Finite difference scheme for a general class of the spatial segregation of reaction-diffusion systems with two population densities *

Avetik Arakelyan, Rafayel Barkhudaryan and Lusine Poghosyan
Institute of Mathematics, National Academy of Sciences of Armenia

Abstract

In the present work we deal with the numerical approximation of equations of stationary states for a general class of the spatial segregation of Reaction-diffusion system with two population densities having disjoint supports. We show that the problem gives rise the generalized version of the so-called two-phase obstacle problem and introduce the notion of viscosity solutions for this new model. Then we use quantitative properties of both, solutions and free boundaries, to develop convergent finite difference scheme.

At the end of paper we present a computational test and discuss numerical result.
Keywords: Free boundary, Two-phase membrane problem, Reaction-diffusion systems, Finite difference
Mathematics Subject Classification 35R35

Introduction

In recent years there have been intense studies of spatial segregation for reaction-diffusion systems. The existence of spatially inhomogeneous solutions for competition models of Lotka-Volterra type in the case of two and more competing densities have been considered in [8, 9, 10]. The aim of this paper is to study the numerical solutions of a certain general class of the Spatial segregation of Reaction-diffusion system with two population densities. Here, for the sake of clarity, we state the problem for general $m \geq 2$ population densities and then explain the problem for the particular case $m = 2$.

Let $\Omega \subset \mathbb{R}^n, (n \geq 2)$ be a connected and bounded domain with smooth boundary and m be a fixed integer. We consider the steady-states of m

*This work was supported by State Committee of Science MES RA, in frame of the research project No. SCS 13YR-1A0038

competing species coexisting in the same area Ω. Let $u_i(x)$ denotes the population density of the i^{th} component with the internal dynamic prescribed by $F_i(x, u_i)$.

We call the m-tuple $U = (u_1, \cdots, u_m) \in (H^1(\Omega))^m$ *a segregated state* if

$$u_i(x) \cdot u_j(x) = 0, \text{ a.e. for } \quad i \neq j, \ x \in \Omega.$$

The problem amounts to

$$\text{Minimize } E(u_1, \cdots, u_m) = \int_\Omega \sum_{i=1}^{m} \left(\frac{1}{2} |\nabla u_i|^2 + F_i(x, u_i) \right) dx, \qquad (1)$$

over the set

$$S = \{(u_1, \ldots, u_m) \in (H^1(\Omega))^m : u_i \geq 0, u_i \cdot u_j = 0, u_i = \phi_i \quad \text{on} \quad \partial\Omega\},$$

where $\phi_i \in H^{\frac{1}{2}}(\partial\Omega)$, $\phi_i \cdot \phi_j = 0$, for $i \neq j$ and $\phi_i \geq 0$ on the boundary $\partial\Omega$.

We assume that

$$F_i(x, s) = \int_0^s f_i(x, v) dv,$$

where $f_i(x, s) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is Lipschitz continuous in s, uniformly continuous in x and $f_i(x, 0) \geq 0$. In the sequel, we assume that the functional (1) is coercive, which will be needed to provide the existence of minimizers. In order to have coercivity for the functional, for instance following [9], one can assume that there exists $b_i(x) \in L_\infty(\Omega)$ such that

$$|f_i(x, s)| \leq b_i(x) \cdot s, \quad \forall x \in \Omega, \quad s \geq s_0 >> 1,$$

and

$$\int_\Omega (|\nabla w(x)|^2 - b_i(x) w^2(x)) dx > 0, \quad \forall w \in H_0^1(\Omega) \setminus \{0\}.$$

Remark 0.1. *By our definition, the functions $f_i(x, s)$'s are defined only for non negative values of s (recall that our densities u_i's are assumed non negative); thus we can arbitrarily extend such functions on the negative semiaxis. For the sake of convenience, when $s \leq 0$ we will let $f_i(x, s) = -f_i(x, -s)$ This extension preserves the continuity, due to conditions on f_i defined above. In the same way, each F_i is extended as an even function.*

For the case of two populations the problem will be reduced to:

$$\text{Minimize } E(u_1, u_2) = \int_\Omega \sum_{i=1}^{2} \left(\frac{1}{2} |\nabla u_i|^2 + F_i(x, u_i) \right) dx, \qquad (2)$$

over the set

$$S = \{(u_1, u_2) \in (H^1(\Omega))^2 : u_i \geq 0, u_1 \cdot u_2 = 0, u_i = \phi_i \quad \text{on} \quad \partial\Omega\}.$$

Here $\phi_i \in H^{\frac{1}{2}}(\partial\Omega)$ with property $\phi_1 \cdot \phi_2 = 0$, $\phi_i \geq 0$ on the boundary $\partial\Omega$. Throughout the paper we will also assume that the functions $F_i(x, s)$ are convex with respect to the variable s.

