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Accelerating the convergence of trigonometric series

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Abstract: A nonlinear method of accelerating both the convergence of Fourier series and trigonometric interpolation is investigated. Asymptotic estimates of errors are derived for smooth functions. Numerical results are represented and discussed.

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1 Introduction

It is well known that, for $f \in L_2(-1,1)$, the series

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i\pi n x}, \ \ f_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi n x} dx \tag{1}$$

is convergent by L_2-norm

$$||f|| = \left(\int_{-1}^{1} |f(x)|^2 dx \right)^{1/2}.$$

For practical purposes, approximations are obtained by using only a finite number of Fourier coefficients $\{f_n\}$, $|n| \le N < \infty$. As is also well known [32], when we approximate f by truncated Fourier series (partial sums)

$$S_N(f) := \sum_{n=-N}^{N} f_n e^{i\pi n x} \tag{2}$$

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or by trigonometric interpolation

$$I_N(f) := \sum_{n=-N}^{N} \widehat{f}_n e^{i\pi nx}, \quad \widehat{f}_n = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) e^{-i\pi n x_k}, \quad x_k = \frac{2k}{2N+1}, \tag{3}$$

the resulting error is strongly dependent on the smoothness of f. Approximating a 2-periodic $f \in C^\infty(R)$ function by S_N or I_N $(N >> 1)$ is highly effective. When the approximated function has a point of discontinuity, the above mentioned approximations lead to the Gibbs phenomenon. The "oscillations" caused by this phenomenon are typically propagated into regions away from the singularity and degrade the quality of the approximations.

Different ways of treating this deficiency have been suggested in the literature (see, for example, [14–17]). The idea of increasing the convergence rate of Fourier series by subtracting a polynomial that represents the discontinuities in the function and some of its first derivatives was suggested by A.Krylov in 1906 [19] and later, in 1964, by Lanczos [20, 21] (see also [2, 23] and [18, with references]). The key problem lies in determining the singularity amplitudes. As formulated by Gottlieb [12], these amplitudes could be found from the first N Fourier coefficients. This idea has been realized by Eckhoff in a series of papers [5–8] where the values of the "jumps" are solutions of the corresponding system of linear equations. Let us refer to this approach as the Krylov-Gottlieb-Eckhoff (KGE) method (see also [3, 11, 13, 22] and, for the multidimensional case, [25, 28]).

Application of Pade approximants [1] to Fourier series has been studied by several investigators. The general form of Fourier-Pade representation has been suggested by Cheney [4], but he does not discuss algorithms for computing coefficients, rates of convergence, and so forth. Geer [10] introduced and studied a class of approximations to a periodic function f that uses the ideas of Pade (rational approximations). While these approximations do not "eliminate" the Gibbs phenomenon, they do mitigate its effect. For eliminating the Gibbs phenomenon, algorithms based on Pade-type approximations were described and studied in [9, 26, 29, 30].

In [24], Pade approximants are applied to the asymptotic expansion of coefficients of Fourier series for piecewise smooth functions, leading to a new kind of approximation. In [27], the corresponding asymptotic estimates of errors of these approximations are investigated. Here, we extend the method to trigonometric interpolations.

The proposed approximations are exact for a system of quasipolynomials while the KGE-method is exact for a subsystem of polynomials. Thus, we obtain a generalization of the latter. The quasipolynomial approach is nonlinear while the KGE-method is (given the exact jumps) linear. If the jumps and Fourier coefficients of the approximated function are known, then the KGE-method can be constructed without any additional calculations. However, for the quasipolynomial method, we also need the values of some parameters that can be determined from a nonlinear system of equations with jumps in the coefficients. This additional complexity in calculation yields round off errors of the approximations that are more precise and more stable. Theorems and numerical examples are presented. Moreover, comparisons between the quasipolynomial and the

KGE-method are made.

We expect that the proposed approximations, especially insofar as they are derived by a tool as flexible as the system of quasipolynomials, should result in new algorithms of increased precision and robustness.

