Analysis and Applications, Vol. 7, No. 2 (2009) 199–211

ASYMPTOTIC BEHAVIOR OF THE KRYLOV–LANCZOS INTERPOLATION

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Received 23 November 2007
Revised 25 September 2008

In this paper the asymptotic behavior of the Krylov–Lanczos interpolation is investigated. Exact asymptotic constants of the errors are obtained on the subintervals where the 2-periodic extension of the approximated function is smooth. In particular, the fast convergence of this method is observed and examined in a special case. Some numerical results are presented.

Keywords: Trigonometric interpolation; Bernoulli polynomials; convergence acceleration; Krylov–Lanczos approximants.

Mathematics Subject Classification 2000: 42A15, 65T40

1. Introduction

It is well known that reconstruction of a smooth on $[-1,1]$ function by truncated Fourier series

$$S_N(f) = \sum_{n=-N}^{N} f_n e^{i\pi nx}, \quad f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}dx \tag{1.1}$$

or by trigonometric interpolation

$$I_N(f) = \sum_{n=-N}^{N} \check{f}_n e^{i\pi nx}, \quad \check{f}_n = \frac{1}{2N+1}\sum_{k=-N}^{N} f(x_k)e^{-i\pi nx_k}, \quad x_k = \frac{2k}{2N+1} \tag{1.2}$$

is noneffective when the 2-periodic extension of the approximated function is discontinuous. The oscillations caused by the Gibbs phenomenon are typically propagated into regions away from the singularities and degrade the quality of approximations. There is ample literature devoted to overcoming this problem; for example, consult $[4, 5, 11–15]$, and references therein.

An efficient approach of convergence acceleration of Fourier partial sum (1.1), by subtracting a polynomial representing the discontinuities in the function and some of its first derivatives ("jumps"), was suggested in 1906 by Krylov [18] and later in 1964 by Lanczos [19, 20]. For further development of this approach see, for example, [2, 3, 21–24, 28]. Inspired by work of Gottlieb *et al.* [1, 6, 16] on problems with discontinuous solutions, Eckhoff *et al.* [7–10] developed a new way to calculate the polynomial terms in the representation suggested by Krylov and Lanczos. In particular, Eckhoff generalizes Krylov–Lanczos representation to piecewise smooth functions and derives a system of linear equations for calculation of "jumps" (see (2.1)).

An interpolation counterpart of the Krylov–Lanczos approach is investigated in [9, 10, 17, 25, 26]. In this paper exploration of Krylov–Lanczos interpolation is continued with restriction to the case when the approximated function is smooth on the interval $[-1, 1]$ with singularities only at the endpoints. In addition, it is assumed that the exact "jumps" are known.

This paper is organized as follows: In Sec. 2, Krylov–Lanczos interpolation is briefly described and the main notations are introduced. In Sec. 3 some preliminary Lemmas are proved. In Sec. 4 the main results are stated; see Theorems 4.2–4.5, where the convergence of Krylov–Lanczos interpolation is investigated on the intervals $[-\varepsilon, \varepsilon]$, $0 < \varepsilon < 1$. In particular, the fast convergence of Krylov–Lanczos interpolation is observed for odd values of q (see (2.4)).

2. Krylov–Lanczos Interpolation

For $f \in C^q[-1, 1]$ denote

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \quad k = 0, \ldots, q. \tag{2.1}$$

The following lemma is crucial for the Krylov–Lanczos approach.

Lemma 2.1 [6]. *Let $f \in C^{q-1}[-1, 1]$ and $f^{(q-1)}$ be absolutely continuous on $[-1, 1]$ for some $q \geq 1$; then, the following expansion is valid for the Fourier coefficients f_n*

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{m=0}^{q-1} \frac{A_m(f)}{(i\pi n)^{m+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^{1} f^{(q)}(x) e^{-i\pi n x} dx, \quad n \neq 0.$$

Proof. The proof can be derived easily from (1.1) by sequential integration by parts. □