1 Segregation problem with two population densities

1.1 Generalized two-phase obstacle problem

This section is devoted to the minimization problem with two population densities. Following [6] it is easy to see that the problem can be treated as a minimization problem subject to the convex set:

$$\text{Minimize: } \mathcal{J}(w) = \int_\Omega \left[\frac{1}{2}|\nabla w|^2 + F_1(x, w^+) + F_2(x, -w^-)\right] dx, \quad (3)$$

over the set $\mathbb{K} = \{w \in H^1(\Omega) : \ \ w - (\phi_1 - \phi_2) \in H_0^1(\Omega)\}$.

The corresponding Euler-Lagrange equation for the minimization problem will be

$$\begin{cases} \Delta w = f_1(x, w) \cdot \chi_{\{w>0\}} - f_2(x, -w) \cdot \chi_{\{w<0\}}, & x \in \Omega, \\ w = \phi_1 - \phi_2, & x \in \partial\Omega, \end{cases} \quad (4)$$

where χ_A stands for the characteristic function of the set A. Inspired by the setting of the two-phase obstacle problem (see [13]) we will call the problem (4) as the *generalized two-phase obstacle problem.* Nowadays, the theory of the two-phase obstacle-like problems (elliptic and parabolic versions) is well-established and for a reference we again address to the book [13]. For the numerical treatment of the same problems we refer to the works [2, 3, 5, 1].

In [2, 3] the authors introduced the so-called Min-Max formulation for the usual Two-phase obstacle problem, which is very useful to define the notion of viscosity solutions. Moreover, it turns out that the introduced viscosity solution is equivalent to the weak solution of the Two-phase obstacle problem. Our aim is to use the same approach for the generalized counterpart. To this end, we need to make some notations.

Let Ω be an open subset of $\mathbb{R}^n$, and for a twice differentiable function $u : \Omega \to \mathbb{R}$ let Du and D^2u denote the gradient and Hessian matrix of u, respectively. Also let the function $G(x, r, p, X)$ be a continuous real-valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n$, with S^n being the space of real symmetric $n \times n$ matrices.

In light of the Min-Max form defined in [2, 3] we introduce the following generalized Min-Max variational equation:

$$\begin{cases} \min\left(-\Delta w + f_1(x, w), \max(-\Delta w - f_2(x, -w), w)\right) = 0, & \text{in } \Omega \\ w = \phi_1 - \phi_2 \equiv g, & \text{on } \partial\Omega. \end{cases} \quad (5)$$

Following the above notations we introduce a function $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ by

$$G(x, r, p, X) = \min(-trace(X) + f_1(x, r), \max(-trace(X) - f_2(x, -r), r)), \quad (6)$$

then the equation in (5) can be rewritten as

$$G(x, w, Dw, D^2 w) = 0 \quad \text{in } \ \Omega. \tag{7}$$

Below we recall the definition of degenerate ellipticity, and prove that the equation (5) is degenerate elliptic.

Definition 1.1. *We call the equation* (7) ***degenerate elliptic*** *if*

$$G(x, r, p, X) \leq G(x, s, p, Y) \quad \textit{whenever} \quad r \leq s \quad \textit{and} \quad Y \leq X,$$

where $Y \leq X$ means that $X - Y$ is a nonnegative definite symmetric matrix.

Lemma 1.1. *The equation* (6) *is degenerate elliptic.*

Proof. Let $X, Y \in S^n$ and $r, s \in \mathbb{R}$ satisfy $Y \leq X$ and $r \leq s$. Then using the fact that $F_i(x,t)$ is convex in t for all $x \in \Omega$, we have $F_i''(x,t) = f_i'(x,t) \geq 0$, where the derivatives are taken with respect to t. Thus,

$$\begin{aligned} - \mathit{trace}(X) + f_1(x, r) \leq -\mathit{trace}(Y) + f_1(x, s), \quad \text{and} \\ \max(-\mathit{trace}(X) - f_2(x, -r), r) \leq \max(-\mathit{trace}(Y) - f_2(x, -s), s). \end{aligned}$$

Therefore

$$\begin{aligned} G(x, r, p, X) &= \min(-\mathit{trace}(X) + f_1(x, r), \max(-\mathit{trace}(X) - f_2(x, -r), r)) \\ &\leq \min(-\mathit{trace}(Y) + f_1(x, s), \max(-\mathit{trace}(Y) - f_2(x, -s), s)) \\ &= G(x, s, p, Y). \end{aligned}$$

□

Now, we are ready to define viscosity solutions for the generalized two-phase obstacle problem. For general background about the theory of viscosity solutions the reader is referred to [11], [7] and references therein.

