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On an auto-correction phenomenon of the Krylov–Gottlieb–Eckhoff method

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We consider the so-called Krylov–Gottlieb–Eckhoff (KGE) approximation of a function f with a discontinuity at a known point. This approximation is based on certain corrections associated with the jumps in the first q derivatives of f. The approximation of the exact jumps is accomplished by the solution of a system of linear equations. We show that, in the regions where the period-2 extension of the approximated function is smooth, the KGE method with approximate values of the jumps converges faster compared with the case where the exact values are used. We call this accelerated convergence the auto-correction phenomenon, which was discovered in the past by numerical experiments. The paper presents a theoretical explanation of the phenomenon with numerical illustrations.

Keywords: Fourier coefficients; Bernoulli polynomials; convergence acceleration; auto-correction phenomenon.

1. Introduction

We consider the problem of approximating a function using a finite number of its Fourier coefficients

$$f_n = \frac{1}{2}\int_{-1}^{1} f(x)\mathrm{e}^{-\mathrm{i}\pi nx}\,\mathrm{d}x, \quad |n| \leqslant N < \infty. \tag{1.1}$$

It is well known that the approximation of a period-2 and smooth function f on the real line by the truncated Fourier series

$$S_N(f) = \sum_{n=-N}^{N} f_n\,\mathrm{e}^{\mathrm{i}\pi nx}$$

is highly effective. When the approximated function has a point of discontinuity, the approximation by the partial sum $S_N(f)$ leads to the Gibbs phenomenon.

Different methods of convergence acceleration have been suggested in the literature. An efficient approach that involved a polynomial representing the discontinuities in the function and some of its first derivatives (jumps) was suggested in 1906 by Krylov (1907) and later in 1964 by Lanczos (1964, 1966) (see also Jones & Hardy, 1970; Lyness, 1974; Lax, 1978; Gottlieb *et al.*, 1981; Cai *et al.*, 1989; Baszenski *et al.*, 1995; Nersessian & Poghosyan, 2000b, 2004; Barkhudaryan *et al.*, 2007; Adcock, 2009; Poghosyan, 2009 and references therein). Hereafter we refer to this approach as the Krylov–Gottlieb–Eckhoff (KGE) method. This method was developed for the applications by Eckhoff (1993,

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1995, 1998) in a series of papers, where the approximate values of the jumps were determined by the solution of a system of linear equations.

Here two different realizations of the KGE approximation, namely, $S_{N,q}(f)$ and $\widetilde{S}_{N,q}(f)$, are discussed. The approximation $S_{N,q}$ uses the exact values of the jumps, while $\widetilde{S}_{N,q}$ uses the approximations of the actual jumps. Throughout the paper we limit our discussion to the smooth function f on $[-1, 1]$. Hence the period-2 extension of f may have discontinuities only at the points $x = 2s + 1$, where $s = 0, \pm 1, \ldots$. We are interested in the asymptotic behaviour of these approximations when $|x| < 1$ (away from the discontinuities).

The approximation $S_{N,q}(f)$ is considered in Section 2 with the main results presented in Subsection 2.2. Here Theorems 2.4 and 2.5 state that, on the interval $|x| < 1$, the rate of convergence of $S_{N,q}(f)$ is $\mathcal{O}(N^{-q-1})$ as $N \to \infty$. In Section 4 the asymptotic behaviour of $\widetilde{S}_{N,q}(f)$ on the interval $|x| < 1$ is investigated and the main results are proved in Subsection 4.2. In particular, Theorems 4.5 and 4.6 consider even values of q ($q = 2m$, where $m = 1, 2, \ldots$) and state that the rate of convergence is $\mathcal{O}(N^{-3m-1})$ as $N \to \infty$. We see that, in comparison with $S_{N,q}(f)$, where the exact values of the jumps are used, we have an improvement in convergence by the factor $\mathcal{O}(N^m)$, where $m = 1, 2, \ldots$. We call this convergence acceleration phenomenon, which is contrary to the slow convergence that might be expected due to the approximate calculation of the jumps, the auto-correction phenomenon of the KGE method. It was first introduced and investigated in Nersessian & Poghosyan (2000a) for the discrete analogues of the approximations $S_{N,q}(f)$ and $\widetilde{S}_{N,q}(f)$. Theorems 4.8 and 4.9 reveal this phenomenon for odd values of q ($q = 2m+1$, where $m = 0, 1, \ldots$). In this case the rate of convergence is $\mathcal{O}(N^{-3m-2})$ as $N \to \infty$. We have an improvement in convergence by the factor $\mathcal{O}(N^m)$, where $m = 0, 1, \ldots$. Note that, for $q = 1$, the auto-correction phenomenon is absent. In Section 5 some numerical demonstrations of this phenomenon are presented.

2. The KGE method with exact values of the jumps

Suppose that $f \in C^q[-1, 1]$. Thus the period-2 extension of f may have singularities only at the points $x = 2s + 1$, where $s = 0, \pm 1, \ldots$. Denote by $A_k(f)$ the exact value of the jump in the kth derivative of f:

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \quad k = 0, \ldots, q.$$

In this section we suppose that, together with $2N + 1$ Fourier coefficients $\{f_n\}_{n=-N}^{N}$, the exact values of the jumps $\{A_k(f)\}_{k=0}^{q-1}$ are also known. In Section 3 we will discuss the methods of approximation of the actual jumps.

2.1 *The accuracy up to the discontinuity*

The basic idea of the KGE method is the representation of the approximated function

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f) B_k(x), \tag{2.1}$$

where B_k are the period-2 extensions of the Bernoulli polynomials with the Fourier coefficients

$$B_{k,n} = \begin{cases} 0, & n = 0, \\ \dfrac{(-1)^{n+1}}{2(\mathrm{i}\pi n)^{k+1}}, & n = \pm 1, \pm 2, \ldots, \end{cases}$$

and F is a period-2 and relatively smooth function on the real line with the Fourier coefficients

$$F_n = f_n - \sum_{k=0}^{q-1} A_k(f) B_{k,n}, \quad |n| \leqslant N < \infty. \tag{2.2}$$

Approximation of F in (2.1) by the truncated Fourier series leads to the approximation

$$S_{N,q}(f) = \sum_{n=-N}^{N} F_n \, \mathrm{e}^{\mathrm{i}\pi n x} + \sum_{k=0}^{q-1} A_k(f) B_k(x) \tag{2.3}$$

with the error

$$R_{N,q}(f) = f(x) - S_{N,q}(f). \tag{2.4}$$

We call the approximation $S_{N,q}(f)$ the KGE method with exact jumps.