The so-called Lanczos representation is based on this lemma

$$f(x) = \sum_{m=0}^{q-1} A_m(f) B_m(x) + F(x), \tag{2.2}$$

where B_m are 2-periodic Bernoulli polynomials with Fourier coefficients

$$B_n(m) = \begin{cases} 0, & n = 0 \\ \dfrac{(-1)^{n+1}}{2(i\pi n)^{m+1}}, & n = \pm 1, \pm 2, \ldots \end{cases} \tag{2.3}$$

and F is a 2-periodic relatively smooth on real line function. Approximation of F in (2.2) by trigonometric interpolation (1.2) leads to Krylov–Lanczos (KL) interpolation

$$I_{N,q}(f) = \sum_{n=-N}^{N} \check{F}_n e^{i\pi nx} + \sum_{m=0}^{q-1} A_m(f)B_m(x), \tag{2.4}$$

where discrete Fourier coefficients $\check{F}_n$ can be calculated from (2.2)

$$\check{F}_n = \check{f}_n - \sum_{m=0}^{q-1} A_m(f)\check{B}_n(m). \tag{2.5}$$

From (2.3) we conclude that $B_0(x) = x/2$, $B_1(x) = -1/12 + x^2/4$ and so on. Hence, discrete Fourier coefficients $\check{B}_n(m)$ have explicit form. For example, here are three of them

$$\check{B}_n(0) = \frac{(-1)^n i}{2(2N+1)\sin\dfrac{\pi n}{2N+1}}, \qquad n \neq 0, \quad \check{B}_0(0) = 0,$$

$$\check{B}_n(1) = \frac{(-1)^n \cos\dfrac{\pi n}{2N+1}}{2(2N+1)^2 \sin^2\dfrac{\pi n}{2N+1}}, \qquad n \neq 0, \quad \check{B}_0(1) = -\frac{1}{12(2N+1)^2},$$

$$\check{B}_n(2) = \frac{(-1)^{n+1} i\left(3 + \cos\dfrac{2\pi n}{2N+1}\right)}{8(2N+1)^3 \sin^3\dfrac{\pi n}{2N+1}}, \quad n \neq 0, \quad \check{B}_0(2) = 0.$$

Use $\|f\|_\varepsilon$ to denote the norm of the space $L_2(-\varepsilon, \varepsilon)$, $0 < \varepsilon \leq 1$, namely

$$\|f\|_\varepsilon = \left(\int_{-\varepsilon}^{\varepsilon} |f(x)|^2 dx\right)^{1/2}.$$

Denote also

$$R_{N,q}(f) = f(x) - I_{N,q}(f).$$

3. Preliminaries

In this section we prove some lemmas required in the next section. Propositions (3.1) and (3.2) in the next Lemma are proved in [27]. Since Proposition (3.3) is a new one, we are repeating the proof from [27].

Lemma 3.1. *Let*

$$\omega_{p,m} = \sum_{s=0}^{p} \binom{p}{s} (-1)^s s^m, \quad m \geq 0;$$

then

$$\omega_{p,m} = 0, \quad 0 \le m < p, \tag{3.1}$$

$$\omega_{p,p} = (-1)^p p!, \tag{3.2}$$

$$\omega_{p,p+1} = (-1)^p \frac{p(p+1)!}{2}. \tag{3.3}$$

Proof. Denote

$$\varphi_{p,0}(z) = (1+z)^p = \sum_{s=0}^{p} \binom{p}{s} z^s,$$

$$\varphi_{p,m}(z) = z\,\varphi'_{p,m-1}(z), \quad m \ge 1$$

and note that

$$\varphi_{p,m}(-1) = \omega_{p,m}.$$

The remaining is obvious. □

Denote

$$\Delta_n^p(f_n) = \sum_{k=0}^{2p} \binom{2p}{k} f_{n+p-k}, \quad p \ge 0.$$

Lemma 3.2. *The following estimate holds for $p \ge 0$ and $m \ge 0$*

$$\Delta_n^p(B_n(m)) = \frac{(-1)^{n+p+1}(m+2p)!}{2(i\pi n)^{m+1} n^{2p} m!} + O(n^{-m-2p-2}), \quad n \to \infty. \tag{3.4}$$

