leqslant \biggl(\int_{-\delta}^{\delta} K_n(t)\int_{\mathbb{T}} |f(x+t)-f(x)|^2\,dx\,dt\biggr)^{1/2}\\ &\qquad+\biggl(\int_{\delta<|t|<\pi} K_n(t)\int_{\mathbb{T}} |f(x+t)-f(x)|^2\,dx\,dt\biggr)^{1/2}\\ &\lesssim \omega_2(\delta,f)+\|f\|_2\biggl(\int_{\delta}^{\infty}\frac{1}{nt^2}\,dt\biggr)^{1/2}\lesssim \omega_2(\delta,f)+\frac{\|f\|_2}{\sqrt{n\delta}}.\end{aligned}$$

Using this inequality for $f=R_k$, $n=N^{66}$ and $\delta=N^{-64}$, as well as the easily checked bound $\omega_2(\delta,R_k)\lesssim \|R_k\|_2/N$, one can obtain

$$\bigl\|Q_k(x)-\exp(2\pi i(N^{66}+1)x)R_k(x)\bigr\|_2\lesssim \frac{\|R_k\|_2}{N}.$$

The latter immediately yields the logarithmic bound (7.3), since we have the same bound for R_k.

Lemma 7.2 is proved.

§ 8. Proof of the main theorem and Corollary 1.1

Lemma 8.1. *Let T be a sublinear operator satisfying*

$$\|T\|_{L^2\to L^{2,\infty}}\leqslant c\sqrt{\log N},\qquad \|T\|_{L^2\to L^\infty}\leqslant N$$

and $c\log N\geqslant 1$. Then

$$\|T\|_{L^2\to L^2}\lesssim c\log N.$$

Proof. For a given function $f\in L^2(\mathbb{T})$, $\|f\|_2\leqslant 1$, we have $\|T(f)\|_\infty\leqslant N$. Denote

$$\varphi(\lambda)=|\{x\colon |Tf(x)|>\lambda\}|.$$

Then we have

$$\begin{cases}\varphi(\lambda)=0, & \lambda>N,\\ \varphi(\lambda)\leqslant \dfrac{\|T\|^2_{L^2\to L^{2,\infty}}}{\lambda^2}\leqslant \dfrac{c^2\log N}{\lambda^2}, & \lambda>0,\end{cases}$$

and so

$$\|T(f)\|_2^2=2\int_0^\infty \lambda\varphi(\lambda)\,d\lambda=2\int_0^N\lambda\varphi(\lambda)d\lambda\leqslant 2+2\int_1^N\lambda\varphi(\lambda)\,d\lambda\lesssim c^2\log^2 N.$$

The lemma is proved.

Proof of Theorem 1.1. The upper bound in (1.7) follows from the Menshov–Rademacher inequality. The lower bound

$$\max_\sigma \|T_{\sigma,N}\|_{L^2\to L^2}\gtrsim \log N$$

easily follows from Lemma 7.2.

Theorem 1.1 is proved.

Proof of Corollary 1.1. A combination of Lemma 8.1 and the lower bound in (1.7) implies (1.8).

§ 9. Proof of Corollary 1.2

The next lemma is based on (1.8). Denote by $\mathscr{P}_N$ the family of one-dimensional trigonometric polynomials of the form

$$\sum_{k=1}^{N} a_k e^{2\pi ik},$$

where the a_k are complex numbers.

Lemma 9.1. *For any $N > N_0$ there exists a polynomial $P \in \mathscr{P}_N$ and a rearrangement $\sigma \in \Sigma_N$ such that $\|P\|_2 \sim 1$ and*

$$\bigl|\bigl\{x \in \mathbb{T}\colon T_{\sigma,N}(x,P) > \sqrt{\log N}\bigr\}\bigr| \gtrsim 1. \tag{9.1}$$

Proof. Let $M = [\sqrt{N}] + 1$. According to (1.8) there is a polynomial $Q \in \mathscr{P}_M$ with $\|Q\|_2 = 1$ and a rearrangement $\tau \in \Sigma_M$ such that

$$|E| = \bigl|\{x \in \mathbb{T}\colon T_{\tau,M}(x,Q) > \lambda_0\}\bigr| \geqslant \frac{c\log M}{\lambda_0^2}$$

for some $\lambda_0 > 0$. Since $|\mathbb{T}| = 1$, we have $\lambda_0 \geqslant \sqrt{c\log M}$ and from $\|Q\|_2 = 1$ it follows that $0 < \lambda_0 \leqslant \sqrt{M}$. Putting $l = [\lambda_0^2/c\log M]$ we have

$$1 \leqslant l \leqslant \frac{M}{c\log M} \leqslant \frac{M}{2} \quad \text{and} \quad |E| > \frac{1}{l+1},$$

for $N > N_0$. Using a well-known argument (see [32], Ch. 13, Lemma 1.24) one can find a sequence of points $x_k \in \mathbb{T}$, $k = 0, 1, \dots, l-1$, such that

$$|F| = \biggl|\bigcup_{k=0}^{l-1}(E + x_k)\biggr| \geqslant 1 - (1 - |E|)^l \geqslant 1 - (1 - (l+1)^{-1})^l \gtrsim 1.$$

Then we consider the polynomial

$$G(x) = \frac{1}{\sqrt{l}}\sum_{k=0}^{l-1} Q_k(x), \quad \text{where } Q_k(x) = Q(x - x_k)e^{2\pi ikMx}.$$