2 KGE-method

We say that $f \in C^q[-1,1]$, $q \geq 0$ if $f^{(q)}$ is continuous in $[-1,1]$. Denote

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \ k = 0, \cdots, q.$$

The idea of the KGE-method is to split the given function $f \in C^q[-1,1]$ into two parts

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{4}$$

where F is a relatively smooth function and $B_k(x)$ are 2-periodic Bernoulli polynomials with Fourier coefficients

$$B_{k,n} = \begin{cases} 0, & n = 0 \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n = \pm 1, \pm 2, ... \end{cases} \tag{5}$$

Approximating the function F by truncated Fourier series leads to the *KGE-approximation*

$$S_{N,q}(f) = \sum_{n=-N}^{N} F_n e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{6}$$

where the coefficients F_n can be expressed by f_n and $B_{k,n}$ from (1), (4), and (5).

Similarly, approximating the function F by trigonometric interpolation leads to *KGE-interpolation* (see (3))

$$I_{N,q}(f) = \sum_{n=-N}^{N} \widehat{F}_n e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{7}$$

where the discrete coefficients $\widehat{F}_n$ can be expressed by $\widehat{f}_n$ and $(\widehat{B_k})_n$ from (4).

Theorem 2.1. *Suppose $f \in C^q[-1,1]$, $q \geq 1$, $f^{(q+1)} \in L_1(-1,1)$. Then*

$$R_{N,q}(f) := f(x) - S_{N,q}(f) = A_q(f) \sum_{|n|>N} \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} e^{i\pi n x} + o(N^{-q}), \ N \to \infty. \tag{8}$$

Proof. By q-fold integration by parts in (1), we have that

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q-1} \frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^{1} f^{(q)}(x) e^{-i\pi n x} dx. \tag{9}$$

Therefore,

$$R_{N,q}(f) = \sum_{|n|>N} F_n e^{i\pi nx}, \tag{10}$$

where

$$F_n = \frac{(-1)^{n+1}}{2} \frac{A_q(f)}{(i\pi n)^{q+1}} + \frac{1}{2(i\pi n)^{q+1}} \int_{-1}^{1} f^{(q+1)}(x) e^{-i\pi nx} dx. \tag{11}$$

Note that, according to the well-known Riemann-Lebesgue theorem [32], the second term is $o(n^{-q-1})$ as $n \to \infty$. This concludes the proof. □

Similarly, we prove the following result:

Theorem 2.2. *Suppose $f \in C^q[-1,1]$, $q \geq 1$, $f^{(q+1)} \in L_1(-1,1)$. Then*

$$r_{N,q}(f) := f(x) - I_{N,q}(f) = A_q(f)\times$$

$$\left(\sum_{|n|>N} \frac{(-1)^{n+1} e^{i\pi nx}}{2(i\pi n)^{q+1}} + \sum_{n=-N}^{N} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{(-1)^{n+s+1} e^{i\pi nx}}{2(i\pi)^{q+1}(n+s(2N+1))^{q+1}} \right) + o(N^{-q}),\ \ N \to \infty. \tag{12}$$

Proof. Just note that, from (11), we have at least $F_n = O(n^{-2}), n \to \infty$, and so

$$\widehat{F}_n = \sum_{s=-\infty}^{\infty} F_{n+s(2N+1)}.$$

□

3 Quasipolynomial (QP-) method

3.1

The essential features of quasipolynomial approximation [24] are both the application of Pade approximants to the asymptotic expansion of f_n (see 9) and the solution of the corresponding system of nonlinear equations (a system that arises in the theory of Pade approximations).

Consider a finite sequence of complex numbers $\theta := \{\theta_k\}_{k=1}^{m}$, $m \geq 1$, and denote

$$\Delta_n^0(\theta) = A_n(f),\ \Delta_n^k(\theta) = \Delta_n^{k-1}(\theta) + \theta_k \Delta_{n-1}^{k-1}(\theta),\ k \geq 1.$$

If $n < 0$, we set $\Delta_n^k(\theta) = 0$.