Denote by $\| f \|_\varepsilon$ the standard norm in the space $L_2(-\varepsilon, \varepsilon)$, where $0 < \varepsilon \leqslant 1$:

$$\| f \|_\varepsilon = \left(\int_{-\varepsilon}^{\varepsilon} |f(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

The next theorem describes the asymptotic behaviour of $S_{N,q}(f)$ on the segment $[-1, 1]$ in the L_2-norm.

THEOREM 2.1 (Nersessian & Poghosyan, 2006; Barkhudaryan *et al.*, 2007) Suppose that $f \in C^q[-1, 1]$ for some $q \geqslant 1$ and $f^{(q)}$ is absolutely continuous on $[-1, 1]$. Then the following estimate holds:

$$\lim_{N \to \infty} N^{q+\frac{1}{2}} \| R_{N,q}(f) \|_1 = |A_q(f)| c(q), \tag{2.5}$$

where

$$c(q) = \frac{1}{\pi^{q+1} \sqrt{2q+1}}. \tag{2.6}$$

2.2 *The accuracy away from the discontinuity*

In this section we investigate the accuracy of $S_{N,q}(f)$ on the interval $|x| < 1$. Note that this is the interval of smoothness of the period-2 extension of the smooth function f on $[-1, 1]$.

LEMMA 2.2 (Nersessian & Poghosyan, 2006) Let

$$\omega_{k,m} = \sum_{s=0}^{k} \binom{k}{s} (-1)^s s^m, \quad 0 \leqslant m.$$

Then

$$\omega_{k,m} = \begin{cases} 0, & m < k, \\ (-1)^k k!, & m = k. \end{cases}$$

We define

$$\Delta_s^0(f_n) = f_s, \quad \Delta_s^k(f_n) = \Delta_s^{k-1}(f_n) + \Delta_{(|s|-1)\mathrm{sgn}(s)}^{k-1}(f_n), \quad k \geqslant 1,$$

where $\mathrm{sgn}(s) = 1$ if $s \geqslant 0$ and $\mathrm{sgn}(s) = -1$ if $s < 0$.

LEMMA 2.3 Let

$$f_n = \frac{(-1)^{n+1}}{2(\mathrm{i}\pi n)^{\alpha+1}}, \quad \alpha = 0, 1, 2, \dots.$$

Then

$$\Delta_n^p(f_n) = f_n \frac{(-1)^p (p+\alpha)!}{|n|^p \alpha!} + \mathcal{O}\left(\frac{1}{n^{p+\alpha+2}}\right), \quad n \to \infty, \quad p = 0, 1, 2, \dots.$$

Proof. For $n > p$ we have that

$$\Delta_n^p(f_n) = \sum_{k=0}^{p} \binom{p}{k} f_{n-k} = f_n \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{\left(1 - \frac{k}{n}\right)^{\alpha+1}} = f_n \sum_{m=0}^{\infty} \binom{m+\alpha}{\alpha} \frac{\omega_{p,m}}{n^m}.$$

The remainder of the assertion follows from Lemma 2.2. The case $n < -p$ can be explored in the same way. □

THEOREM 2.4 Suppose that $f \in C^{q+1}[-1, 1]$ for some $q \geqslant 1$ and $f^{(q+1)}$ is absolutely continuous on $[-1, 1]$. Then the following estimates hold for $|x| < 1$:

$$R_{N,q}(f) = A_q(f) \frac{(-1)^{N+\frac{q}{2}}}{2(\pi N)^{q+1}} \frac{\sin \pi \left(N + \frac{1}{2}\right)x}{\cos \frac{\pi x}{2}} + o(N^{-q-1}), \quad N \to \infty, \tag{2.7}$$

for even values of q and

$$R_{N,q}(f) = A_q(f) \frac{(-1)^{N+\frac{q+1}{2}}}{2(\pi N)^{q+1}} \frac{\cos \pi \left(N + \frac{1}{2}\right)x}{\cos \frac{\pi x}{2}} + o(N^{-q-1}), \quad N \to \infty, \tag{2.8}$$

for odd values of q.

Proof. Equations (2.1), (2.3) and (2.4) imply that

$$R_{N,q}(f) = R_{N,q}^+(f) + R_{N,q}^-(f),$$

where

$$R_{N,q}^+(f) = \sum_{n=N+1}^{\infty} F_n \, \mathrm{e}^{\mathrm{i}\pi n x}, \quad R_{N,q}^-(f) = \sum_{n=-\infty}^{-N-1} F_n \, \mathrm{e}^{\mathrm{i}\pi n x}.$$

We first estimate $R_{N,q}^+(f)$. The following transformation can be checked easily ($|x| < 1$):

$$R_{N,q}^+(f) = -\frac{\mathrm{e}^{\mathrm{i}\pi(N+1)x}}{1 + \mathrm{e}^{\mathrm{i}\pi x}} F_N + \frac{1}{1 + \mathrm{e}^{\mathrm{i}\pi x}} \sum_{n=N+1}^{\infty} \Delta_n(F_n) \mathrm{e}^{\mathrm{i}\pi n x}.$$

After reapplying this transformation for the second term, we obtain that

$$R^{+}_{N,q}(f) = -\frac{e^{i\pi(N+1)x}}{1+e^{i\pi x}}F_N - \frac{e^{i\pi(N+1)x}}{(1+e^{i\pi x})^2}\Delta_N(F_n) + \frac{1}{(1+e^{i\pi x})^2}\sum_{n=N+1}^{\infty}\Delta_n^2(F_n)e^{i\pi nx}. \tag{2.9}$$