Definition 1.2. *A bounded uniformly continuous function $w : \Omega \to \mathbb{R}$ is called a viscosity subsolution (resp. supersolution) to* (5)*, if for each $\varphi \in C^2(\Omega)$ and local maximum point of $w - \varphi$ (respectively minimum) at $x_0 \in \Omega$, we have*

$$\min\left(-\Delta\varphi(x_0) + f_1(x_0, w(x_0)), \max(-\Delta\varphi(x_0) - f_2(x_0, -w(x_0)), w(x_0))\right) \leq 0.$$

(respectively

$$\min\left(-\Delta\varphi(x_0) + f_1(x_0, w(x_0)), \max(-\Delta\varphi(x_0) - f_2(x_0, -w(x_0)), w(x_0))\right) \geq 0.)$$

The function $w : \Omega \to \mathbb{R}$ is said to be a **viscosity solution** of (5), if it is both a viscosity subsolution and supersolution for (5).

1.2 Finite difference scheme

We will make the notations for the one-dimensional and two-dimensional cases parallely.

For the sake of simplicity, we will assume that $\Omega = (-1, 1)$ in one-dimensional case and $\Omega = (-1, 1) \times (-1, 1)$ in two-dimensional case in the rest of the paper, keeping in mind that the method works also for more complicated domains.

Let $N \in \mathbb{N}$ be a positive integer, $h = 2/N$ and

$$x_i = -1 + ih, \; y_i = -1 + ih, \quad i = 0, 1, ..., N.$$

We use the notation u_i and $u_{i,j}$ (or simply u_α, where α is one- or two-dimensional index) for finite-difference scheme approximation to $u(x_i)$ and $u(x_i, y_j)$,

$$g_i = \phi_1(i) - \phi_2(i) = \phi_1(x_i) - \phi_2(x_i)$$

and

$$g_{i,j} = \phi_1(i,j) - \phi_2(i,j) = \phi_1(x_i, y_j) - \phi_2(x_i, y_j),$$

in one- and two-dimensional cases, respectively, assuming that the function $\phi_1 - \phi_2$ is extended to be zero everywhere outside the boundary $\partial\Omega$.

In this paper we will use also notations $u = (u_\alpha)$, $g = (g_\alpha)$ (not to be confused with functions u, g).

Denote

$$\mathcal{N} = \{i : \; 0 \le i \le N\} \quad \text{or} \quad \mathcal{N} = \{(i,j) : \; 0 \le i, j \le N\},$$

$$\mathcal{N}^o = \{i : \; 1 \le i \le N-1\} \quad \text{or} \quad \mathcal{N}^o = \{(i,j) : \; 1 \le i, j \le N-1\},$$

in one- and two- dimensional cases, respectively, and

$$\partial\mathcal{N} = \mathcal{N} \setminus \mathcal{N}^o.$$

In one-dimensional case we consider the following approximation for Laplace operator: for any $i \in \mathcal{N}^o$,

$$\Delta_h u_i \equiv L_h u_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},$$

and for two-dimensional case we introduce the following 5-point stencil approximation for Laplacian:

$$\Delta_h u_{i,j} \equiv L_h u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2}$$

for any $(i,j) \in \mathcal{N}^o$.

Now with the use of the generalized Min-Max variational equation (5) we define appropriate finite difference scheme as follows:

$$
\begin{cases}
\min(-L_h u_\alpha + f_1(x_\alpha, u_\alpha)\,, \\
\qquad\qquad\qquad \max(-L_h u_\alpha - f_2(x_\alpha, -u_\alpha)\,,\, u_\alpha)) = 0, & \alpha \in \mathcal{N}^o, \\
u_\alpha = g_\alpha, & \alpha \in \partial\mathcal{N}.
\end{cases} \tag{8}
$$

Theorem 1.1. *The nonlinear system* (8) *has a unique solution.*

For convergence analysis of the difference scheme (8) we apply Barles-Souganidis theory (see [4]) developed for viscosity solutions. To this aim, we define a uniform structured grid on the domain Ω as a directed graph consisting of a set of points $x_i \in \Omega$, $i = 1, ..., N$, each endowed with a number of neighbors K. A grid function is a real valued function defined on the grid, with values $u_i = u(x_i)$. The typical examples of such grid are 3-point and 5-point stencil discretization for the spaces of one dimension and two dimension, respectively. Here we recall degenerate elliptic schemes introduced by Oberman (see [12]).