It is easy to verify that, for $x \neq -1/\theta_1$,

$$\sum_{k=0}^{q-1} A_k x^k = x^q \frac{A_{q-1}\theta_1}{1+\theta_1 x} + \frac{1}{1+\theta_1 x} \sum_{k=0}^{q-1} (A_k + \theta_1 A_{k-1}) x^k. \tag{13}$$

Note that, for $\theta_1 = 0$, the sum on the left side of (13) remains unchanged. Iterating this transformation up to m times yields the following formula ($x \neq -1/\theta_k;\ k = 1, \cdots, m;\ m \leq q-1$):

$$\sum_{k=0}^{q-1} A_k x^k = x^q \sum_{k=1}^{m} \frac{\theta_k \Delta_{q-1}^{k-1}(\theta)}{\prod_{s=1}^{k}(1+\theta_s x)} + \frac{1}{\prod_{s=1}^{m}(1+\theta_s x)} \sum_{k=0}^{q-1} \Delta_k^m(\theta) x^k. \tag{14}$$

Suppose $f \in C^q[-1,1]$. Applying transformation (14) to the first term of (9) with $(i\pi n)^{-1}$ instead of x, we derive

$$f_n = Q_n + P_n, \ \ n \neq 0, \tag{15}$$

where

$$Q_n = \frac{(-1)^{n+1}(i\pi n)^m}{2\prod_{s=1}^{m}(i\pi n + \theta_s)} \sum_{k=0}^{q-m-1} \frac{\Delta_k^m(\theta)}{(i\pi n)^{k+1}} \tag{16}$$

and

$$P_n = \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \sum_{k=1}^{m} \frac{\theta_k \Delta_{q-1}^{k-1}(\theta)(i\pi n)^k}{\prod_{s=1}^{k}(i\pi n + \theta_s)} +$$

$$+ \frac{(-1)^{n+1}(i\pi n)^m}{2\prod_{k=1}^{m}(i\pi n + \theta_k)} \sum_{k=q-m}^{q-1} \frac{\Delta_k^m(\theta)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^{1} f^{(q)}(t)\, e^{-i\pi n t} dt. \tag{17}$$

Inasmuch as (15) holds, we can split the function f into two parts

$$f(x) = Q(x) + P(x), \tag{18}$$

where

$$Q(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Q_n e^{i\pi n x}, \ \ P(x) = \sum_{n=-\infty}^{\infty} P_n e^{i\pi n x}, \ \ P_0 = f_0. \tag{19}$$

Approximating P by the truncated Fourier series leads to

$$S_{N,q,m}(f) = Q(x) + \sum_{n=-N}^{N} (f_n - Q_n) e^{i\pi n x} \tag{20}$$

and approximating P by trigonometric interpolation leads to

$$I_{N,q,m}(f) = Q(x) + \sum_{n=-N}^{N} (\widehat{f}_n - \widehat{Q}_n) e^{i\pi n x}. \tag{21}$$

It is important to note that, for $\theta_1 = \theta_2 = \cdots = \theta_m = 0$, approximations $S_{N,q,m}(f)$ and $I_{N,q,m}(f)$ coincide with the KGE-approximation and the KGE- interpolation, respectively.

Approximation properties of $S_{N,q,m}$ and $I_{N,q,m}$ are strongly connected with the smoothness of the function P or, put another way, with the rate of convergence of P_n to zero. This condition leads to the following system of nonlinear equations (see the second term in (17)) for the unknown vector θ:

$$\Delta_k^m(\theta) = 0, \ \ k = q-m, \cdots, q-1. \tag{22}$$

Note that, if θ is a solution of (22) and $f \in C^q[-1,1]$, $f^{(q+1)} \in L_1(-1,1)$, then $P_n = O(n^{-q-1})$, $n \to \infty$. We call approximations (20) and (21), together with (22), *QP-approximation* and *QP-interpolation*, respectively. Actually, we apply the Pade approximation [q+m-1/m] to the sum on the left side of (14) [1].

3.2

We are interested in the asymptotic behavior of

$$R_{N,q,m}(f) := f(x) - S_{N,q,m}(f) \;\; and \;\; r_{N,q,m}(f) := f(x) - I_{N,q,m}(f).$$

By $\gamma_k(m), k = 0, \cdots, m$ we denote the coefficients of the polynomial

$$\prod_{k=1}^{m}(1+\theta_k x) \equiv \sum_{k=0}^{m} \gamma_k(m) x^k.$$

Note that

$$\Delta_k^m(\theta) = A_k(f) + \sum_{s=1}^{m} \gamma_s(m) A_{k-s}(f).$$

Hence, the system (22) can be written in the form

$$\sum_{s=1}^{m} \gamma_s(m) A_{k-s+q-m-1}(f) = -A_{k+q-m-1}(f), \;\; k = 1, \cdots, m. \tag{23}$$