Taking into account that

$$F_n = A_q(f)\frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} + A_{q+1}(f)\frac{(-1)^{n+1}}{2(i\pi n)^{q+2}} + o(n^{-q-2}), \quad n \to \infty,$$

in view of Lemma 2.3, we conclude that

$$\Delta_n^2(F_n) = o(n^{-q-2}), \quad n \to \infty,$$

and

$$\Delta_N(F_n) = \mathcal{O}(N^{-q-2}), \quad N \to \infty.$$

Hence the third term in (2.9) is $o(N^{-q-1})$ as $N \to \infty$, and the second term is $\mathcal{O}(N^{-q-2})$ as $N \to \infty$. Substituting all these into (2.9), we derive that

$$R^{+}_{N,q}(f) = A_q(f)\frac{(-1)^N}{2(i\pi N)^{q+1}}\frac{e^{i\pi(N+1)x}}{1+e^{i\pi x}} + o(N^{-q-1}), \quad N \to \infty.$$

We estimate $R^{-}_{N,q}(f)$ similarly as follows:

$$R^{-}_{N,q}(f) = A_q(f)\frac{(-1)^N}{2(-i\pi N)^{q+1}}\frac{e^{-i\pi(N+1)x}}{1+e^{-i\pi x}} + o(N^{-q-1}), \quad N \to \infty.$$

Therefore

$$R_{N,q}(f) = A_q(f)\frac{(-1)^N}{(\pi N)^{q+1}}\mathrm{Re}\left(\frac{1}{i^{q+1}}\frac{e^{i\pi(N+1)x}}{1+e^{i\pi x}}\right) + o(N^{-q-1}), \quad N \to \infty.$$

This concludes the proof. □

In the next theorem the norm $\|R_{N,q}(f)\|_\varepsilon$ for every $0 < \varepsilon < 1$ is estimated.

THEOREM 2.5 Let the conditions of Theorem 2.4 be valid. Then the following estimate holds:

$$\lim_{N\to\infty} N^{q+1}\|R_{N,q}(f)\|_\varepsilon = \frac{|A_q(f)|}{\pi^{q+1}\sqrt{2\pi}}\left(tg\frac{\pi\varepsilon}{2}\right)^{1/2}, \quad 0 < \varepsilon < 1. \tag{2.10}$$

Proof. Equation (2.7) yields that

$$\begin{aligned} N^{q+1}\|R_{N,q}(f)\|_\varepsilon &= \frac{|A_q(f)|}{2\pi^{q+1}}\left(\int_{-\varepsilon}^{\varepsilon}\frac{\sin^2\pi\left(N+\frac{1}{2}\right)x}{\cos^2\frac{\pi x}{2}}dx\right)^{1/2} + o(1) \\ &= \frac{|A_q(f)|}{2\pi^{q+1}}\left(\int_{-\varepsilon}^{\varepsilon}\frac{1-\cos 2\pi\left(N+\frac{1}{2}\right)x}{2\cos^2\frac{\pi x}{2}}dx\right)^{1/2} + o(1) \\ &= \frac{|A_q(f)|}{2\pi^{q+1}}\left(\int_{-\varepsilon}^{\varepsilon}\frac{dx}{2\cos^2\frac{\pi x}{2}}\right)^{1/2} + o(1), \quad N \to \infty, \end{aligned}$$

where the last integral can be calculated explicitly.

Equation (2.8) implies the same estimate. □

3. The KGE method with approximate values of the jumps

In order to determine the approximate values $\widetilde{A}_k(f)$ for $A_k(f)$, the fact that the coefficients F_n asymptotically ($n \to \infty$) decay faster than the coefficients f_n is used and can therefore be discarded for large $|n|$. Hence equation (2.2) implies that

$$f_n = \sum_{k=0}^{q-1} \widetilde{A}_k(f) B_{k,n}, \quad n = n_1, n_2, \ldots, n_q. \tag{3.1}$$

Thus, for any given N, assume that we have chosen q different integer indices

$$n_1 = n_1(N), \quad n_2 = n_2(N), \quad \ldots, \quad n_q = n_q(N)$$

for evaluating system (3.1). The next theorem shows how well the values $\widetilde{A}_k(f)$ approximate the actual jumps $A_k(f)$ depending on the choice of the indices n_s.

By the multiplicity of some number x in a sequence $x_1, \ldots, x_m$, we mean the number of indices i for which $x_i = x$.

THEOREM 3.1 (Barkhudaryan *et al.*, 2007) Suppose that the indices $n_s = n_s(N)$ in (3.1) are chosen such that

$$\lim_{N\to\infty} \frac{n_s}{N} = c_s \neq 0, \quad s = 1, \ldots, q. \tag{3.2}$$

Let α be the greatest multiplicity of a number in the sequence $c_1, c_2, \ldots, c_q$. Then, for $f \in C^{q+\alpha-1}[-1, 1]$ such that $f^{(q+\alpha-1)}$ is absolutely continuous on $[-1, 1]$, the following estimate holds:

$$\widetilde{A}_j(f) = A_j(f) - A_q(f)\frac{\chi_j}{(\mathrm{i}\pi N)^{q-j}} + o(N^{-q+j}), \quad N \to \infty, \quad j = 0, \ldots, q-1, \tag{3.3}$$

where the constants χ_j are the coefficients of the polynomial

$$\prod_{s=1}^{q}\left(x - \frac{1}{c_s}\right) = \sum_{s=0}^{q} \chi_s x^s. \tag{3.4}$$

As in (2.2)–(2.4), let us write that

$$\widetilde{F}_n = f_n - \sum_{k=0}^{q-1} \widetilde{A}_k(f) B_{k,n}, \tag{3.5}$$

$$\widetilde{S}_{N,q}(f) = \sum_{n=-N}^{N} \widetilde{F}_n \, \mathrm{e}^{\mathrm{i}\pi nx} + \sum_{k=0}^{q-1} \widetilde{A}_k B_k(x) \tag{3.6}$$

and

$$\widetilde{R}_{N,q}(f) = f(x) - \widetilde{S}_{N,q}(f). \tag{3.7}$$

We call the approximation by $\widetilde{S}_{N,q}(f)$, where the approximate jumps are calculated from (3.1), the KGE method with approximate values of the jumps. Note that, instead of (2.1), we now use the other representation of the approximated function

$$f(x) = \widetilde{F}(x) + \sum_{k=0}^{q-1} \widetilde{A}_k(f) B_k(x) \tag{3.8}$$

and the asymptotic behaviour of the KGE method with approximate jumps is highly dependent on the smoothness of the function $\widetilde{F}$ or, in other words, the asymptotic behaviour of the Fourier coefficients $\widetilde{F}_n$ as $n \to \infty$.