Clearly, $G \in \mathscr{P}_{lM} \subset \mathscr{P}_N$ since $lM \leqslant N$. Define a rearrangement $\sigma \in \Sigma_N$ by

$$\sigma(n) = \begin{cases} \tau(n - Mk) + Mk, & Mk < n \leqslant M(k+1), \quad k = 0, 1, \dots, l-1, \\ n, & lM < n \leqslant N. \end{cases}$$

One can check that $\|G\|_2 = 1$. Any partial sum of the σ-rearrangement of the polynomial $Q_k/\sqrt{l}$ can be written as a difference of two partial sums of G. This implies that

$$T_{\sigma,N}(x,G) \geqslant \frac{1}{2\sqrt{l}} T_{\tau,M}(x - x_k, Q), \qquad x \in \mathbb{T}.$$

Thus, for any $x \in E + x_k$ we have

$$T_{\sigma,N}(x,G) \geqslant \frac{1}{2\sqrt{l}} T_{\tau,M}(x - x_k, Q) > \frac{\lambda_0}{2\sqrt{l}} \geqslant \frac{\sqrt{c \log M}}{2} \gtrsim \sqrt{\log N}.$$

Hence $T_{\sigma,N}(x,G) \gtrsim \sqrt{\log N}$ whenever $x \in F$. Since $|F| \gtrsim 1$, a polynomial $P(x) = cG(x)$ with a suitable absolute constant c may become our desired polynomial.

The lemma is proved.

Proof of Corollary 1.2. Using (1.9) one can define integers $N_k \geqslant 1$, $k = 1, 2, \dots,$ such that

$$N_{k+1} > 2N_k, \quad w(2N_k) \leqslant \frac{\log N_k}{k^2}, \qquad k = 1, 2, \dots .$$

Applying Lemma 9.1, we find polynomials $P_k \in \mathscr{P}_{N_k}$ and rearrangements $\sigma_k \in \Sigma_{N_k}$ such that $\|P_k\|_2 \sim 1$ and the sets

$$E_k = \left\{ x \in \mathbb{T} \colon T_{\sigma_k}(x, P_k) > \sqrt{\log N_k} \right\}$$

satisfy $|E_k| > c > 0$. It is well known that this condition provides a sequence $t_k \in \mathbb{T}$ such that

$$\left| \bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} (E_n + t_n) \right| = 1$$

(see [32], Ch. 13, Lemma 1.24). Consider the trigonometric series

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{w(2N_k)}} P_k(x - t_k) e^{2\pi i N_k x} = \sum_{n=1}^{\infty} c_n e^{2\pi i n x}. \tag{9.2}$$

We have

$$\sum_{n=1}^{\infty} |c_n|^2 w(n) \leqslant \sum_{k=1}^{\infty} \frac{\|P_k\|_2^2}{k^2} < \infty. \tag{9.3}$$

Define a permutation σ of $\mathbb{N}$ as follows:

$$\sigma(n) = \begin{cases} \sigma_k(n - N_k) + N_k, & N_k < n \leqslant 2N_k, \quad k = 1, 2, \dots, \\ n, & n \notin \bigcup_{k \geqslant 1} (N_k, 2N_k]. \end{cases}$$

If $x \in \bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} (E_n + t_n)$, then $x \in E_k + t_k$ for infinitely many k. For $x \in E_k + t_k$ we have

$$\max_{N_k < m \leqslant 2N_k} \left| \sum_{n=N_k+1}^{m} c_{\sigma(n)} e^{2\pi i \sigma(n) x} \right| \geqslant \frac{T_{\sigma_k}(x - t_k, P_k)}{k\sqrt{w(2N_k)}} \geqslant \frac{\sqrt{\log N_k}}{k\sqrt{w(2N_k)}} > 1.$$

Thus we get that the series (9.2) is almost everywhere divergent. Combining this with (9.3) we complete the proof.

The corollary is proved.

§ 10. Proof of Corollary 1.3

Lemma 10.1. *Let f_k, $k=1,2,\dots,n$, be a sequence of complex-valued functions on an interval Δ. Then for any $\lambda>0$ we have the inequality*

$$\biggl|\biggl\{x\in\Delta\colon \max_{1\leqslant m\leqslant n}\biggl(\operatorname{Re}\sum_{k=1}^{m}\alpha f_k(x)\biggr)>\frac{\lambda}{\sqrt{2}}\biggr\}\biggr| \geqslant\frac14\biggl|\biggl\{x\in\Delta\colon \max_{1\leqslant m\leqslant n}\biggl|\sum_{k=1}^{m}f_k(x)\biggr|>\lambda\biggr\}\biggr|$$

for some $\alpha=e^{\pi is/2}$, $s=0,1,2,3$.

Proof. One can check that

$$\max_{1\leqslant m\leqslant n}\biggl|\sum_{k=1}^{m}f_k(x)\biggr|>\lambda$$

yields at least one of the following four inequalities

$$\max_{1\leqslant m\leqslant n}\biggl(\operatorname{Re}\sum_{k=1}^{m}f_k(x)\biggr)>\frac{\lambda}{\sqrt{2}},\qquad \min_{1\leqslant m\leqslant n}\biggl(\operatorname{Re}\sum_{k=1}^{m}f_k(x)\biggr)<-\frac{\lambda}{\sqrt{2}},$$
$$\max_{1\leqslant m\leqslant n}\biggl(\operatorname{Im}\sum_{k=1}^{m}f_k(x)\biggr)>\frac{\lambda}{\sqrt{2}},\qquad \min_{1\leqslant m\leqslant n}\biggl(\operatorname{Im}\sum_{k=1}^{m}f_k(x)\biggr)<-\frac{\lambda}{\sqrt{2}}.$$

This immediately gives the required inequality for some α.