Proof. According to the definition of $\Delta_n^p(B_n(m))$, we derive

$$\begin{aligned}
\Delta_n^p(B_n(m)) &= \sum_{k=0}^{2p} \binom{2p}{k} B_{n+p-k}(m) \\
&= \frac{(-1)^{p+n+1}}{2(i\pi n)^{m+1}} \sum_{k=0}^{2p} \binom{2p}{k} \frac{(-1)^k}{\left(1 + \dfrac{p-k}{n}\right)^{m+1}} \\
&= \frac{(-1)^{p+n+1}}{2(i\pi n)^{m+1}} \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \sum_{s=m}^{\infty} \binom{s}{m} \frac{(-1)^{s-m}}{n^{s-m}} (p-k)^{s-m} \\
&= \frac{(-1)^{p+n+1}}{2(i\pi n)^{m+1}} \sum_{s=0}^{\infty} \binom{m+s}{m} \frac{(-1)^s}{n^s} \sum_{j=0}^{s} \binom{s}{j} (-1)^j p^{s-j} \omega_{2p,j}.
\end{aligned}$$

This concludes the proof according to Lemma 3.1. □

Lemma 3.3. *The following estimate holds for $p \geq 0$ and $m \geq 0$*

$$\Delta_n^p(\check{B}_n(m) - B_n(m)) = \frac{(-1)^{n+p+1}(m+2p)!}{2(i\pi N)^{m+1}N^{2p}m!}\sum_{s\neq 0}\frac{(-1)^s}{\left(2s+\frac{n}{N}\right)^{m+2p+1}}$$
$$+ O(N^{-m-2p-2}), \quad |n| \leq N, \quad N \to \infty. \tag{3.5}$$

Proof. Taking into account that for $m \geq 0$

$$\check{B}_n(m) = \sum_{s=-\infty}^{\infty} B_{n+s(2N+1)}(m) = B_n(m) + \sum_{s\neq 0} B_{n+s(2N+1)}(m), \tag{3.6}$$

we derive for $p \geq 0$ and $|n| \leq N$

$$\Delta_n^p(\check{B}_n(m) - B_n(m))$$
$$= \frac{(-1)^{n+p+1}}{2(i\pi N)^{m+1}}\sum_{k=0}^{p}\binom{2p}{k}(-1)^k\sum_{s\neq 0}\frac{(-1)^s}{\left(2s+\frac{n}{N}\right)^{m+1}}\frac{1}{\left(1+\frac{p-k+s}{N\left(2s+\frac{n}{N}\right)}\right)^{m+1}}$$
$$= \frac{(-1)^{n+p+1}}{2(i\pi N)^{m+1}}\sum_{t=0}^{\infty}\binom{m+t}{m}\frac{(-1)^t}{N^t}\sum_{j=0}^{t}\binom{t}{j}(-1)^j\omega_{2p,j}\sum_{s\neq 0}\frac{(-1)^s(p+s)^{t-j}}{\left(2s+\frac{n}{N}\right)^{m+t+1}}.$$

Now (3.5) follows from Lemma 3.1. ☐

Lemma 3.4. *Let $m \geq 0$ be an even number*; *then*, *the following estimate holds*

$$\Delta_{\pm N}^p(\check{B}_n(m)) = \pm\frac{(-1)^{N+p+1}(m+2p)!}{2(i\pi N)^{m+1}N^{2p}m!}\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2p+m+1}}$$
$$+ O(N^{-m-2p-2}), \quad N \to \infty, \quad p \geq 0. \tag{3.7}$$