Taking into account (3.5), system (3.1) can be rewritten in the form

$$\widetilde{F}_n = 0, \quad n = n_1, n_2, \ldots, n_q. \tag{3.9}$$

The next theorem formulates the analogue of Theorem 2.1 for $\widetilde{S}_{N,q}(f)$.

THEOREM 3.2 (Barkhudaryan *et al.*, 2007) Suppose that the conditions of Theorem 3.1 are valid. Then the following estimate holds:

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} \|\widetilde{R}_{N,q}(f)\|_1 = |A_q(f)| d(q), \tag{3.10}$$

where

$$d(q) = \frac{1}{\sqrt{2}\pi^{q+1}} \left(\int_{-1}^{1} \prod_{s=1}^{q} \left(x - \frac{1}{c_s} \right)^2 \mathrm{d}x \right)^{1/2}. \tag{3.11}$$

Theorems 2.1 and 3.2 explore the asymptotic behaviour of the approximations $S_{N,q}$ and $\widetilde{S}_{N,q}$ on the whole interval of approximation $[-1, 1]$. Recall that the points $x = \pm 1$ are the possible singularities of the period-2 extension of f. Note also that the rate of convergence in Theorem 2.1 is the same as that in Theorem 3.2, i.e., the approximate calculation of the jumps does not reduce the convergence rate.

4. The auto-correction phenomenon

Hereafter we consider the following special choice of the indices n_s in system (3.1):

$$n_s = N - s + 1, \quad s = 1, \ldots, m, \quad n_s = -(N - s + m + 1), \quad s = m + 1, \ldots, 2m, \tag{4.1}$$

for even values of q, that is, $q = 2m$, where $m = 1, 2, \ldots$, and

$$n_s = N - s + 1, \quad s = 1, \ldots, m + 1, \quad n_s = -(N - s + m + 2), \quad s = m + 2, \ldots, 2m + 1, \tag{4.2}$$

for odd values of q, that is, $q = 2m + 1$, where $m = 0, 1, \ldots$.

As will be shown below, these choices lead to the fast convergence of the approximation $\widetilde{S}_{N,q}(f)$ in the regions where the period-2 extension of f is smooth.

4.1 *The accuracy up to the discontinuity*

In the next theorems we investigate the accuracy of the approximation of the actual jumps and its impact on the precision of the KGE method with approximate jumps.

THEOREM 4.1 Let q be an even number, that is, $q = 2m$, where $m = 1, 2, \dots$, and let the indices $n_s = n_s(N)$ be chosen as in (4.1). Then, for $f \in C^{3m-1}[-1, 1]$ such that $f^{(3m-1)}$ is absolutely continuous on $[-1, 1]$, the following estimates hold:

$$\widetilde{A}_{2j}(f) = A_{2j}(f) - A_{2m}(f)\frac{\binom{m}{j}(-1)^{m+j}}{(\mathrm{i}\pi N)^{2m-2j}} + o(N^{-2m+2j}), \quad j = 0, \dots, m-1, \quad N \to \infty, \tag{4.3}$$

and

$$\widetilde{A}_{2j+1}(f) = A_{2j+1}(f) + o(N^{-2m+2j+1}), \quad j = 0, \dots, m-1, \quad N \to \infty. \tag{4.4}$$

Proof. It is easy to check that for the numbers c_s (see (3.2)) we have

$$c_s = 1, \quad s = 1, \dots, m, \quad c_s = -1, \quad s = m+1, \dots, 2m. \tag{4.5}$$

Hence the greatest multiplicity of a number in the sequence $c_1, c_2, \dots, c_{2m}$ is equal to m. Then replacing α by m in Theorem 3.1, according to the definition of the coefficients χ_k, we derive that

$$\sum_{s=0}^{2m} \chi_s x^s = \prod_{s=1}^{2m}\left(x - \frac{1}{c_s}\right) = \prod_{s=1}^{m}(x-1)\prod_{s=1}^{m}(x+1)$$
$$= (x^2-1)^m = \sum_{s=0}^{m}\binom{m}{s}(-1)^{m+s}x^{2s}.$$

Therefore

$$\chi_{2s} = \binom{m}{s}(-1)^{m+s}, \quad s = 0, \dots, m, \tag{4.6}$$

and

$$\chi_{2s+1} = 0, \quad s = 0, \dots, m-1. \tag{4.7}$$

Substitution of (4.6) and (4.7) into (3.3) completes the proof. □

THEOREM 4.2 Let q be an odd number, that is, $q = 2m+1$, where $m = 0, 1, \dots$, and let the indices $n_s = n_s(N)$ be chosen as in (4.2). Then, for $f \in C^{3m+1}[-1, 1]$ such that $f^{(3m+1)}$ is absolutely continuous on $[-1, 1]$, the following estimates hold:

$$\widetilde{A}_{2j}(f) = A_{2j}(f) - A_{2m+1}(f)\frac{\binom{m}{j}(-1)^{m+j+1}}{(\mathrm{i}\pi N)^{2m-2j+1}} + o(N^{-2m+2j-1}), \quad j = 0, \dots, m, \quad N \to \infty, \tag{4.8}$$

and

$$\widetilde{A}_{2j+1}(f) = A_{2j+1}(f) - A_{2m+1}(f)\frac{\binom{m}{j}(-1)^{m+j}}{(\mathrm{i}\pi N)^{2m-2j}} + o(N^{-2m+2j}), \quad j = 0, \dots, m-1, \quad N \to \infty. \tag{4.9}$$

Proof. We have

$$c_s = 1, \ \ s = 1, \dots, m+1, \quad c_s = -1, \ \ s = m+2, \dots, 2m+1.$$

Therefore the greatest multiplicity of a number in the sequence $c_1, c_2, \dots, c_{2m+1}$ is equal to $m+1$. Substituting this into Theorem 3.1 instead of α, we proceed as follows:

$$\sum_{s=0}^{2m+1} \chi_s x^s = \prod_{s=1}^{2m+1} \left(x - \frac{1}{c_s} \right) = (x^2-1)^m (x-1)$$

$$= \sum_{s=0}^{m} \binom{m}{s} (-1)^{m+s} x^{2s+1} - \sum_{s=0}^{m} \binom{m}{s} (-1)^{m+s} x^{2s}.$$

Thus

$$\chi_{2s} = \binom{m}{s} (-1)^{m+s+1}, \quad s = 0, \dots, m, \tag{4.10}$$

and

$$\chi_{2s+1} = \binom{m}{s} (-1)^{m+s}, \quad s = 0, \dots, m. \tag{4.11}$$

Substitution of (4.10) and (4.11) into (3.3) concludes the proof. □

Now we reformulate Theorem 3.2 for the choices (4.1) and (4.2).