The lemma is proved.

The spectrum of a trigonometric polynomial

$$U(x)=\sum_{k=m}^{n}a_ke^{2\pi ikx}$$

will be denoted by $\operatorname{spec}(U)=\{k\colon a_k\neq0\}$.

Lemma 10.2. *Let $\Delta\subset\mathbb{T}$ be an arbitrary interval and let the integer $N>N_0$ satisfy*

$$\frac{1}{\sqrt{N}}\leqslant|\Delta|. \tag{10.1}$$

Then for any positive integer l there exists a sequence of nonoverlapping trigonometric polynomials U_n, $n=1,2,\dots,N$, such that

$$\operatorname{spec}(U_n)\subset(l,l+N^5],$$
$$\biggl\|\sum_{n=1}^{N}U_n\biggr\|_{L^2(\mathbb{T})}\lesssim\sqrt{|\Delta|},$$
$$\sum_{n=1}^{N}|U_n(x)|\lesssim\frac1N,\qquad x\in\mathbb{T}\setminus\Delta,$$

and

$$\left|\left\{x\in\Delta\colon \max_{1\leqslant m\leqslant N}\operatorname{Re}\left(\sum_{n=1}^{m}U_n(x)\right)>\sqrt{\log N}\right\}\right|\gtrsim|\Delta|.$$

Proof. Suppose $\Delta=[a,b]$. In view of (10.1) we consider the polynomial

$$R(x)=R_N(x)=\frac{1}{\pi}\int_{a+1/(4\sqrt{N})}^{b-1/(4\sqrt{N})}K_{[N^3/3]}(x-t)\,dt=\sum_{k=-[N^3/3]}^{[N^3/3]}c_ke^{2\pi ikx},$$

where K_n is the Fejér kernel of order n. The standard properties of the Fejér kernel imply that

$$0\leqslant R(x)\lesssim\frac{1}{N^2},\qquad x\in\mathbb{T}\setminus\Delta, \tag{10.2}$$

$$1\geqslant R(x)\geqslant\frac{1}{2},\qquad x\in\widetilde{\Delta}=\left[a+\frac{1}{3\sqrt{N}},b-\frac{1}{3\sqrt{N}}\right], \tag{10.3}$$

and

$$1\geqslant R(x)\geqslant 0,\qquad x\in\Delta\setminus\widetilde{\Delta},$$

where (10.3) holds for N large enough. Applying Lemma 9.1, we find a polynomial

$$P(x)=\sum_{n=1}^{N}b_ne^{2\pi inx}$$

satisfying the conditions in that lemma. In light of (10.1), from (9.1) it easily follows that

$$\left|\left\{x\in\widetilde{\Delta}\colon \max_{1\leqslant m\leqslant N}\left|\sum_{n=1}^{m}b_{\sigma(n)}e^{2\pi i\sigma(n)N^3x}\right|>\sqrt{\log N}\right\}\right|\gtrsim|\Delta|, \tag{10.4}$$

where σ is the permutation in (9.1). Consider the polynomials

$$p_n(x)=4e^{2\pi ilx}R(x)b_{\sigma(n)}e^{2\pi i\sigma(n)N^3x},\qquad n=1,2,\dots,N,$$

whose spectra lie in $(l,l+N^5]$ and are nonoverlapping. Using (10.2) and (10.3) we conclude that

$$\left\|\sum_{n=1}^{N}p_n\right\|_{L^2(\mathbb{T})}\lesssim\|P(N^3x)\|_{L^2(\Delta)}+\frac{1}{N^2}\lesssim\|P\|_2\sqrt{|\Delta|}\lesssim\sqrt{|\Delta|}.$$

Using (10.2) and $\sum_{n=1}^{N}|b_n|\leqslant\sqrt{N}\|P\|_2\lesssim\sqrt{N}$ we get

$$\sum_{n=1}^{N}|p_n(x)|\lesssim\sqrt{N}|R(x)|\lesssim\frac{1}{N}\quad\text{for all } x\in\mathbb{T}\setminus\Delta.$$

Then we have

$$\max_{1\leqslant m\leqslant N}\left|\sum_{n=1}^{m}p_n(x)\right|=4R(x)\max_{m}\left|\sum_{n=1}^{m}b_{\sigma(n)}e^{2\pi i\sigma(n)N^3x}\right|,$$

and therefore, by (10.3) and (10.4) we get

$$\left|\left\{x\in\Delta\colon \max_{1\leqslant m\leqslant N}\left|\sum_{n=1}^{m}p_n(x)\right|>2\sqrt{\log N}\right\}\right|$$
$$\geqslant\left|\left\{x\in\widetilde{\Delta}\colon \max_{1\leqslant m\leqslant N}\left|\sum_{n=1}^{m}b_{\sigma(n)}e^{2\pi i\sigma(n)N^3x}\right|>\sqrt{\log N}\right\}\right|\gtrsim|\Delta|.$$

According to Lemma 10.1 the polynomials $U_n(x)=\pm e^{\pi is/2}p_n(x)$ with an appropriate choice of $s=0,1,2,3$ and the sign $\pm$ can serve as the desired sequence. Clearly, they satisfy the conditions of the lemma.