Proof. Taking into account (3.6), we derive

$$\Delta_{\pm N}^p(\check{B}_n(m))$$
$$= \pm\frac{(-1)^{N+p+1}}{2(i\pi N)^{m+1}}\sum_{k=0}^{2p}\binom{2p}{k}(-1)^k\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{m+1}\left(1+\frac{s+p-k}{N(2s+1)}\right)^{m+1}}$$
$$= \pm\frac{(-1)^{N+p+1}}{2(i\pi N)^{m+1}}\sum_{t=0}^{\infty}\binom{m+t}{m}\frac{(-1)^t}{N^t}\sum_{j=0}^{t}\binom{t}{j}(-1)^j\omega_{2p,j}\sum_{s=-\infty}^{\infty}\frac{(-1)^s(s+p)^{t-j}}{(2s+1)^{t+m+1}}.$$

This concludes the proof. ☐

Lemma 3.5. *Let $m \geq 1$ be an odd number; then, the following estimate holds*

$$\Delta^p_{\pm N}(\check{B}_n(m)) = \frac{(-1)^{N+p}(m+2p+1)!}{2(i\pi N)^{m+1}N^{2p+1}m!}\sum_{s=-\infty}^{\infty}\frac{(-1)^s s}{(2s+1)^{2p+m+2}} + O(N^{-m-2p-3}), \quad N\to\infty, \quad p\geq 0. \tag{3.8}$$

Proof. We proceed as in the proof of Lemma 3.4 and derive

$$\Delta^p_{\pm N}(\check{B}_n(m)) = \pm\frac{(-1)^{N+p+1}(m+2p)!}{2(i\pi N)^{m+1}\,m!\,N^{2p}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2p+m+1}} + \frac{(-1)^{N+p}(m+2p+1)!}{2(i\pi N)^{m+1}m!\,N^{2p+1}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s s}{(2s+1)^{2p+m+2}} + O(N^{-2p-m-3}), \quad N\to\infty.$$

From the fact that

$$\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2m}} = 0, \quad m = 1, 2, \dots$$

we get (3.8). □

4. Asymptotics of KL-Interpolation

The next theorem reveals the asymptotic behavior of KL-interpolation on the whole interval of approximation. The proof can be found also in [26] in a more general context.

Theorem 4.1. *Let $f \in C^q[-1,1]$ and $f^{(q)}$ be absolutely continuous on $[-1,1]$ for some $q \geq 1$; then, the following estimate holds*

$$\lim_{N\to\infty} N^{q+\frac{1}{2}}\|R_{N,q}(f)\|_1 = |A_q(f)|a(q), \tag{4.1}$$

where

$$a(q) = \frac{1}{\sqrt{2}\,\pi^{q+1}}\left(\frac{2}{2q+1} + \int_{-1}^{1}\left|\sum_{s\neq 0}\frac{(-1)^s}{(2s+x)^{q+1}}\right|^2 dx\right)^{1/2}.$$

Proof. From Lanczos representation (2.2) we get

$$R_{N,q}(f) = \sum_{n=-N}^{N}(F_n - \check{F}_n)e^{i\pi nx} + \sum_{|n|>N}F_n e^{i\pi nx}.$$

Therefore

$$\|R_{N,q}(f)\|_1 = \sqrt{2}\left(\sum_{n=-N}^{N}|F_n - \check{F}_n|^2 + \sum_{|n|>N}|F_n|^2\right)^{1/2}. \tag{4.2}$$

According to the smoothness of f, from Lemma 2.1 and representation (2.2), we derive

$$F_n = A_q(f)\frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} + o(n^{-q-1}), \quad n \to \infty. \tag{4.3}$$

Taking into account that

$$\check{F}_n = F_n + \sum_{s\neq 0} F_{n+s(2N+1)} \tag{4.4}$$

from (4.3) and Lemma 3.3 we get

$$F_n - \check{F}_n = A_q(f)\frac{(-1)^N}{2(i\pi N)^{q+1}} \sum_{s\neq 0} \frac{(-1)^s}{\left(2s + \frac{n}{N}\right)} + o(N^{-q-1}), \quad N \to \infty. \tag{4.5}$$

Substituting (4.3) and (4.5) into (4.2), letting N tend to infinity and replacing the Riemann sums by corresponding integrals, we derive (4.1). □

In Table 1 we show the numerical values of the constant $a(q)$ for different values of q. It is interesting to see also the values of the ratio $a(q)/a(q+1)$. The corresponding results are presented in Table 2. It is easy to check that $a(q)/a(q+1) \to \pi$ as $q \to \infty$.