THEOREM 4.3 Let the conditions of Theorem 4.1 be valid. Then the following estimate holds:

$$\lim_{N \to \infty} N^{2m+\frac{1}{2}} \| \widetilde{R}_{N,2m}(f) \|_1 = |A_{2m}(f)| d(2m), \tag{4.12}$$

where

$$d(2m) = \frac{(2m)! 2^{2m}}{\pi^{2m+1} \sqrt{(4m+1)!}}. \tag{4.13}$$

Proof. Equations (3.11) and (4.5) yield that

$$d(2m) = \frac{1}{\sqrt{2}\pi^{2m+1}} \left(\int_{-1}^{1} \prod_{s=1}^{2m} \left(x - \frac{1}{c_s} \right)^2 \mathrm{d}x \right)^{1/2}$$

$$= \frac{1}{\sqrt{2}\pi^{2m+1}} \left(\int_{-1}^{1} (1-x^2)^{2m} \, \mathrm{d}x \right)^{1/2}$$

$$= \frac{1}{\sqrt{2}\pi^{2m+1}} \left(\frac{\Gamma(2m+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{3}{2}\right)} \right)^{1/2}.$$

This ends the proof. □

THEOREM 4.4 Let the conditions of Theorem 4.2 be valid. Then the following estimate holds:

$$\lim_{N\to\infty} N^{2m+\frac{3}{2}} \|\widetilde{R}_{N,2m+1}(f)\|_1 = |A_{2m+1}(f)| d(2m+1), \tag{4.14}$$

where

$$d(2m+1) = \frac{(2m)!2^{2m+1}\sqrt{(2m+1)(2m+2)}}{\pi^{2m+2}\sqrt{(4m+3)!}}. \tag{4.15}$$

Proof. Equations (3.11) and (4.1) yield that

$$\begin{aligned} d(2m+1) &= \frac{1}{\sqrt{2}\pi^{2m+2}}\left(\int_{-1}^{1}\prod_{s=1}^{2m+1}\left(x-\frac{1}{c_s}\right)^2 \mathrm{d}x\right)^{1/2} \\ &= \frac{1}{\sqrt{2}\pi^{2m+2}}\left(\int_{-1}^{1}(1-x^2)^{2m}(1-x)^2\,\mathrm{d}x\right)^{1/2} \\ &= \frac{1}{\sqrt{2}\pi^{2m+2}}\left(\frac{2^{4m+3}\Gamma(2m+1)\Gamma(2m+3)}{\Gamma(4m+4)}\right)^{1/2}. \end{aligned}$$

This finalizes the proof. □

4.2 *The accuracy away from the discontinuity*

The next theorems reveal the theoretical basis of the auto-correction phenomenon.

THEOREM 4.5 Let q be an even number, that is $q = 2m$, where $m = 1, 2, \ldots$, and let the indices $n_s = n_s(N)$ be chosen as in (4.1). Suppose that $f \in C^{3m+1}[-1, 1]$ for some $m \geqslant 1$ and $f^{(3m+1)}$ is absolutely continuous on $[-1, 1]$. Then the following estimate holds for $|x| < 1$ and $N \to \infty$:

$$\widetilde{R}_{N,2m}(f) = \frac{A_{2m}(f)(-1)^{N+m}}{2^{m+1}N^{3m+1}\pi^{2m+1}}\frac{\sin\frac{\pi x}{2}(2N-m+1)}{\cos^{m+1}\frac{\pi x}{2}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(m+2k)!}{(2k)!} + o(N^{-3m-1}). \tag{4.16}$$

Proof. In view of equations (3.5)–(3.8), we conclude that

$$\widetilde{R}_{N,2m}(f) = \widetilde{R}^{+}_{N,2m}(f) + \widetilde{R}^{-}_{N,2m}(f),$$

where

$$\widetilde{R}^{+}_{N,2m}(f) = \sum_{n=N+1}^{\infty}\widetilde{F}_n\,\mathrm{e}^{\mathrm{i}\pi n x}, \quad R^{-}_{N,q}(f) = \sum_{n=-\infty}^{-N-1}\widetilde{F}_n\,\mathrm{e}^{\mathrm{i}\pi n x}.$$

We first estimate $\widetilde{R}^{+}_{N,2m}(f)$. Recall (see (3.9)) that for this case we solve the system

$$\widetilde{F}_n = 0, \quad n = -N, \ldots, -N+m-1, \tag{4.17}$$

$$\widetilde{F}_n = 0, \quad n = N-m+1, \ldots, N. \tag{4.18}$$

Utilization of Abel's transformation as in the proof of Theorem 2.4 leads to the following expansion of the error:

$$\widetilde{R}^{+}_{N,2m}(f) = -\mathrm{e}^{\mathrm{i}\pi(N+1)x}\sum_{k=1}^{m+2}\frac{\Delta_N^{k-1}(\widetilde{F}_n)}{(1+\mathrm{e}^{\mathrm{i}\pi x})^k} + \frac{1}{(1+\mathrm{e}^{\mathrm{i}\pi x})^{m+2}}\sum_{n=N+1}^{\infty}\Delta_n^{m+2}(\widetilde{F}_n)\mathrm{e}^{\mathrm{i}\pi nx}. \tag{4.19}$$