A function $F^h : \mathbb{R}^N \to \mathbb{R}^N$, which is regarded as a map from grid functions to grid functions, is a *finite difference scheme* if

$$
F^h[u]^i = F^i[u_i, u_i - u_{i_1}, \dots, u_i - u_{i_K}] \quad (i = 1, \dots, N),
$$

where $\{i_1, i_2, \dots, i_K\}$ are the neighbor points of a grid point i. Denote

$$
F^i[u] \equiv F^i[u_i, u_i - u_{i_j}|_{j=\overline{1,K}}] \equiv F^i[u_i, u_i - u_j], \quad i = 1, ..., N,
$$

where u_j is shorthand for the list of neighbors $u_{i_j}|_{j=\overline{1,K}}$.

Definition 1.3. *The scheme F is* degenerate elliptic *if each component F^i is nondecreasing in each variable, i.e. each component of the scheme F^i is a nondecreasing function of u_i and the differences $u_i - u_{i_j}$ for $j = 1, ..., K$.*

Since the grid is uniformly structured, we denote $h > 0$ be the size of the mesh. Then, for the nonlinear system (8) we have

$$
F^i[u_i, u_i - u_j] = \min(-L_h u_i + f_1(x_i, u_i)\,,\, \max(-L_h u_i - f_2(x_i, -u_i)\,,\, u_i)), \tag{9}
$$

where

$$
L_h u_i = \sum_{j=1}^{K} \frac{1}{h^2}(u_{i_j} - u_i), \quad i = 1, ..., N. \tag{10}
$$

Since the functions $f_i(x, s)$ are monotone non-decreasing with respect to s, we clearly see that $F^i[u_i, u_i - u_j]$ is non-decreasing with respect to u_i and $u_i - u_j$ as well. Therefore the finite difference scheme (8) is a *degenerate elliptic scheme.* But we know that the degenerate elliptic schemes are *monotone* and *stable* (see [12]). The consistency of the system (8) is obvious. Thus, we show that the nonlinear system (8) fulfills all the necessary properties for the Barles-Souganidis framework, namely it is stable, monotone and consistent and therefore, due to Barles - Souganidis theorem, the solution to the discrete nonlinear system (8) converges locally uniformly to the unique viscosity solution of the generalized two-phase obstacle problem (5).

2 Numerical example

In this section we present a numerical simulation for two competing densities with internal dynamics f_i. We consider the following minimization problem:

$$\text{Minimize} \int_\Omega \sum_{i=1}^{2} \left(\frac{1}{2} |\nabla u_i|^2 + F_i(x, u_i) \right) dx, \tag{11}$$

over the set

$$S = \{(u_1, u_2) \in (H^1(\Omega))^2 : u_i \geq 0, u_1 \cdot u_2 = 0, u_i = \phi_i \quad \text{on} \quad \partial\Omega\}.$$

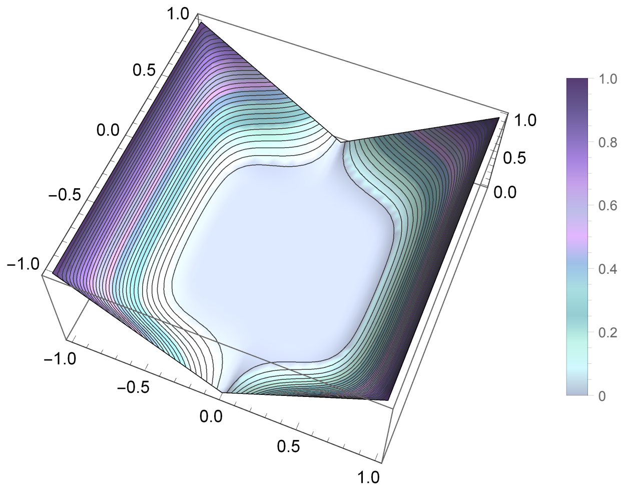

Figure 1: Numerical solution with 50×50 discretization points.