Lemma 10.2 is proved.

In the rest of the paper we consider the sequence

$$\nu_0=1,\qquad \nu_k=2^{50^k},\qquad k=1,2,\dots. \tag{10.5}$$

Lemma 10.3. *If an increasing sequence of numbers $w(n)$ satisfies (1.4), then there exists a set of integers $G\subset\mathbb{N}$ such that*

$$w(\nu_{k+1})<100w(\nu_k),\qquad k\in G, \tag{10.6}$$
$$\sum_{k\in G}\frac{50^k}{w(\nu_k)}=\infty, \tag{10.7}$$

where ν_k is the sequence (10.5).

Proof. First observe that it follows from (1.4) that

$$\sum_{k=1}^{\infty}\frac{50^k}{w(\nu_k)}=\infty. \tag{10.8}$$

Let G be the set of integers k satisfying (10.6). If

$$\sum_{k\in\mathbb{N}\setminus G}\frac{50^k}{w(\nu_k)}<\infty, \tag{10.9}$$

then (10.7) immediately follows from (10.8) and the lemma is proved. So we can suppose that the series in (10.9) is divergent. Clearly, G is infinite and the set $\mathbb{N}\setminus G$ can be written in the form

$$\mathbb{N}\setminus G=\bigcup_j\{m_{2j}+1,m_{2j}+2,\dots,m_{2j+1}\}$$

where $m_{2j}\in G$ for any $j=1,2,\dots$. We have

$$w(\nu_{k+1})\geqslant 100w(\nu_k)\quad \text{for all } k=m_{2j}+1,m_{2j}+2,\dots,m_{2j+1},$$

which implies that

$$\sum_{k=m_{2j}+1}^{m_{2j+1}}\frac{50^k}{w(\nu_k)}\leqslant\frac{50^{m_{2j}+1}}{w(\nu_{m_{2j}+1})}\left(\frac12+\frac1{2^2}+\cdots\right)\leqslant\frac{50^{m_{2j}+1}}{w(\nu_{m_{2j}})}.$$

Thus we get

$$\sum_{k\in G}\frac{50^k}{w(\nu_k)}\geqslant\sum_{j=1}^{\infty}\frac{50^{m_{2j}}}{w(\nu_{m_{2j}})}\geqslant\frac{1}{50}\sum_{k\in\mathbb{N}\setminus G}\frac{50^k}{w(\nu_k)}=\infty.$$

The lemma is proved.

One can find the following lemma in [8], Ch. 9, in the proof of Theorem 6.

Lemma 10.4. *If $E_k\subset(0,1)$ are stochastically independent sets such that $|E_k|>c>0$ and the sequence $b_k>0$ satisfies $\sum_{k=1}^{\infty}b_k=\infty$, then*

$$\sum_{k=1}^{\infty}b_k\mathbf{1}_{E_k}(x)=\infty\quad\textit{almost everywhere.}\tag{10.10}$$

Proof. Let $0<c_k\leqslant b_k$ satisfy $\sum_k c_k=\infty$ and $\sum_k c_k^2<\infty$. Observe that $\varphi_k(x)=\mathbf{1}_{E_k}(x)-|E_k|$ form a stochastically independent system of orthogonal functions. It is well known that any series

$$\sum_{k=1}^{\infty}c_k\varphi_k(x)\quad\text{with}\ \sum_k c_k^2<\infty$$

in such a system is almost everywhere convergent (see [8], Ch. 2, Theorem 9). Combining this with the relation $\sum_{k=1}^{\infty}c_k|E_k|=\infty$, we get the divergence of $\sum_{k=1}^{\infty}c_k\mathbf{1}_{E_k}(x)$ almost everywhere and so (10.10).
The lemma is proved.

Lemma 10.5. *If $P\in\mathscr{P}_N$ is a polynomial of degree N and $\Delta\subset\mathbb{T}$ is an interval, then*

$$\mathrm{OSC}_\Delta(P)=\sup_{x,y\in\Delta}|P(x)-P(y)|\lesssim N^{3/2}|\Delta|\,\|P\|_2.$$

Proof. Suppose

$$P(x)=\sum_{k=1}^{N}a_ke^{2\pi ikx}.$$

Applying Hölder's inequality, for $x,y\in\Delta$ we get

$$\begin{aligned}|P(x)-P(y)|&\leqslant\left(\sum_{k=1}^{N}|a_k|^2\right)^{1/2}\left(\sum_{k=1}^{N}|e^{2\pi ikx}-e^{2\pi iky}|^2\right)^{1/2}\\&\lesssim N^{3/2}|\Delta|\,\|P\|_2.\end{aligned}$$

The lemma is proved.

Proof of Corollary 1.3. Applying Lemma 10.3 we find a set of indices $G\subset\mathbb{N}$ satisfying (10.6) and (10.7). For the sake of simplicity and without loss of generality we can suppose that $G=\mathbb{N}$. Indeed, considering the general case when $G=\{r_k\}$, one just needs to replace everywhere the sum $\sum_{k=1}^{\infty}$ by $\sum_{k\in G}$ and $\sum_{k=k_0}^{\infty}$ by $\sum_{k\in G\cap[k_0,\infty)}$.