Taking into account that

$$\Delta_N^{k-1}(\widetilde{F}_n) = \sum_{s=0}^{k-1}\binom{k-1}{s}\widetilde{F}_{N-s},$$

from (4.18) we see that $\Delta_N^s(\widetilde{F}_n) = 0$ for $s = 0, \ldots, m-1$. Hence (4.19) can be rewritten in the form

$$\widetilde{R}^{+}_{N,2m}(f) = -\mathrm{e}^{\mathrm{i}\pi(N+1)x}\frac{\Delta_N^{m}(\widetilde{F}_n)}{(1+\mathrm{e}^{\mathrm{i}\pi x})^{m+1}} - \mathrm{e}^{\mathrm{i}\pi(N+1)x}\frac{\Delta_N^{m+1}(\widetilde{F}_n)}{(1+\mathrm{e}^{\mathrm{i}\pi x})^{m+2}}$$
$$+ \frac{1}{(1+\mathrm{e}^{\mathrm{i}\pi x})^{m+2}}\sum_{n=N+1}^{\infty}\Delta_n^{m+2}(\widetilde{F}_n)\mathrm{e}^{\mathrm{i}\pi nx}. \tag{4.20}$$

Equation (3.5) results in the following:

$$\widetilde{F}_n = f_n - \sum_{k=0}^{q-1}\widetilde{A}_k(f)B_{k,n}$$
$$= \frac{(-1)^{n+1}}{2}\sum_{k=0}^{2m-1}\frac{A_k(f)-\widetilde{A}_k(f)}{(\mathrm{i}\pi n)^{k+1}}$$
$$+ \frac{(-1)^{n+1}}{2}\sum_{k=2m}^{3m+1}\frac{A_k(f)}{(\mathrm{i}\pi n)^{k+1}} + o(n^{-3m-2}), \quad n\to\infty. \tag{4.21}$$

Lemma 2.3 implies that

$$\Delta_n^p(\widetilde{F}_n) = \frac{(-1)^{p+n+1}}{2n^p}\sum_{k=0}^{2m-1}\frac{A_k(f)-\widetilde{A}_k(f)}{(\mathrm{i}\pi n)^{k+1}}\frac{(p+k)!}{k!}$$
$$+ \frac{(-1)^{p+n+1}}{2n^p}\sum_{k=2m}^{3m+1}\frac{A_k(f)}{(\mathrm{i}\pi n)^{k+1}}\frac{(p+k)!}{k!} + o(n^{-3m-2}), \quad n\to\infty. \tag{4.22}$$

Application of Theorem 4.1, in view of (4.22) and $n > N$, yields that

$$\Delta_n^{m+2}(\widetilde{F}_n) = \frac{o(1)}{N^{2m}n^{m+2}}, \quad N\to\infty. \tag{4.23}$$

Hence the third term in (4.20) is $o(N^{-3m-1})$ as $N\to\infty$. In a similar way it can be derived that the second term is $\mathcal{O}(N^{-3m-2})$ as $N\to\infty$. Finally, we obtain that

$$\widetilde{R}^{+}_{N,2m}(f) = -\Delta_N^{m}(\widetilde{F}_n)\frac{\mathrm{e}^{\mathrm{i}\pi(N+1)x}}{(1+\mathrm{e}^{\mathrm{i}\pi x})^{m+1}} + o(N^{-3m-1}), \quad N\to\infty. \tag{4.24}$$

Similarly, we get that

$$\varDelta_N^m(\widetilde{F}_n) = A_{2m}(f)\frac{(-1)^{N+1}}{2(\mathrm{i}\pi)^{2m+1}N^{3m+1}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(m+2k)!}{(2k)!} + o(N^{-3m-1}), \quad N \to \infty.$$

Substituting this into (4.24) and making the same statements for $R_{N,2m}^{-}(f)$, we complete the proof. □

The next result is an immediate consequence of Theorem 4.5.

THEOREM 4.6 Let the conditions of Theorem 4.5 be valid. Then the following estimate holds:

$$\lim_{N\to\infty} N^{3m+1}\|\widetilde{R}_{N,2m}(f)\|_{\varepsilon} = |A_{2m}(f)|\frac{I_{m,\varepsilon}}{2^{m+1}\pi^{2m+1}\sqrt{2}}\left|\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(m+2k)!}{(2k)!}\right|, \quad 0 < \varepsilon < 1, \tag{4.25}$$

where

$$I_{m,\varepsilon} = \left(\int_{-\varepsilon}^{\varepsilon}\frac{\mathrm{d}x}{\cos^{2m+2}\frac{\pi x}{2}}\right)^{1/2}. \tag{4.26}$$

REMARK 4.7 Note that the integral in (4.26) can be calculated explicitly by the following recurrent relation:

$$\int\frac{\mathrm{d}x}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1}x} + \frac{n-2}{n-1}\int\frac{\mathrm{d}x}{\cos^{n-2}x}.$$

A comparison of Theorems 2.4 and 2.5 with Theorems 4.5 and 4.6 reveals the fast convergence ($\mathcal{O}(N^{-3m-1})$ as $N \to \infty$) of the KGE method with approximate jumps compared to the convergence rate ($\mathcal{O}(N^{-2m-1})$ as $N \to \infty$) of the KGE method with exact jumps. We call this improvement in the convergence rate the auto-correction phenomenon. The magnitude of the phenomenon for even values of q ($q = 2m$, where $m = 1, 2, \ldots$) is exactly m powers of N.

Now we formulate the analogues of Theorems 4.5 and 4.6 for odd values of q.

