Anal. Theory Appl.
Vol. 26, No. 3 (2010), 236–260
DOI10.1007/s10496-010-0236-3

ASYMPTOTIC BEHAVIOR OF THE ECKHOFF METHOD FOR CONVERGENCE ACCELERATION OF TRIGONOMETRIC INTERPOLATION*

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Received Oct. 14, 2009

Abstract. Convergence acceleration of the classical trigonometric interpolation by the Eckhoff method is considered, where the exact values of the "jumps" are approximated by solution of a system of linear equations. The accuracy of the "jump" approximation is explored and the corresponding asymptotic error of interpolation is derived. Numerical results validate theoretical estimates.

Key words: *Fourier series, trigonometric interpolation, convergence acceleration, Bernoulli polynomials*

AMS (2010) subject classification: 42A15, 65T40, 97N50

1 Introduction

The problem of a function reconstruction by its finite number of Fourier coefficients

$$f_n := \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}\mathrm{d}x, \qquad |n| \le N, N \ge 1$$

or discrete Fourier coefficients

$$\check{f}_n := \frac{1}{2N+1}\sum_{k=-N}^{N} f(x_k)e^{-i\pi nx_k}, \qquad x_k := \frac{2k}{2N+1}, |n| \le N, N \ge 1$$

Supported in part by grant PS 1867 from the Armenian National Science and Education Fund (ANSEF) based in New York, USA

is considered.

It is well known that approximation of a 2-periodic and smooth function on the real line by truncated Fourier series

$$S_N(f) := \sum_{n=-N}^{N} f_n e^{i\pi nx}$$

or trigonometric interpolation

$$I_N(f) := \sum_{n=-N}^{N} \check{f}_n e^{i\pi nx}$$

is highly effective. When the approximated function has a point of discontinuity, the approximation by $S_N(f)$ or $I_N(f)$ leads to the Gibbs phenomenon which degrades the quality of approximations.

An efficient approach of convergence acceleration of $S_N(f)$ and $I_N(f)$ by subtracting a polynomial representing the discontinuities (“jumps") in the function and some of its first derivatives was suggested by Krylov[10] in 1906 and later in 1964 by Lanczos[11],[12]. The latter was developed for applications by Eckhoff in a series of papers [5]-[7], where the approximate values of the “jumps" were determined by solution of a system of linear equations. The Eckhoff method was developed and generalized by a number of authors, see [1], [2], [4], [13]-[18] with references therein.

In [2] the theoretical background of the Eckhoff method was investigated, where the reconstruction of function by its finite number of Fourier coefficients was considered. Therein, the asymptotic behavior of the approximate “jumps” was studied and the asymptotic L_2 constant of the rate of convergence was computed.

In this paper we continue investigations started in [2] and consider the Eckhoff method for function reconstruction by discrete Fourier coefficients. Section 2 describes the Krylov-Lanczos and the Eckhoff approximations. Asymptotic estimates of the “jumps" approximation and function reconstruction are presented. The main results are coming from [2]. Section 3 investigates the Krylov-Lanczos interpolation and the corresponding asymptotic error. The main results are coming from [18]. Also a closed form of the discrete Fourier coefficients of the Bernoulli polynomials is derived. In Section 4 the Eckhoff interpolation is considered. Following Eckhoff we explore a system of linear equations for the “jumps" approximation. In particular, we calculate the determinant of the corresponding system and obtain the conditions providing the system with unique solution. Moreover, we modify the system to the equivalent one with the upper triangular matrix. This modification is important for applications as it reduces the complexity and leads to more robust procedure. Section 5 studies the precision of the “jumps" approximation and the corresponding asymptotic error of interpolation. Numerical results in Section 6 accomplishes theoretical investigations.