Denote

$$U_r^m = [A_{k-s+r}(f)], \;\; k, \; s = 1, \cdots, m.$$

Theorem 3.1. *[27] Suppose $f \in C^q[-1,1]$, $q \geq 1$, $f^{(q+1)} \in L_1[-1,1]$ and*

$$detU_{q-m-1}^m \neq 0.$$

Then, with θ from (22), the following holds:

$$R_{N,\,q,m}(f) = (-1)^m \frac{detU_{q-m}^{m+1}}{detU_{q-m-1}^m} \sum_{|n|>N} \frac{(-1)^{n+1} e^{i\pi n x}}{2(i\pi n)^{q+1}} + o(N^{-q}), \;\; N \to \infty. \tag{24}$$

Proof. From (18) and (20), we have

$$R_{N,\,q,m}(f) = \sum_{|n|>N} P_n e^{i\pi n x}, \tag{25}$$

where (see (17))

$$P_n = \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \left(A_q(f) + \sum_{k=1}^{m} \frac{\theta_k \Delta_{q-1}^{k-1}(\theta)}{\prod_{s=1}^{k} \left(1 + \frac{\theta_s}{i\pi n}\right)} \right) + o(n^{-q-1}), \;\; n \to \infty. \tag{26}$$

It now follows that

$$R_{N,q,m}(f) = \Delta_q^m(\theta) \sum_{|n|>N} \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} e^{i\pi nx} + o(N^{-q}),\ N \to \infty. \tag{27}$$

Here, we use the fact that

$$\Delta_q^m(\theta) = \Delta_q^{m-1}(\theta) + \theta_m \Delta_{q-1}^{m-1}(\theta) = \Delta_q^{m-2}(\theta) + \theta_{m-1}\Delta_{q-1}^{m-2}(\theta) +$$

$$+\theta_m \Delta_{q-1}^{m-1}(\theta) = \Delta_q^0(\theta) + \sum_{k=1}^{m} \theta_k \Delta_{q-1}^{k-1}(\theta) = A_q(f) + \sum_{k=1}^{m} \theta_k \Delta_{q-1}^{k-1}(\theta).$$

According to Cramer's rule,

$$\gamma_s(m) = \frac{M_s}{detU_{q-m-1}^m},\ s = 1, \cdots, m,$$

where $\{M_s\}$ are the corresponding minors. Consequently,

$$\Delta_q^m(\theta) = A_q(f) + \sum_{s=1}^{m} \gamma_s(m) A_{q-s}(f) =$$

$$= A_q(f) + \frac{1}{detU_{q-m-1}^m} \sum_{s=1}^{m} M_s A_{q-s}(f) = (-1)^m \frac{detU_{q-m}^{m+1}}{detU_{q-m-1}^m}.$$

□

Theorem 3.2. *Suppose $f \in C^q[-1,1]$, $q \geq 1$, $f^{(q+1)} \in L_1[-1,1]$, and $detU_{q-m-1}^m \neq 0$. Then, for θ from (22), the following holds:*

$$r_{N,q,m}(f) = (-1)^m \frac{detU_{q-m}^{m+1}}{detU_{q-m-1}^m} \times$$

$$\left(\sum_{|n|>N} \frac{(-1)^{n+1} e^{i\pi nx}}{2(i\pi n)^{q+1}} + \sum_{n=-N}^{N} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{(-1)^{n+s+1} e^{i\pi nx}}{2(i\pi)^{q+1}(n+s(2N+1))^{q+1}} \right) + o(N^{-q}),\ N \to \infty. \tag{28}$$

Proof. Using the relation (at least $P_n = O(n^{-2}), n \to \infty$)

$$\widehat{P}_n = \sum_{s=-\infty}^{\infty} P_{n+s(2N+1)},$$

we obtain

$$r_{N,q,m}(f) = \sum_{|n|>N} P_n e^{i\pi nx} - \sum_{n=-N}^{N} e^{i\pi nx} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} P_{n+s(2N+1)}.$$

Proceeding as in the proof of theorem 3.1, this concludes the proof. □

3.3

For a practical realization of the QP-method, we need the explicit form of the function $Q(x)$.