THEOREM 4.8 Let q be an odd number, that is $q = 2m+1$, where $m = 0, 1, \ldots$, and let the indices $n_s = n_s(N)$ be chosen as in (4.2). Suppose that $f \in C^{3m+1}[-1,1]$ for some $m \geqslant 0$ and $f^{(3m+1)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds for $|x| < 1$:

$$\widetilde{R}_{N,2m+1}(f) = A_{2m+1}(f)\frac{(-1)^{N+m+1}}{2N^{3m+2}\pi^{2m+2}}\frac{\mathrm{e}^{-\mathrm{i}\pi(N+1)x}}{(1+\mathrm{e}^{-\mathrm{i}\pi x})^{m+1}}$$
$$\times\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(m+2k)!(4k+m+2)}{(2k+1)!} + o(N^{-3m-2}), \quad N \to \infty. \tag{4.27}$$

Proof. We proceed as in the proof of Theorem 4.5. The additional equation

$$\widetilde{F}_{N-m} = 0$$

implies that

$$\widetilde{R}_{N,2m+1}(f) = \widetilde{R}_{N,2m+1}^{-}(f) + o(N^{-3m-1}) = -\varDelta_{-N}^m(\widetilde{F}_n)\frac{\mathrm{e}^{-i\pi(N+1)x}}{(1+\mathrm{e}^{-\mathrm{i}\pi x})^{m+1}} + o(N^{-3m-2}), \quad N \to \infty.$$

The remainder of the proof is obvious. □

THEOREM 4.9 Let the conditions of Theorem 4.8 be valid. Then the following estimate holds for $0 < \varepsilon < 1$:

$$\lim_{N\to\infty} N^{3m+2}\|\widetilde{R}_{N,2m+1}(f)\|_{\varepsilon} = |A_{2m+1}(f)|\frac{I_{m,\varepsilon}}{2^{m+2}\pi^{2m+2}}\left|\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(m+2k)!(4k+m+2)}{(2k+1)!}\right|, \tag{4.28}$$

where $I_{m,\varepsilon}$ is defined by (4.26).

The comparison of Theorems 2.4 and 2.5 with Theorems 4.8 and 4.9 shows that the magnitude of the auto-correction phenomenon for odd values of q ($q = 2m+1$, $m = 0, 1, \ldots$) is exactly m powers of N. We see that for $m = 0$ the phenomenon is absent.

5. Numerical illustrations

In this section we illustrate the convergence properties of the KGE method by some numerical examples. The auto-correction phenomenon is also explored.

Consider the simple function

$$f(x) = \sin(x-1). \tag{5.1}$$

In Table 1 we present the L_2-errors of the approximations $S_{N,q}(f)$ and $\widetilde{S}_{N,q}(f)$ on the interval $[-1, 1]$. The approximation of the jumps are calculated from (3.1) with the indices n_s given by (4.1) and (4.2). It can be seen that $S_{N,q}(f)$ is more precise than $\widetilde{S}_{N,q}(f)$ for all values of N and q. We came to the same conclusion by comparing Theorem 2.1 with Theorems 4.3 and 4.4. After simple calculations, we derive that

$$\frac{d(2m)}{c(2m)} = \frac{(2m)!4^m}{\sqrt{(4m)!}} > 1, \qquad \frac{d(2m+1)}{c(2m+1)} = \frac{2^{2m+1}(2m)!\sqrt{(2m+1)(2m+2)}}{\sqrt{(4m+2)!}} > 1.$$

The results from numerical experiments on the interval $[-0.7, 0.7]$ are presented in Table 2. Here one can see that the situation is reversed for the region where the period-2 extension of the approximated function is smooth. In spite of our expectations, the approximation $\widetilde{S}_{N,q}(f)$ is more precise than $S_{N,q}(f)$.

From Table 2 we get that

$$\frac{\|R_{32,2}(f)\|_{0.7}}{\|R_{64,2}(f)\|_{0.7}} = 8, \qquad \frac{\|R_{64,2}(f)\|_{0.7}}{\|R_{128,2}(f)\|_{0.7}} = 7.8, \qquad \frac{\|R_{128,2}(f)\|_{0.7}}{\|R_{256,2}(f)\|_{0.7}} = 7.9.$$

TABLE 1 *L_2-errors while approximating the function (5.1) by $S_{N,q}(f)$ and $\widetilde{S}_{N,q}(f)$ on the interval $[-1, 1]$ when the indices (4.1) and (4.2) are considered*

	$N = 32$	$N = 64$	$N = 128$	$N = 256$
$\|R_{N,2}(f)\|_1$	2.2×10^{-6}	3.9×10^{-7}	7.0×10^{-8}	1.2×10^{-8}
$\|\widetilde{R}_{N,2}(f)\|_1$	3.7×10^{-6}	6.5×10^{-7}	1.2×10^{-7}	2.0×10^{-8}
$\|R_{N,3}(f)\|_1$	2.8×10^{-8}	2.5×10^{-9}	2.3×10^{-10}	2.0×10^{-11}
$\|\widetilde{R}_{N,3}(f)\|_1$	6.3×10^{-8}	5.5×10^{-9}	4.8×10^{-10}	4.2×10^{-11}
$\|R_{N,4}(f)\|_1$	1.6×10^{-10}	7.1×10^{-12}	3.2×10^{-13}	1.4×10^{-14}
$\|\widetilde{R}_{N,4}(f)\|_1$	3.4×10^{-10}	1.5×10^{-11}	6.3×10^{-13}	2.8×10^{-14}

TABLE 2 *L_2-errors while approximating the function* (5.1) *by $S_{N,q}(f)$ and $\widetilde{S}_{N,q}(f)$ on the interval* $[-0.7, 0.7]$ *when the indices* (4.1) *and* (4.2) *are considered*

	$N = 32$	$N = 64$	$N = 128$	$N = 256$
$\|R_{N,2}(f)\|_{0.7}$	4.8×10^{-7}	6.0×10^{-8}	7.7×10^{-9}	9.7×10^{-10}
$\|\widetilde{R}_{N,2}(f)\|_{0.7}$	2.3×10^{-8}	1.5×10^{-9}	9.3×10^{-11}	5.7×10^{-12}
$\|R_{N,3}(f)\|_{0.7}$	7.2×10^{-9}	4.7×10^{-10}	3.0×10^{-11}	1.9×10^{-12}
$\|\widetilde{R}_{N,3}(f)\|_{0.7}$	5.2×10^{-10}	1.6×10^{-11}	5.1×10^{-13}	1.6×10^{-14}
$\|R_{N,4}(f)\|_{0.7}$	4.6×10^{-11}	1.5×10^{-12}	4.7×10^{-14}	1.5×10^{-15}
$\|\widetilde{R}_{N,4}(f)\|_{0.7}$	2.6×10^{-13}	2.0×10^{-15}	1.5×10^{-17}	1.2×10^{-19}