The integers r_k will also appear in indices in some places. So we suppose that $G = \mathbb{N}$. Clearly there is a sequence of positive numbers $q_k \nearrow \infty$ such that

$$\frac{50^k}{q_k w(\nu_k)} \leqslant 1, \tag{10.11}$$
$$\sum_{k=1}^{\infty} \frac{50^k}{q_k w(\nu_k)} = \infty$$

and

$$\sum_{k=1}^{\infty} \frac{50^k}{q_k^2 w(\nu_k)} < \infty. \tag{10.12}$$

Consider the intervals

$$\Delta_{k,j} = \left[\frac{j-1}{\nu_k}, \frac{j}{\nu_k}\right), \qquad 1 \leqslant j \leqslant \nu_k, \quad k = 1, 2, \ldots .$$

Applying Lemma 10.2 with $N = (\nu_k)^2$ and $\Delta = \Delta_{k,j}$, $k \geqslant k_0$, we find a sequence of nonoverlapping polynomials $U_{k,j,n}(x)$, $n = 1, 2, \ldots, (\nu_k)^2$, such that

$$\operatorname{spec}(U_{k,j,n}) \subset (j(\nu_k)^{10}, (j+1)(\nu_k)^{10}], \tag{10.13}$$
$$\sum_{n=1}^{(\nu_k)^2} \|U_{k,j,n}\|_{L^2(\mathbb{T})}^2 \lesssim |\Delta_{k,j}| = \frac{1}{\nu_k}, \tag{10.14}$$
$$\sum_{n=1}^{(\nu_k)^2} |U_{k,j,n}(x)| \lesssim \frac{1}{(\nu_k)^2}, \qquad x \in \mathbb{T} \setminus \Delta_{k,j}, \tag{10.15}$$

and

$$\left|\left\{x \in \Delta_{k,j} \colon \max_{1 \leqslant m \leqslant (\nu_k)^2} \operatorname{Re}\left(\sum_{n=1}^{m} U_{k,j,n}(x)\right) > \sqrt{50^k}\right\}\right| \gtrsim |\Delta_{k,j}| = \frac{1}{\nu_k}. \tag{10.16}$$

Observe that if

$$\Delta_{k+1,i} \cap \left\{x \in \Delta_{k,j} \colon \max_{1 \leqslant m \leqslant (\nu_k)^2} \operatorname{Re}\left(\sum_{n=1}^{m} U_{k,j,n}(x)\right) > \sqrt{50^k}\right\} \neq \varnothing, \tag{10.17}$$

then one can find an integer $m = m(k+1, j)$ such that $1 \leqslant m \leqslant (\nu_k)^2$ and

$$\operatorname{Re}\left(\sum_{n=1}^{m(k+1,i)} U_{k,j,n}(x)\right) > \frac{\sqrt{50^k}}{2} \quad \text{for all } x \in \Delta_{k+1,i}, \tag{10.18}$$

since by (10.13) any sum $\sum_{n=1}^{m} U_{k,j,n}$ is a polynomial of degree at most $(\nu_k)^{15}$ and, using Lemma 10.5, its oscillation on $\Delta_{k+1,i}$ can roughly be estimated by

$$\left\|\sum_{n=1}^{m} U_{k,j,n}\right\|_2 (\nu_k)^{45/2} |\Delta_{k+1,i}| \leqslant 1.$$

This and (10.17) immediately imply (10.18). From (10.16) it follows that the measure of the union of all the intervals $\Delta_{k+1,i}$ satisfying (10.17) has a lower bound $c|\Delta_{k,j}|$, where $0 < c < 1$ is an absolute constant. Thus one can determine a set $E_{k,j} (\subset \Delta_{k,j})$ which is a union of some intervals $\Delta_{k+1,i}$ satisfying (10.17) and such that

$$|E_{k,j}| = d_k |\Delta_{k,j}|, \qquad 0 < c_1 < d_k < 1,$$

where c_1 is another absolute constant, while the constant d_k is common for all the indices $j = 1, 2, \ldots, \nu_k$. Thus, the sets

$$E_k = \bigcup_{1 \leqslant j \leqslant \nu_k} E_{k,j}, \qquad k \geqslant k_0,$$

are stochastically independent, and applying Lemma 10.4 we get

$$\sum_{k=k_0}^{\infty} \frac{50^k}{q_k w(\nu_k)} \mathbf{1}_{E_k}(x) = \infty \quad \text{almost everywhere.}$$

Using this, one can choose an increasing sequence of integers $k_0 < k_1 < k_2 < \cdots$ such that

$$\left| \left\{ x \in \mathbb{T} \colon \sum_{k=k_s+1}^{k_{s+1}} \frac{50^k}{q_k w(\nu_k)} \mathbf{1}_{E_k}(x) > s \right\} \right| > 1 - \frac{1}{s}.$$