2 The Krylov-Lanczos and the Eckhoff Approximations

Throughout the paper we limit our discussion to the smooth function f on $[-1,1]$. Suppose $f \in C^q[-1,1]$ and denote by $A_k(f)$ the exact value of the “jump" in the k-th derivative of f

$$A_k(f) := f^{(k)}(1) - f^{(k)}(-1), \qquad k = 0,\dots,q.$$

The following lemma is crucial for the Krylov-Lanczos approach.

Lemma 2.1. *Let $f \in C^{q-1}[-1,1]$. Assume that $d\, f^{(q-1)}$ is absolutely continuous on $[-1,1]$ for some $q \geq 1$. Then the following expansion is valid*

$$f_n = \frac{(-1)^{n+1}}{2}\sum_{k=0}^{q-1}\frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q}\int_{-1}^{1} f^{(q)}(x)e^{-i\pi n x}\mathrm{d}x, \qquad n \neq 0.$$

Proof. The proof is trivial due to integration by parts.

Lemma 2.1 implies the representation

$$f(x) = \sum_{k=0}^{q-1} A_k(f)B_k(x) + F(x), \tag{2.1}$$

where B_k are 2-periodic Bernoulli-like polynomials with the Fourier coefficients

$$B_{k,n} := \begin{cases} 0, & n = 0, \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n \neq 0 \end{cases} \tag{2.2}$$

and F is a 2-periodic and relatively smooth function on the real line ($F \in C^{q-1}(R)$) with the Fourier coefficients

$$F_n = f_n - \sum_{k=0}^{q-1} A_k(f)B_{k,n}. \tag{2.3}$$

Approximation of F by the truncated Fourier series leads to the *Krylov-Lanczos (KL-) approximation*

$$S_{N,q}(f) := \sum_{n=-N}^{N}\left(f_n - \sum_{k=0}^{q-1} A_k(f)B_{k,n}\right)e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f)B_k(x).$$

Denote by $\|\cdot\|$ the standard norm in the space $f\in L_2(-1,1)$

$$||f||:=\left(\int_{-1}^{1}|f(x)|^2\mathrm{d}x\right)^{1/2}.$$

We put

$$R_{N,q}(f):=f(x)-S_{N,q}(f)$$

and below present an asymptotic behavior of the KL-approximation.

Theorem 2.2[2]**.** *Let $f\in C^q[-1,1]$. Assume that $f^{(q)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds*

$$\lim_{N\to\infty}(2N+1)^{q+\frac{1}{2}}\|R_{N,q}(f)\|=|A_q(f)|\frac{2^{q+\frac{1}{2}}}{\pi^{q+1}\sqrt{2q+1}}.$$

In [5]-[7] Eckhoff suggested to compute approximate "jump" values $A_k^a(f,N)$ for $A_k(f)$ directly from the Fourier coefficients f_n. As the Fourier coefficients F_n asymptotically ($n\to\infty$) decay faster than the coefficients f_n, and can therefore be discarded for large $|n|$. Hence, from (2.3) we derive the following system of linear equations for determining the approximate "jumps"

$$f_n=\sum_{k=0}^{q-1}A_k^a(f,N)B_{k,n},\qquad n=n_1,n_2,\ldots,n_q. \tag{2.4}$$

Thus, for any given N we assume to have chosen q different integer indices

$$n_1=n_1(N),\quad n_2=n_2(N),\ldots,n_q=n_q(N)$$

for evaluating the system (2.4).

By $\widetilde{S}_{N,q}(f)$ the *Eckhoff approximation* is denoted where the approximate "jumps" $A_k^a(f,N)$ are used instead of the exact ones

$$\widetilde{S}_{N,q}(f):=\sum_{n=-N}^{N}\left(f_n-\sum_{k=0}^{q-1}A_k^a(f,N)B_{k,n}\right)e^{i\pi nx}+\sum_{k=0}^{q-1}A_k^a(f,N)B_k(x).$$

Below we present some results from [2] that reveal the properties of the Eckhoff approximation.

Definition 2.3. By the multiplicity of some number x in a sequence $x_1,\ldots,x_m$ we mean the number of indices i for which $x_i=x$.