These results coincide with the statement of Theorem 2.5, where $\|R_{N,2}\|_\varepsilon = \mathcal{O}(N^{-3})$ as $N \to \infty$. This yields asymptotically that

$$\frac{\|R_{2^z,2}(f)\|_{0.7}}{\|R_{2^{z+1},2}(f)\|_{0.7}} = 8.$$

In view of Theorem 4.5, we have that $\widetilde{R}_{N,2} = \mathcal{O}(N^{-4})$ as $N \to \infty$, which implies asymptotically that

$$\frac{\|\widetilde{R}_{2^z,2}(f)\|_{0.7}}{\|\widetilde{R}_{2^{z+1},2}(f)\|_{0.7}} = 16.$$

This theoretical estimate coincides with the results in Table 2 as follows:

$$\frac{\|\widetilde{R}_{32,2}(f)\|_{0.7}}{\|\widetilde{R}_{64,2}(f)\|_{0.7}} = 15.3, \qquad \frac{\|\widetilde{R}_{64,2}(f)\|_{0.7}}{\|\widetilde{R}_{128,2}(f)\|_{0.7}} = 16.1, \qquad \frac{\|\widetilde{R}_{128,2}(f)\|_{0.7}}{\|\widetilde{R}_{256,2}(f)\|_{0.7}} = 16.3.$$

Consequently, the theoretical and the numerical estimates coincide—the magnitude of the auto-correction phenomenon for $q = 2$ is 1 power of N.

Similarly, for $q = 4$, Theorem 2.5 implies the estimate

$$\frac{\|R_{2^z,4}(f)\|_{0.7}}{\|R_{2^{z+1},4}(f)\|_{0.7}} = 32,$$

while the numerical experiments yield that

$$\frac{\|R_{32,4}(f)\|_{0.7}}{\|R_{64,4}(f)\|_{0.7}} = 30.7, \qquad \frac{\|R_{64,4}(f)\|_{0.7}}{\|R_{128,4}(f)\|_{0.7}} = 31.9, \qquad \frac{\|R_{128,4}(f)\|_{0.7}}{\|R_{256,4}(f)\|_{0.7}} = 31.3.$$

For $\widetilde{R}_{N,4}(f)$, Theorem 4.5 implies that

$$\frac{\|\widetilde{R}_{2^z,4}(f)\|_{0.7}}{\|\widetilde{R}_{2^{z+1},4}(f)\|_{0.7}} = 128$$

and Table 2 confirms that

$$\frac{\|\widetilde{R}_{32,4}(f)\|_{0.7}}{\|\widetilde{R}_{64,4}(f)\|_{0.7}} = 130, \qquad \frac{\|\widetilde{R}_{64,4}(f)\|_{0.7}}{\|\widetilde{R}_{128,4}(f)\|_{0.7}} = 133.3, \qquad \frac{\|\widetilde{R}_{128,4}(f)\|_{0.7}}{\|\widetilde{R}_{256,4}(f)\|_{0.7}} = 125.$$

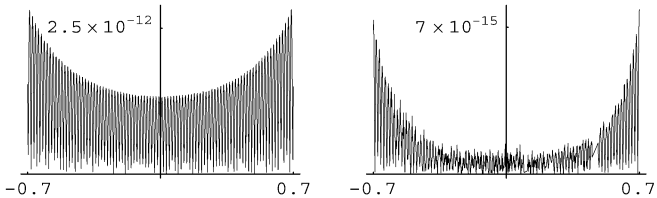

FIG. 1. Plots of $|R_{N,q}(f)|$ (left) and $|\widetilde{R}_{N,q}(f)|$ (right) while approximating the function (5.1) on the interval $[-0.7, 0.7]$ for $q = 4$ and $N = 64$.

Again, the theoretical and the numerical estimates coincide—the magnitude of the auto-correction phenomenon for $q = 4$ is 2 powers of N.

For the value $q = 3$, Theorem 2.5 implies that

$$\frac{\|R_{2^z,3}(f)\|_{0.7}}{\|R_{2^{z+1},3}(f)\|_{0.7}} = 16.$$

This result coincides with calculations from Table 2 as follows:

$$\frac{\|R_{32,3}(f)\|_{0.7}}{\|R_{64,3}(f)\|_{0.7}} = 15.3, \quad \frac{\|R_{64,3}(f)\|_{0.7}}{\|R_{128,3}(f)\|_{0.7}} = 15.7, \quad \frac{\|R_{128,3}(f)\|_{0.7}}{\|R_{256,3}(f)\|_{0.7}} = 15.8.$$

According to Theorem 4.9, we have

$$\frac{\|\widetilde{R}_{2^z,4}(f)\|_{0.7}}{\|\widetilde{R}_{2^{z+1},4}(f)\|_{0.7}} = 32.$$

Table 2 gives that

$$\frac{\|\widetilde{R}_{32,3}(f)\|_{0.7}}{\|\widetilde{R}_{64,3}(f)\|_{0.7}} = 32.5, \quad \frac{\|\widetilde{R}_{64,3}(f)\|_{0.7}}{\|\widetilde{R}_{128,3}(f)\|_{0.7}} = 31.4, \quad \frac{\|\widetilde{R}_{128,3}(f)\|_{0.7}}{\|\widetilde{R}_{256,3}(f)\|_{0.7}} = 31.9.$$

Also this time, the theoretical and the numerical estimates coincide—the magnitude of the auto-correction phenomenon for $q = 3$ is 1 power of N.

In Fig. 1 we visually show the auto-correction phenomenon while approximating the function (5.1) on the interval $[-0.7, 0.7]$ when $q = 4$ and $N = 64$.

6. Conclusion

We have investigated the asymptotic behaviour of the KGE method. In particular, we considered two families of approximations $S_{N,q}(f)$ and $\widetilde{S}_{N,q}(f)$ that use the exact and approximate values of the jumps, respectively. The approximations to the actual jumps were carried out according to the procedure described in Subsection 4.1.

The main results of the paper are the asymptotic behaviour of $S_{N,q}(f)$ and $\widetilde{S}_{N,q}(f)$ for $|x| < 1$, where the period-2 extension of the approximated function f is smooth. The comparison of these results shows that the approximation by the $\widetilde{S}_{N,q}(f)$ gives higher accuracy by about $q/2$ powers of N. This is the 'auto-correction phenomenon'—the approximate jumps give better accuracy than the exact ones.

The numerical results confirm the statements of the corresponding theorems.

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