Denote (for $A_q(f) \neq 0$)

$$\tilde{a}_N(q,f) = \frac{N^{q+1/2}}{A_q(f)} \|R_{N,q}(f)\|_1.$$

According to Theorem 4.1 $\tilde{a}_N(q,f) \to a(q)$ as $N \to \infty$.

Now consider the following simple example

$$f(x) = \sin(x-1). \tag{4.6}$$

In Table 3 the values of $\tilde{a}_N(q,f)$ are presented when $N = 32$ and (4.6) is approximated. We see that these values are rather close to the theoretical values from Table 1.

Table 1. Numerical values of the constant $a(q)$ for different values of q.

q	1	2	3	4	5	6	7
$a(q)$	0.08389	0.01899	0.00554	0.00152	$4.4 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$	$4 \cdot 10^{-5}$
q	8	9	10	11	12	13	14
$a(q)$	$1 \cdot 10^{-5}$	$4 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	$3 \cdot 10^{-7}$	$1 \cdot 10^{-7}$	$3 \cdot 10^{-8}$	$1 \cdot 10^{-8}$

Table 2. Numerical values of the ratio $a(q)/a(q+1)$.

q	1	2	3	4	10	15	20
$a(q)/a(q+1)$	4.42	3.43	3.64	3.42	3.29	3.24	3.22

Table 3. Numerical values of the constant $\tilde{a}_N(q,f)$ for different values of q and $N = 32$ when (4.6) is approximated.

q	1	2	3	4	5	6	7
$\tilde{a}_N(q,f)$	0.08195	0.01826	0.00525	0.00142	0.00041	0.00012	0.00004

In the next four theorems the asymptotic behavior of KL-interpolation is investigated on the interval $[-\varepsilon, \varepsilon]$ when $0 < \varepsilon < 1$. First, even values of q are considered.

Theorem 4.2. *Let $q \geq 1$ be an even number, $f \in C^{q+2}[-1,1]$ and $f^{(q+2)}$ be absolutely continuous on $[-1,1]$; then, the following estimate holds as $N \to \infty$ and $|x| < 1$ is fixed*

$$R_{N,q}(f) = A_q(f)\frac{(-1)^{N+\frac{q}{2}}}{2(\pi N)^{q+1}} \frac{\sin \dfrac{\pi x}{2}(2N+1)}{\cos \dfrac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+1}} + O(N^{-q-2}). \quad (4.7)$$

Proof. The following transformation can be easily checked for $|x| < 1$

$$\begin{aligned} R_{N,q}(f) &= \check{F}_N \frac{e^{-i\pi N x} - e^{i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})} + \check{F}_{-N} \frac{e^{i\pi N x} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})} \\ &\quad + \frac{1}{(1+e^{i\pi x})(1+e^{-i\pi x})} \sum_{n=-N}^{N} \Delta_n^1(F_n - \check{F}_n) e^{i\pi n x} \\ &\quad + \frac{1}{(1+e^{i\pi x})(1+e^{-i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^1(F_n) e^{i\pi n x} \\ &\quad + \frac{1}{(1+e^{i\pi x})(1+e^{-i\pi x})} \sum_{n=-\infty}^{-N-1} \Delta_n^1(F_n) e^{i\pi n x}. \end{aligned} \quad (4.8)$$

Taking into account the smoothness of f and representation (2.2), we write

$$F_n = \sum_{m=q}^{q+2} A_m(f) B_n(m) + o(n^{-q-3}), \quad n \to \infty.$$

From the definition of $\Delta_n^1(F_n)$ the following is derived

$$\Delta_n^1(F_n) = \sum_{m=q}^{q+2} A_m(f) \Delta_n^1(B_n(m)) + o(n^{-q-3}), \quad n \to \infty.$$

Now, according to Lemma 3.2, we obtain

$$\Delta_n^1(F_n) = O(n^{-q-3}), \quad n \to \infty.$$

Hence, the last two terms in (4.8) are of the order $O(N^{-q-2})$ as $N \to \infty$.