Consider the case $m+1 \leq q \leq 2m$ and $\theta_s \neq 0$, $\theta_s \neq \theta_k$, $s,k = 1,\cdots,m$. Given the relation

$$\frac{(i\pi n)^{m-k-1}}{\prod_{s=1}^{m}(i\pi n+\theta_s)} = \sum_{j=1}^{m} \frac{(-1)^{m-k-1}\theta_j^{m-k-1}}{(i\pi n+\theta_j)\prod_{\substack{s=1\\ s\neq j}}^{m}(\theta_s-\theta_j)},\ k=0,\cdots,q-m-1,$$

we expand Q_n into simple fractions

$$Q_n = \frac{(-1)^n}{2}\sum_{j=1}^{m}\frac{1}{(i\pi n+\theta_j)\prod_{\substack{s=1\\ s\neq j}}^{m}(\theta_s-\theta_j)}\sum_{k=0}^{q-m-1}\Delta_k^m(\theta)(-1)^{m-k}\theta_j^{m-k-1}. \tag{29}$$

According to the representation

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{i\pi n x}}{i\pi n+\theta_j} = \frac{1}{sh\theta_j}e^{-\theta_j x}$$

from (29), for the case $m+1 \leq q < 2m$, we derive

$$Q(x) = \sum_{j=1}^{m}\frac{e^{-\theta_j x}}{2\,sh(\theta_j)\prod_{\substack{s=1\\ s\neq j}}^{m}(\theta_s-\theta_j)}\sum_{k=0}^{q-m-1}\Delta_k^m(\theta)(-1)^{m-k}\theta_j^{m-k-1}. \tag{30}$$

If $q=2m$, then

$$Q(x) = \frac{\Delta_{m-1}^m(\theta)}{2\prod_{k=1}^{m}\theta_k}+\sum_{j=1}^{m}\frac{e^{-\theta_j x}}{2sh\,\theta_j\ \prod_{\substack{s=1\\ s\neq j}}^{m}(\theta_s-\theta_j)}\sum_{k=0}^{m-1}\Delta_k^m(\theta)(-1)^{m-k}\theta_j^{m-k-1}. \tag{31}$$

The explicit form of $Q(x)$ in other cases can be calculated similarly. In general, we have the following representation:

Lemma 3.3. [24]. *Let* $\{\alpha_s\}, s=1,...,, 1\leq \ <\infty$ *, be a finite set of complex numbers and* $\Upsilon \subseteq \{\alpha_s\}$ *a subset of integers. Then*

$$\sum_{\substack{k=-\infty\\ k\notin\Upsilon}}^{\infty}\frac{(-1)^{k+1}p(k)e^{i\pi k x}}{\prod_{s=1}(k-\alpha_s)^{\beta_s}} = \pi\sum_{r=1}\underset{z=\alpha_r}{Res}\,\frac{p(z)e^{i\pi z x^{\ddagger}}}{sin(\pi z)\prod_{s=1}(z-\alpha_s)^{\beta_s}},$$

where $\{\beta_s\}$, $s=1,...,$ *is a set of positive integers,* $p(z)$ *is a polynomial of degree less than* $\sum_{s=1}\beta_s - 1$*, and* $x^{\ddagger} = (x+1)\,(mod2)-1$, $-1 < x^{\ddagger} < 1$.

Now it follows that, in general, $Q(x)$ is a quasipolynomial of the form

$$Q(x) = \sum_k a_k x^{p_k} e^{i\omega_k x},$$

where $\omega_k \in \mathcal{C}$ and where $\{p_k\}$ is a set of nonnegative integers.

We have calculated the explicit form of $Q(x)$ for specific cases. Calculations are carried out using the MATHEMATICA software package [31]. For a given q and m, denote $Q_{q,m}(x) = Q(x)$. Some of them are

$$Q_{2,1}(x) = \frac{A_0}{2\theta_1} - \frac{A_0}{2\,sh\,\theta_1} e^{-\theta_1 x},$$

$$Q_{3,1}(x) = -\frac{A_1}{2\theta_1^2} + x\frac{A_1 + A_0\theta_1}{2\theta_1} + \frac{A_1}{2\,\theta_1\,sh\,\theta_1} e^{-\theta_1 x},$$