Hence, for almost every $x \in \mathbb{T}$ the relation

$$\sum_{k=k_s+1}^{k_{s+1}} \frac{50^k}{q_k w(\nu_k)} \mathbf{1}_{E_k}(x) > s \tag{10.19}$$

holds for infinitely many s. Our desired trigonometric series is

$$\sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \sum_{j=1}^{\nu_k} \sum_{n=1}^{(\nu_k)^2} U_{k,j,n}(x), \tag{10.20}$$

where each $U_{k,j,n}$ is considered in its trigonometric form. Note that some of the coefficients of the above trigonometric series are zeros. Let us show that the coefficients of this series satisfy condition (1.2). Indeed, in light of (10.6) and (10.13) we have $w(s) \leqslant 100 w(\nu_k)$ for any $s \in \operatorname{spec}(U_{k,j,n}) \subset (\nu_k, \nu_{k+1}]$. Thus (1.2) may be easily deduced from (10.14), (10.12) and the bound

$$\sum_{k=k_0}^{\infty} \left(\frac{\sqrt{50^k}}{q_k w(\nu_k)} \right)^2 w(\nu_k) \sum_{j=1}^{\nu_k} \sum_{n=1}^{(\nu_k)^2} \|U_{k,j,n}\|_2^2 \lesssim \sum_{k=k_0}^{\infty} \frac{50^k}{q_k^2 w(\nu_k)} < \infty.$$

We construct an appropriate rearrangement of the series (10.20) as follows. The collections of trigonometric terms of our series (10.20) which are contained in the groups

$$U_{k,j,n}, \qquad k_s < k \leqslant k_{s+1}, \quad 1 \leqslant j \leqslant \nu_k, \quad 1 \leqslant n \leqslant (\nu_k)^2, \tag{10.21}$$

are arranged in increasing order with respect to s. We just need to determine the location of each polynomial $U_{k,j,n}$ within the group. We do it using induction with respect to the index k in (10.21). We leave the first group of polynomials

$$\{U_{k_s+1,j,n}\colon 1\leqslant j\leqslant \nu_{k_s+1},\ 1\leqslant n\leqslant (\nu_{k_s+1})^2\}$$

in its original order. Then suppose we have already rearranged all the polynomials $U_{k,j,n}$ corresponding to indices $k = k_s+1, k_s+2,\dots,l-1$, so that the polynomials $U_{l-1,j,n}$, $n=1,2,\dots,(\nu_{l-1})^2$, are arranged consecutively. We describe the procedure of how to locate the polynomials in the next collection $\{U_{l,j,n}\colon 1\leqslant j\leqslant \nu_l,\ 1\leqslant n\leqslant (\nu_l)^2\}$. Denote by $\Delta_{l-1,\bar{j}}$ the unique $(l-1)$st-order interval containing the given interval $\Delta_{l,j}$ of order l. The following two cases are possible.

1) If $\Delta_{l,j}\subset \Delta_{l-1,\bar{j}}\setminus E_{l-1,\bar{j}}$, then we locate the polynomials $U_{l,j,n}$, $n=1,2,\dots,(\nu_l)^2$, immediately after $U_{l-1,\bar{j},(\nu_{l-1})^2}$.

2) If $\Delta_{l,j}\subset E_{l-1,\bar{j}}$, then by the definition of $E_{l-1,\bar{j}}$ and by (10.17) and (10.18) for some $m=m(l,j)$ we have

$$\mathrm{Re}\left(\sum_{n=1}^{m(l,j)} U_{l-1,\bar{j},n}(x)\right) > \frac{\sqrt{50^{l-1}}}{2},\qquad x\in\Delta_{l,j}. \tag{10.22}$$

In this case we locate the polynomials $U_{l,j,n}$, $n=1,2,\dots,(\nu_l)^2$, immediately after $U_{l-1,\bar{j},m}$. This completes the induction procedure and so the construction of the required rearrangement. It remains to prove the a.e. divergence of the series (10.20) after the described rearrangement of the terms. For a given point $x\in\mathbb{T}$ there is a unique decreasing sequence of intervals $\Delta_{k,j_k(x)}$ containing x. Hence our series (10.20) can be split into two subseries

$$\sum_{k=k_0}^{\infty}\frac{\sqrt{50^k}}{q_k w(\nu_k)}\sum_{n=1}^{(\nu_k)^2} U_{k,j_k(x),n}(x) + \sum_{k=k_0}^{\infty}\frac{\sqrt{50^k}}{q_k w(\nu_k)}\sum_{j=1}^{\nu_k}\sum_{n=1}^{(\nu_k)^2} U_{k,j,n}(x)\mathbf{1}_{\mathbb{T}\setminus\Delta_{k,j}}(x). \tag{10.23}$$

From (10.11) and (10.15) it follows that

$$\sum_{k=k_0}^{\infty}\frac{\sqrt{50^k}}{q_k w(\nu_k)}\sum_{j=1}^{\nu_k}\sum_{n=1}^{(\nu_k)^2} |U_{k,j,n}(x)|\mathbf{1}_{\mathbb{T}\setminus\Delta_{k,j}}(x)\leqslant \sum_{k=k_0}^{\infty}\frac{1}{\nu_k}<\infty.$$

Thus we conclude that the second series in (10.23) converges absolutely for any $x\in\mathbb{T}$. Our rearrangement of the basic series produces a rearrangement of the first subseries in (10.23), and it remains to prove that for almost every $x\in\mathbb{T}$ such a rearranged series diverges. Denote

$$A_s(x)=\{k\in\mathbb{N}\colon k_s<k\leqslant k_{s+1},\ x\in E_{k,j_k(x)}\}$$

and

$$\begin{aligned}B_s(x)&=\{k\in\mathbb{N}\colon k_s<k\leqslant k_{s+1},\ x\in \Delta_{k,j_k(x)}\setminus E_{k,j_k(x)}\}\\&=\{k_s+1,k_s+2,\dots,k_{s+1}\}\setminus A_s(x).\end{aligned}$$