In the same way, we get

$$\Delta_n^1(\check{F}_n - F_n) = \sum_{m=q}^{q+2} A_m(f)\Delta_n^1(\check{B}_n(m) - B_n(m)) + o(N^{-q-3}), \quad |n| \le N, \quad N \to \infty.$$

According to Lemma 3.3

$$\Delta_n^1(\check{F}_n - F_n) = O(N^{-q-3}), \quad |n| \le N, \quad N \to \infty.$$

From here we conclude that the third term in (4.8) is also of the order $O(N^{-q-2})$ as $N \to \infty$.

Finally, we get

$$R_{N,q}(f) = \check{F}_N \frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})} + \check{F}_{-N} \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})} + O(N^{-q-2}), \quad N \to \infty. \tag{4.9}$$

From Lemmas 3.4 and 3.5, when $p = 0$, we have

$$\check{F}_N = A_q(f)\frac{(-1)^{N+1}}{2(i\pi N)^{q+1}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+1}} + O(N^{-q-2}), \quad N \to \infty$$

and

$$\check{F}_{-N} = -\check{F}_N.$$

Substituting these last two estimates into (4.9), we derive (4.7). $\square$

From (4.7) the L_2-norm of the error can be derived on the interval $[-\varepsilon, \varepsilon]$ when $0 < \varepsilon < 1$.

Theorem 4.3. *Let $q \ge 1$ be an even number, $f \in C^{q+2}[-1,1]$ and $f^{(q+2)}$ be absolutely continuous on $[-1,1]$; then, the following estimate holds for $0 < \varepsilon < 1$*

$$\lim_{N\to\infty} N^{q+1}\|R_{N,q}(f)\|_\varepsilon = |A_q(f)| b(q,\varepsilon), \tag{4.10}$$

where

$$b(q,\varepsilon) = \frac{1}{\sqrt{2}\pi^{q+3/2}} \left| \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+1}} \right| tg^{1/2}\frac{\pi\varepsilon}{2}.$$

Proof. The proof follows immediately from (4.7). $\square$

In Table 4, the numerical values of $b(q,\varepsilon)$ are presented for $\varepsilon = 0.7$.

Now we carry out similar investigations for odd values of q.

Table 4. Numerical values of $b(q,\varepsilon)$ for $\varepsilon = 0.7$.

q	2	4	6	8	10	12
$b(q,\varepsilon)$	0.03493	0.00364	0.00037	0.00004	$3.8 \cdot 10^{-6}$	$3.8 \cdot 10^{-7}$

Theorem 4.4. *Let $q \geq 1$ be an odd number, $f \in C^{q+3}[-1,1]$ and $f^{(q+3)}$ be absolutely continuous on $[-1,1]$; then, the following estimate holds as $N \to \infty$ and $|x| < 1$ is fixed*

$$R_{N,q}(f) = \frac{(-1)^{N+\frac{q+1}{2}}}{2\pi^{q+1}N^{q+2}} \frac{\sin\dfrac{\pi x}{2}(2N+1)}{\cos\dfrac{\pi x}{2}}$$

$$\times \left(A_q(f)(q+1)tg\frac{\pi x}{2} \sum_{s=-\infty}^{\infty} \frac{(-1)^s s}{(2s+1)^{q+2}} \right.$$

$$\left. + \frac{A_{q+1}(f)}{\pi} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+2}} \right) + O(N^{-q-3}). \tag{4.11}$$