$$Q_{3,2}(x) = -\frac{A_0\theta_1}{2\,(\theta_1-\theta_2)\,sh\,\theta_1} e^{-\theta_1 x} + \frac{A_0\theta_2}{2\,(\theta_1-\theta_2)\,sh\,\theta_2} e^{-\theta_2 x},$$

$$Q_{4,1}(x) = -\frac{A_1\theta_1^3 + A_2(\theta_1^2-6)}{12\theta_1^3} + \frac{x}{2}\left(A_0 - \frac{A_2}{\theta_1^2}\right) + x^2\frac{A_2 + A_1\theta_1}{4\theta_1} - \frac{A_2}{2\,\theta_1^2\,sh\,\theta_1} e^{-\theta_1 x},$$

$$Q_{4,2}(x) = \frac{A_1 + A_0\theta_2}{2\,(\theta_1-\theta_2)\,sh\,\theta_1} e^{-\theta_1 x} - \frac{A_1 + A_0\theta_1}{2\,(\theta_1-\theta_2)\,sh\,\theta_2} e^{-\theta_2 x} + \frac{A_1 + A_0(\theta_1+\theta_2)}{2\theta_1\theta_2},$$

$$Q_{4,3}(x) = -\frac{A_0\theta_1^2}{2\,(\theta_1-\theta_2)(\theta_1-\theta_3)\,sh\,\theta_1} e^{-\theta_1 x} - \frac{A_0\theta_2^2}{2\,(\theta_2-\theta_1)(\theta_2-\theta_3)\,sh\,\theta_2} e^{-\theta_2 x} -$$
$$-\frac{A_0\theta_3^2}{2\,(\theta_3-\theta_1)(\theta_3-\theta_2)\,sh\,\theta_3} e^{-\theta_3 x}.$$

Similar calculations can be carried out for multiple θ. For example, when $\theta_1 = \theta_2 = \theta$, $q = 3, m = 2$, we have

$$Q_{3,2}(x) = \frac{A_0}{2sh\,\theta}(2\theta cth\,\theta - 1 + 2x\theta)e^{-\theta x}.$$

If $\theta_1 = \theta_2 = \theta_3 = \theta$, $q = 4, m = 3$, we derive

$$Q_{4,3}(x) = \frac{A_0 e^{-\theta x}}{4sh\,\theta}(2x\theta(5 - 3\theta cth\,\theta) - 3x^2\theta^2 - 6\theta^2 cth^2\,\theta + 3\theta^2 + 10\theta cth\,\theta - 2).$$

For $m = 1$, system (22) can easily be solved symbolically. In particular, the explicit forms of some of the quasipolynomials are derived to be

$$Q_{2,1}(x) = \frac{A_0}{2sh\frac{A_1}{A_0}} e^{\frac{A_1}{A_0}x} - \frac{A_0^2}{2A_1},$$

$$Q_{3,1}(x) = -\frac{A_1^3}{2A_2^2} + x\left(\frac{A_0}{2} - \frac{A_1^2}{2A_2}\right) + \frac{A_1^2}{2A_2 sh\frac{A_2}{A_1}} e^{\frac{A_2}{A_1}x},$$

$$Q_{4,1}(x) = -\frac{A_1}{12} - \frac{A_2^4}{2A_3^3} + \frac{A_2^2}{12A_3} + x\left(\frac{A_0}{2} - \frac{A_2^3}{2A_3^2}\right) + x^2\left(\frac{A_1}{4} - \frac{A_2^2}{4A_3}\right) + \frac{A_2^3 e^{\frac{A_3}{A_2}x}}{2A_3^2 sh\frac{A_3}{A_2}}.$$

4 Numerical results

For a given f, q, and m, we put

$$a_{q,m}(f) = \left| A_q \frac{det(U^m_{q-m-1})}{det(U^{m+1}_{q-m})} \right|. \tag{32}$$

The constant $a_{q,m}(f)$ describes the effectiveness of the QP-approximation (Theorem 3.1) compared to the KGE-approximation (Theorem 2.1) as well as the effectiveness of the QP-interpolation (Theorem 3.2) compared to the KGE-interpolation (Theorem 2.2) when $N >> 1$. Let us consider two typical examples. All calculations are carried out using MATHEMATICA software on a Pentium 4 computer.