We take $\Omega = [0, 1] \times [0, 1]$, and the internal dynamics such that

$$f_1(x, y, u_1) = 7(x^2 + y^2) + (1 + u_1)^2,$$

and

$$f_2(x, y, u_2) = 10(x^2 + y^2) + (1 + u_2)^2,$$

with the boundaries $\phi_1(x, y)$ and $\phi_2(x, y)$ defined as follows:

$$\phi_1(x, \pm 1) = \begin{cases} x & 0 \leq x \leq 1, \\ 0 & -1 \leq x \leq 0, \end{cases}$$

$$\phi_1(1, y) = 1, \qquad \phi_1(-1, y) = 0,$$

and

$$\phi_2(x, \pm 1) = \begin{cases} 0 & 0 \leq x \leq 1, \\ -x & -1 \leq x \leq 0, \end{cases}$$

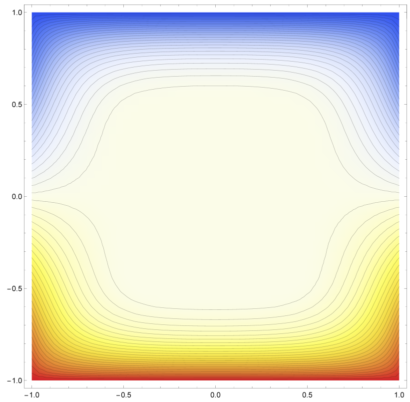

Figure 2: Level sets of the numerical solution.

$$\phi_2(-1, y) = 1, \qquad \phi_2(1, y) = 0.5,$$

In Figure 1 we clearly see the zero set between densities u_i, which is due to taken large internal dynamics $f_i(x, y, u_i)$ of the system.

In Figure 2 the level sets of the numerical solution are depicted. We see the tangential touch between two competing densities u_1 and u_2 close to the fixed boundary of Ω.

References

[1] Arakelyan, A. A finite difference method for two-phase parabolic obstacle-like problem. *Armen. J. Math. 7*, 1 (2015), 32–49.

[2] Arakelyan, A., Barkhudaryan, R., and Poghosyan, M. Finite difference scheme for two-phase obstacle problem. *Dokl. Nats. Akad. Nauk Armen. 111*, 3 (2011), 224–231.

[3] Arakelyan, A., Barkhudaryan, R., and Poghosyan, M. Numerical Solution of the Two-Phase Obstacle Problem by Finite Difference Method. *Armen. J. Math. 7*, 2 (2015), 164–182.

[4] Barles, G., and Souganidis, P. E. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal. 4*, 3 (1991), 271–283.

[5] Bozorgnia, F. Numerical solutions of a two-phase membrane problem. *Applied Numerical Mathematics 61*, 1 (2011), 92–107.

[6] Bozorgnia, F., and Arakelyan, A. Numerical algorithms for a variational problem of the spatial segregation of reaction–diffusion systems. *Applied Mathematics and Computation 219*, 17 (2013), 8863–8875.

[7] Caffarelli, L. A., and Cabré, X. *Fully nonlinear elliptic equations*, vol. 43 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1995.

[8] Conti, M., Terracini, S., and Verzini, G. Asymptotic estimates for the spatial segregation of competitive systems. *Adv. Math. 195*, 2 (2005), 524–560.

[9] Conti, M., Terracini, S., and Verzini, G. A variational problem for the spatial segregation of reaction-diffusion systems. *Indiana Univ. Math. J. 54*, 3 (2005), 779–815.

[10] Conti, M., Terracini, S., and Verzini, G. Uniqueness and least energy property for solutions to strongly competing systems. *Interfaces Free Bound. 8*, 4 (2006), 437–446.

[11] Crandall, M. G., Ishii, H., and Lions, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.) 27*, 1 (1992), 1–67.

[12] Oberman, A. M. Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems. *SIAM J. Numer. Anal. 44*, 2 (2006), 879–895 (electronic).

[13] Petrosyan, A., Shahgholian, H., and Ural'ceva, N. N. *Regularity of free boundaries in obstacle-type problems*, vol. 136. American Mathematical Soc., 2012.

Avetik Arakelyan
Institute of Mathematics NAS of Armenia,
Baghramyan 24/5, 0019 Yerevan, Armenia
avetik@instmath.sci.am

Rafayel Barkhudaryan
Institute of Mathematics NAS of Armenia,
Baghramyan 24/5, 0019 Yerevan, Armenia
rafayel@instmath.sci.am

Lusine Poghosyan
Institute of Mathematics NAS of Armenia,
Baghramyan 24/5, 0019 Yerevan, Armenia
lusine@instmath.sci.am