According to the construction of the rearrangement, one can observe that there is a 'restricted' partial sum (a sum of the form $\sum_p^q$) of the rearranged first subseries of (10.23) which contains completely all sums of the forms

$$\frac{\sqrt{50^k}}{q_k w(\nu_k)} \sum_{n=1}^{m(k+1,j_{k+1}(x))} U_{k,j_k(x),n}(x), \qquad k \in A_s(x), \tag{10.24}$$

and

$$\frac{\sqrt{50^k}}{q_k w(\nu_k)} \sum_{n=1}^{(\nu_k)^2} U_{k,j_k(x),n}(x), \qquad k \in B_s(x), \tag{10.25}$$

and contains no other terms. If (10.19) holds, then according to (10.22), for the sum of the elements (10.24) we obtain

$$\begin{aligned} \operatorname{Re}\Bigg(\sum_{k\in A_s(x)} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \sum_{n=1}^{m(k+1,j_{k+1}(x))} U_{k,j_k(x),n}(x)\Bigg) \\ \geqslant \frac{1}{2}\sum_{k=k_s+1}^{k_{s+1}} \frac{50^k}{q_k w(\nu_k)} \mathbf{1}_{E_k}(x) > \frac{s}{2}. \end{aligned} \tag{10.26}$$

As for the elements (10.25), they form an a.e. absolutely convergence series. Indeed, we have a pointwise bound

$$\begin{aligned} \sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \Bigg|\sum_{n=1}^{(\nu_k)^2} U_{k,j_k(x),n}(x)\Bigg| &= \sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \Bigg|\sum_{n=1}^{(\nu_k)^2} U_{k,j_k(x),n}(x)\Bigg| \mathbf{1}_{\Delta_{k,j_k(x)}}(x) \\ &\leqslant \sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \sum_{j=1}^{\nu_k}\Bigg|\sum_{n=1}^{(\nu_k)^2} U_{k,j,n}(x)\Bigg| \mathbf{1}_{\Delta_{k,j}}(x) = \sum_{k=k_0}^{\infty} R_k(x), \end{aligned} \tag{10.27}$$

and then using (10.11), (10.14) and an orthogonality argument we obtain

$$\begin{aligned} \sum_{k=k_0}^{\infty} \|R_k\|_2 &= \sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \Bigg\| \sum_{j=1}^{\nu_k} \Bigg| \sum_{n=1}^{(\nu_k)^2} U_{k,j,n}(x)\Bigg| \mathbf{1}_{\Delta_{k,j}}(x)\Bigg\|_2 \\ &= \sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \Bigg(\sum_{j=1}^{\nu_k} \Bigg\| \sum_{n=1}^{(\nu_k)^2} U_{k,j,n}(x)\mathbf{1}_{\Delta_{k,j}}(x)\Bigg\|_2^2\Bigg)^{1/2} \\ &\leqslant \sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} \Bigg(\sum_{j=1}^{\nu_k} \Bigg\| \sum_{n=1}^{(\nu_k)^2} U_{k,j,n}\Bigg\|_2^2\Bigg)^{1/2} \\ &\lesssim \sum_{k=k_0}^{\infty} \frac{\sqrt{50^k}}{q_k w(\nu_k)} = \sum_{k=k_0}^{\infty} \frac{1}{\sqrt{50^k}} \frac{50^k}{q_k w(\nu_k)} \leqslant \sum_{k=k_0}^{\infty} \frac{1}{\sqrt{50^k}} < \infty \end{aligned}$$

which implies the a.e. convergence of the series (10.27). Combining (10.26) with the a.e. absolute convergence of the series consisting of the terms (10.25), we conclude that the first subseries in (10.23) diverges for a.e. $x \in \mathbb{T}$. This completes the proof of Corollary 1.3.

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Bibliography

[1] S. V. Bočkarev (Bochkarev), "Rearrangements of Fourier-Walsh series", *Izv. Akad. Nauk SSSR Ser. Mat.* **43**:5 (1979), 1025–1041; English transl. in *Math. USSR-Izv.* **15**:2 (1980), 259–275.

[2] S. V. Bočkarev (Bochkarev), "On a majorant of the partial sums for a rearranged Walsh system", *Dokl. Akad. Nauk SSSR* **239**:3 (1978), 509–510; English transl. in *Soviet Math. Dokl.* **19**:2 (1978), 341–342.

[3] C. Demeter, "Singular integrals along N directions in $\mathbb{R}^2$", *Proc. Amer. Math. Soc.* **138**:12 (2010), 4433–4442.

[4] G. G. Gevorkyan, "Weyl multipliers for the unconditional convergence of series in a Franklin system", *Mat. Zametki* **41**:6 (1987), 789–797; English transl. in *Math. Notes* **41**:6 (1987), 446–451.

[5] S. Sh. Galstyan, "On convergence and unconditional convergence of Fourier series", *Dokl. Akad. Nauk* **323**:2 (1992), 216–218; English transl. in *Dokl. Math.* **45**:2 (1992), 286–289.

[6] G. A. Karagulyan, "On systems of non-overlapping Haar polynomials", *Ark. Mat.* **58**:1 (2020), 121–131.