First, consider the Bessel function

$$f(x) = J_0(14x - 1). \tag{33}$$

In Figure 1, the graphics of $a_{q,m}(f)$ for (33) are represented when $q = 8, 9, 10$ and $1 \leq m \leq q-1$. We observe that the QP-method is more precise than the KGE-method almost 250 times for $q = 8; m = 4$ and more than 300 times for $q = 10; m = 4, 6$. Figure 1 also shows the optimal values of m when parameter q is fixed.

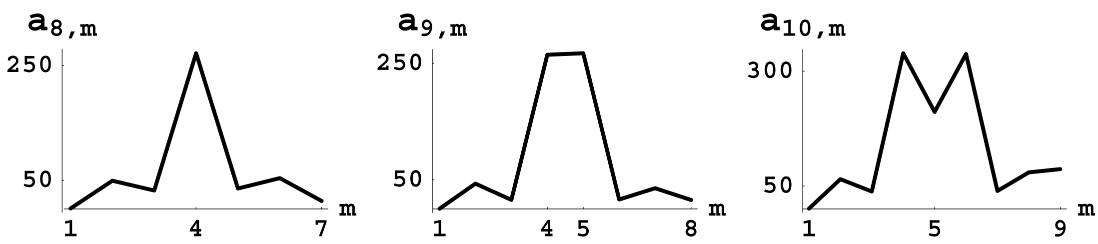

Fig. 1 Graphics of $a_{q,m}(f)$ for (33) when $q = 8, 9, 10$ and $1 \leq m \leq q - 1$.

Results in Figure 1 are asymptotic ($N \gg 1$). It is interesting to see the numerical behavior of the QP-method for both small and moderate values of N. We illustrate the results for just the QP-interpolation because our experiments show that, in general, the behavior of the QP-approximation (see [24] and [27] for details) is very similar to that of the QP-interpolation. The actual effectiveness (in a uniform metric) of the QP-interpolation compared to the KGE-interpolation can be represented by the ratio

$$a_{N,q,m}(f) = \frac{\max_{|x|\leq 1} |r_{N,q}(f)|}{\max_{|x|\leq 1} |r_{N,q,m}(f)|}. \tag{34}$$

In Table 1, approximate values of $a_{N,8,4}$ are shown for (33). Calculations are carried out with 64 digits of precision.

Comparison with the theoretical value $a_{8,4} = 271.1$ shows that experimental and theoretical estimates are rather close for $N \geq 32$. In Figure 2, the uniform errors are scaled logarithmically. Here, we compare the QP- and the KGE-interpolations for $q = 8$

Table 1 Approximate values of $a_{N,8,4}$ for different N while interpolating the function (33).

N	8	16	32	64	128	256	512
$a_{N,8,4}$	97	206	250	264	269	270	271

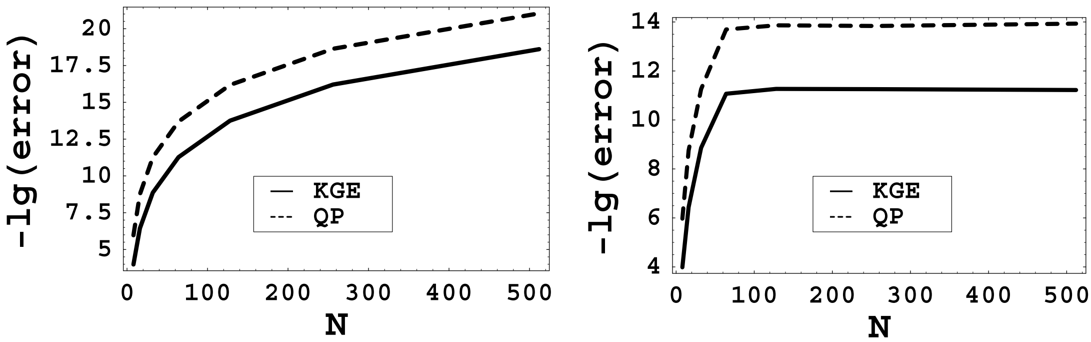

Fig. 2 Uniform errors in log scale, f defined by (33), $q = 8, m = 4$, $N \leq 512$. Left: with 64 digits of precision, Right: with standard precision.

and $m = 4$. The left figure corresponds to calculations with 64 digits of precision; the right figure we obtain from standard MATHEMATICA precision calculations.