Proof. Reiteration of transformation (4.8) leads to the similar expansion of the error when $|x| < 1$

$$R_{N,q}(f) = \check{F}_N \frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})} + \check{F}_{-N} \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})}$$

$$+ \Delta_N^1(\check{F}_n) \frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{(1+e^{i\pi x})^2(1+e^{-i\pi x})^2} + \Delta_{-N}^1(\check{F}_n) \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})^2(1+e^{-i\pi x})^2}$$

$$+ \frac{1}{(1+e^{i\pi x})^2(1+e^{-i\pi x})^2} \sum_{n=-N}^{N} \Delta_n^2(F_n - \check{F}_n) e^{i\pi nx}$$

$$+ \frac{1}{(1+e^{i\pi x})^2(1+e^{-i\pi x})^2} \sum_{n=N+1}^{\infty} \Delta_n^2(F_n) e^{i\pi nx}$$

$$+ \frac{1}{(1+e^{i\pi x})^2(1+e^{-i\pi x})^2} \sum_{n=-\infty}^{-N-1} \Delta_n^2(F_n) e^{i\pi nx}. \tag{4.12}$$

In view of the smoothness of f the Fourier coefficients F_n have the following asymptotic expansion

$$F_n = \sum_{m=q}^{q+3} A_m(f) B_n(m) + o(n^{-q-4}), \quad n \to \infty.$$

Hence

$$\Delta_n^2(F_n) = \sum_{m=q}^{q+3} A_m(f)\Delta_n^2(B_n(m)) + o(n^{-q-4}), \quad n \to \infty.$$

Now, from Lemma 3.2, we get

$$\Delta_n^2(F_n) = o(N^{-q-4}), \quad N \to \infty.$$

Therefore, the last two terms in (4.12) are of the order $o(N^{-q-3})$ as $N \to \infty$. According to Lemma 3.3 the same estimate is valid also for the fifth term. The third and the fourth terms are of the order $O(N^{-q-3})$ as $N \to \infty$ according to Lemmas 3.4 and 3.5, when $p = 1$. Finally, we get

$$\begin{aligned} R_{N,q}(f) &= \check{F}_N \frac{e^{-i\pi Nx} - e^{i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})} \\ &\quad + \check{F}_{-N} \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})} + O(N^{-q-3}), \quad N \to \infty. \end{aligned} \tag{4.13}$$

From Lemmas 3.4 and 3.5, when $p = 0$, we derive

$$\begin{aligned} \check{F}_{\pm N} &= A_q(f)\frac{(-1)^N(q+1)}{2(i\pi N)^{q+1}N} \sum_{s=-\infty}^{\infty} \frac{(-1)^s s}{(2s+1)^{q+2}} \\ &\quad \pm A_{q+1}(f)\frac{(-1)^{N+1}}{2(i\pi N)^{q+2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+2}} + O(N^{-q-3}), \quad N \to \infty \end{aligned}$$

Substituting this last estimate into (4.13), we obtain (4.11). □

Theorem 4.5. *Let $q \geq 1$ be an odd number, $f \in C^{q+3}[-1,1]$ and $f^{(q+3)}$ be absolutely continuous on $[-1,1]$; then, the following estimate holds for $0 < \varepsilon < 1$*

$$\begin{aligned} &\lim_{N\to\infty} N^{q+2}\|R_{N,q}(f)\|_\varepsilon \\ &\quad = \left(\frac{|A_q(f)|^2(q+1)^2}{6\pi^{2q+3}} \left(\sum_{s=-\infty}^{\infty} \frac{(-1)^s s}{(2s+1)^{q+2}} \right)^2 tg^3\frac{\pi\varepsilon}{2} \right. \\ &\qquad \left. + \frac{|A_{q+1}(f)|^2}{2\pi^{2q+5}} \left(\sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+2}} \right)^2 tg\frac{\pi\varepsilon}{2} \right)^{\frac{1}{2}}. \end{aligned} \tag{4.14}$$

Proof. The proof follows immediately from (4.11). □

From Theorem 4.1 we conclude that it makes no difference whether odd or even values are used for KL-interpolation if asymptotic behavior is estimated on the whole interval of approximation. Hence, the larger the value of the parameter q the more precise is the corresponding interpolation. Note only that larger values of parameter q lead to round off errors and usually it is recommended to use values $q \leq 10$. Meanwhile, if we are interested in precision on the subintervals where the approximated function is smooth, then according to the last four Theorems the odd values are preferable.

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