The next theorem shows how well the values $A_k^a(f,N)$ approximate the exact "jumps" $A_k(f)$.

Theorem 2.4[2]**.** *Suppose the indices $n_s = n_s(N)$ are chosen such that*

$$\lim_{N\to\infty} \frac{n_s}{N} = c_s \neq 0, \qquad s = 1, \ldots, q.$$

Let α be the greatest multiplicity of a number in the sequence $c_1, c_2, \ldots, c_q$. Then for $f \in C^{q+\alpha-1}[-1,1]$ such that $f^{(q+\alpha-1)}$ is absolutely continuous on $[-1,1]$, the following estimate holds

$$A_j^q(f,N) = A_j(f) - A_q(f)\frac{\chi_j}{(i\pi N)^{q-j}} + o(N^{-q+j}), N \to \infty, \qquad j = 0, \ldots, q-1,$$

where the constants χ_j are the coefficients of the polynomial

$$\prod_{s=1}^{q}\left(x - \frac{1}{c_s}\right) = \sum_{s=0}^{q} \chi_s x^s.$$

We put

$$\widetilde{R}_{N,q}(f) := f(x) - \widetilde{S}_{N,q}(f).$$

The following result addresses the accuracy of the Eckhoff approximation.

Theorem 2.5[2]**.** *Suppose that the conditions of Theorem* 2.4 *are valid. Then the following estimate holds*

$$\lim_{N\to\infty} (2N+1)^{q+\frac{1}{2}} \|\widetilde{R}_{N,q}(f)\| = |A_q(f)| \frac{2^{q+\frac{1}{2}}}{\sqrt{2}\,\pi^{q+1}} \left(\int_{-1}^{1} \prod_{s=1}^{q}\left(x - \frac{1}{c_s}\right)^2 \mathrm{d}x\right)^{1/2}.$$

Note that the rate of convergence in Theorem 2.2 is the same as in Theorem 2.5, i.e. approximate calculation of the “jumps" does not degrade the rate of convergence.

In the next sections we present the analogs of Theorems 2.2, 2.4, and 2.5 when the discrete Fourier coefficients are used for function reconstruction.

3 The Krylov-Lanczos Interpolation

Representation (2.1) allows to calculate the discrete Fourier coefficients of F as well

$$\check{F}_n = \check{f}_n - \sum_{k=0}^{q-1} A_k(f)\check{B}_{k,n}. \tag{3.1}$$

Approximation of F by $I_N(f)$ in (2.1), leads to the *Krylov-Lanczos (KL-) interpolation*

$$I_{N,q}(f) := \sum_{n=-N}^{N}\left(\check{f}_n - \sum_{k=0}^{q-1} A_k(f)\check{B}_{k,n}\right) e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f) B_k(x).$$

We set

$$r_{N,q}(f) := f(x) - I_{N,q}(f)$$

and investigate the asymptotic behavior of $r_{N,q}(f)$ in the next theorem.

Theorem 3.1[18]**.** *Let $f \in C^q[-1,1]$. Assume that $f^{(q)}$ is absolutely continuous on $[-1,1]$. Then*

$$\lim_{N\to\infty} (2N+1)^{q+\frac{1}{2}} \|r_{N,q}(f)\| = |A_q(f)| a_1(q),$$

where

$$a_1(q) := \frac{1}{\pi^{q+1}} \left(\frac{2^{2q+1}}{2q+1} + \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{s\neq 0} \frac{(-1)^s}{(s+x)^{q+1}} \right|^2 dx \right)^{1/2}.$$

We conclude from (2.2) that

$$B_0(x) = \frac{x}{2},\ B_k(x) = \int B_{k-1}(x)dx,\ \int_{-1}^{1} B_k(x)\mathrm{d}x = 0.$$