We see that, even in standard machine precision, the QP-interpolation is much more precise than the KGE-interpolation. For $N \geq 100$, the QP-method is nearly 10^3 (compare this with the theoretically-predicted value of 271) times more precise than the KGE-method. Furthermore, the QP-method is less sensitive to round-off errors.

Now consider the second example

$$f(x) = \frac{1}{1.1 - x}. \tag{35}$$

This function has the greatest increase of A_k jumps within the class of analytic functions in the neighborhood of the interval $[-1, 1]$. In Figure 3, the graphics of $a_{q,m}$ are represented. Approximate values of $a_{N,8,4}$ are displayed in Table 2.

For this example, when $N \geq 256$, experimental results $a_{N,8,4}$ are close to the theoretical estimate $a_{8,4} = 72.9$.

Table 2 Approximate values of $a_{N,8,4}$ for different N while interpolating the function (35).

N	8	16	32	64	128	256	512	1024
$a_{N,8,4}$	3414	433	237	62	44	62	68	71

In Figure 4, the logarithms of the uniform errors are represented. The left figure corresponds to calculations with 64 digits of precision; the right figure is obtained from stan-

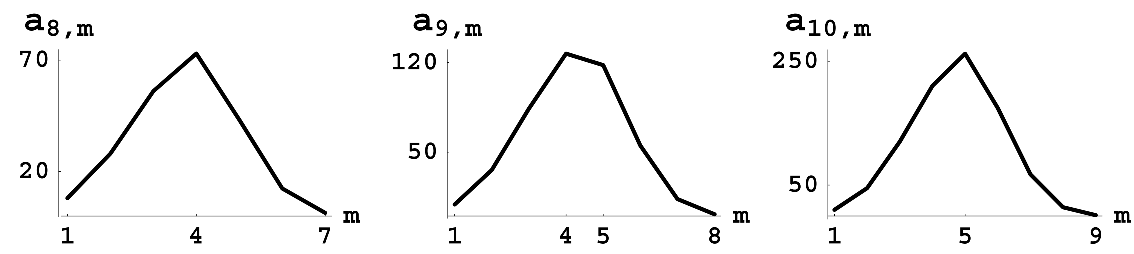

Fig. 3 The graphs of $a_{q,m}$ for fixed values of q ($q = 8, 9, 10$) and $1 \leq m \leq q$ when (35) is approximated.

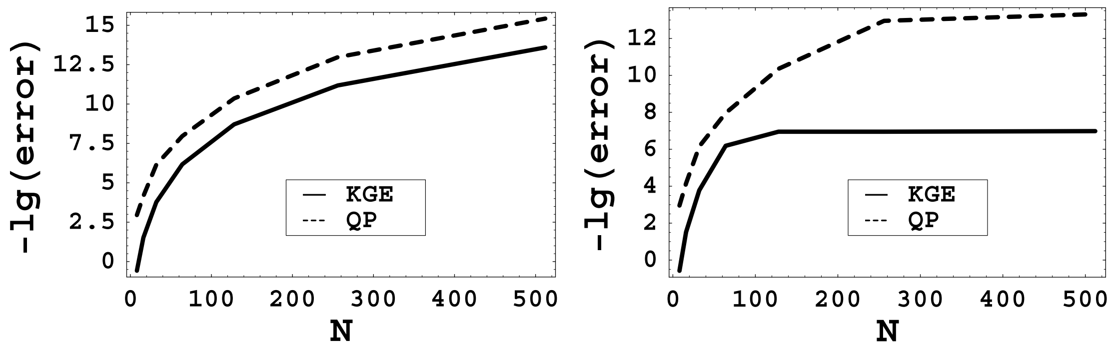

Fig. 4 Uniform errors in log scale, f defined by (35), $q = 8, m = 4$, $N \leq 512$. Left: with 64 digits of precision, Right: with standard precision.

dard precision calculations. With standard precision and $N \geq 200$ the QP-interpolation is 10^5 times more precise than the KGE-interpolation.

For practical application of the QP-method, the numerical values of jumps $A_k(f)$ are also needed. These values can be recovered from Fourier coefficients or from discrete Fourier coefficients as shown in [5–8]. Numerical experiments show [24] that the application of this procedure to the QP-method is acceptable and, in general, maintains all characteristic features.

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