[7] S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Monogr. Mat., vol. 6, Subwencji funduszu kultury narodowej, Warszawa–Lwow 1935, vi+298 pp.

[8] B. S. Kashin and A. A. Saakyan, *Orthogonal series*, 2nd ed., Izdatel'stvo Nauchno-Issledovatel'skogo Aktuarno-Finansovogo Tsentra, Moscow 1999, x+550 pp.; English transl. of 1st ed. in Transl. Math. Monogr., vol. 75, Amer. Math. Soc., Providence, RI 1989, xii+451 pp.

[9] A. Kolmogoroff (Kolmogorov) and D. Menchoff (Menshov), "Sur la convergence des séries de fonctions orthogonales", *Math. Z.* **26**:1 (1927), 432–441.

[10] D. E. Menchoff (Menshov), "Sur les séries de fonctions orthogonales. I", *Fund. Math.* **4** (1923), 82–105.

[11] F. Móricz, "On the convergence of Fourier series in every arrangement of the terms", *Acta Sci. Math. (Szeged)* **31** (1970), 33–41.

[12] S. Nakata, "On the divergence of rearranged Fourier series of square integrable functions", *Acta Sci. Math. (Szeged)* **32** (1971), 59–70.

[13] S. Nakata, "On the divergence of rearranged trigonometric series", *Tohoku Math. J.* (2) **27**:2 (1975), 241–246.

[14] S. Nakata, "On the unconditional convergence of Walsh series", *Anal. Math.* **5**:3 (1979), 201–205.

[15] S. Nakata, "On the divergence of rearranged Walsh series", *Tohoku Math. J.* (2) **24**:2 (1972), 275–280.

[16] A. M. Olevskii, "Divergent Fourier series", *Izv. Akad. Nauk SSSR Ser. Mat.* **27**:2 (1963), 343–366.

[17] W. Orlicz, "Zur Theorie der Orthogonalreihen", *Bull. Intern. Acad. Sci. Polon. Cracovie*, 1927, 81–115.

[18] S. N. Poleščuk (Poleshchuk), "On the unconditional convergence of orthogonal series", *Anal. Math.* **7**:4 (1981), 265–275.

[19] H. Rademacher, "Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen", *Math. Ann.* **87**:1–2 (1922), 112–138.

[20] B. Sendov, B. Boyanov, D. Vacov, R. Maleev, S. Markov and T. Boyanov (eds.), *Constructive function theory*'81 (Varna 1981), Publ. House Bulgar. Acad. Sci., Sofia 1983, 598 pp.

[21] K. Tandori, "Über die orthogonalen Funktionen. X. (Unbedingte Konvergenz.)", *Acta Sci. Math. (Szeged)* **23** (1962), 185–221.

[22] K. Tandori, "Über die Divergenz der Walshschen Reihen", *Acta Sci. Math. (Szeged)* **27** (1966), 261–263.

[23] P. L. Ul'yanov, "Divergent Fourier series", *Uspekhi Mat. Nauk* **16**:3(99) (1961), 61–142; English transl. in *Russian Math. Surveys* **16**:3 (1961), 1–75.

[24] P. L. Ul'yanov, "Divergent Fourier series of class L^p ($p \geqslant 2$)", *Dokl. Akad. Nauk SSSR* **137**:4 (1961), 786–789; English transl. in *Soviet Math. Dokl.* **2** (1961), 350–354.

[25] P. L. Ul'yanov, "Weyl factors for unconditional convergence", *Mat. Sb. (N.S.)* **60(102)**:1 (1963), 39–62.

[26] P. L. Ul'yanov, "On Weyl multipliers for unconditional convergence of orthogonal series", *Dokl. Akad. Nauk SSSR* **235**:5 (1977), 1038–1041; English transl. in *Soviet Math. Dokl.* **18** (1977), 1110–1113.

[27] P. L. Ul'yanov, "Exact Weil convergence factors for unconditional convergence", *Dokl. Akad. Nauk SSSR* **141**:5 (1961), 1048–1049; English transl. in *Soviet Math. Dokl.* **2** (1961), 1621–1622.

[28] P. L. Ul'yanov, "Solved and unsolved problems in the theory of trigonometric and orthogonal series", *Uspekhi Mat. Nauk* **19**:1(115) (1964), 3–69; English transl. in *Russian Math. Surveys* **19**:1 (1964), 1–62.

[29] P. L. Ul'yanov, "Kolmogorov and divergent Fourier series", *Uspekhi Mat. Nauk* **38**:4(232) (1983), 51–90; English transl. in *Russian Math. Surveys* **38**:4 (1983), 57–100.

[30] P. L. Ul'yanov, "Works of D. E. Men'shov on the theory of orthogonal series and their further development", *Vestnik Moskov. Univ. Ser. 1. Mat. Mekh.*, 1992, no. 4, 8–24; English transl. in *Moscow Univ. Math. Bull.* **47**:4 (1992), 8–20.

[31] Z. Zahorski, "Une série de Fourier permutée d'une fonction de classe L^2 divergente presque partout", *C. R. Acad. Sci. Paris* **251** (1960), 501–503.

[32] A. Zygmund, *Trigonometric series*, vol. II, 2nd ed., reprinted with corr. and some additions, Cambridge Univ. Press, London–New York 1968, vii+364 pp.

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