Whence, the discrete Fourier coefficients $\check{B}_{k,n}$ can be calculated explicitly. For example, here are three of them

$$\begin{aligned}
\check{B}_{0,n} &= \frac{(-1)^n\, i}{2(2N+1)\sin\frac{\pi n}{2N+1}}, && n\neq 0,\ \check{B}_{0,0}=0,\\
\check{B}_{1,n} &= \frac{(-1)^n\, \cos\frac{\pi n}{2N+1}}{2(2N+1)^2\sin^2\frac{\pi n}{2N+1}}, && n\neq 0,\ \check{B}_{1,0}=-\frac{1}{12(2N+1)^2},\\
\check{B}_{2,n} &= \frac{(-1)^{n+1}\, i\left(3+\cos\frac{2\pi n}{2N+1}\right)}{8(2N+1)^3\sin^3\frac{\pi n}{2N+1}}, && n\neq 0,\ \check{B}_{2,0}=0.
\end{aligned}$$

It is possible to get a closed form of $\check{B}_{k,n}$ for every $k \geq 0$. Theorem 3.4 does exactly that. First we prove some auxiliary relations. The next lemma presents the particular case of the Faa di Bruno formula [8], [21].

Lemma 3.2. *Let $f \in C^p[-1,1]$. Then the following identity is valid*

$$f^{(p)}(e^{i\pi x}) = (i\pi)^p \sum_{k=0}^{p} S(p,k) f^{(k)}(e^{i\pi x}) e^{i\pi kx}, \qquad p \geq 0,$$

where $S(p,k)$ is the Stirling number of the second kind [20].

Proof. We fulfill the proof by mathematical induction. For $p=0$ the formula is obvious as $S(0,0)=1$. Suppose it is true for $p=n$. For $p=n+1$ we have

$$\begin{aligned}
f^{(n+1)}(e^{i\pi x}) &= \frac{\mathrm{d}}{\mathrm{d}x} f^{(n)}(e^{i\pi x}) \\
&= (i\pi)^{n+1} \sum_{k=0}^{n} kS(n,k) e^{i\pi kx} f^{(k)}(e^{i\pi x}) \\
&+ (i\pi)^{n+1} \sum_{k=0}^{n} S(n,k) e^{i\pi(k+1)x} f^{(k+1)}(e^{i\pi x}) \\
&= (i\pi)^{n+1} \sum_{k=1}^{n} (kS(n,k)+S(n,k-1)) e^{i\pi kx} f^{(k)}(e^{i\pi x}) \\
&+ (i\pi)^{n+1} e^{i\pi(n+1)x} f^{(n+1)}(e^{i\pi x}).
\end{aligned}$$

Note that

$$kS(n,k)+S(n,k-1)=S(n+1,k),$$

and

$$S(n+1,0)=0,$$

we complete the proof.

Lemma 3.3. *The following identity holds*

$$\left(\frac{1}{\sin\pi x}\right)^{(k)} = \frac{\pi^k}{2^k(\sin\pi x)^{k+1}} \sum_{j=0}^{k} \alpha_{k,j} e^{i\pi(k-2j)x}, \qquad k\ge 0,$$

where

$$\alpha_{k,j} := \sum_{\ell=0}^{j} (-1)^{\ell} \sum_{n=0}^{k} n!(-1)^n S(k,n) \binom{k-n}{\ell} \binom{n+1}{2j-2\ell}.$$

Proof. We have

$$\frac{1}{\sin\pi x} = \frac{2i}{e^{i\pi x}-e^{-i\pi x}} = i\left(\frac{1}{1+e^{i\pi x}} - \frac{1}{1-e^{i\pi x}}\right).$$

In view of Lemma 3.2

$$\begin{aligned}
\left(\frac{1}{\sin\pi x}\right)^{(k)} &= i(i\pi)^k \sum_{n=0}^{k} e^{i\pi nx} S(k,n) \left(\frac{n!(-1)^n}{(1+e^{i\pi x})^{n+1}} - \frac{n!}{(1-e^{i\pi x})^{n+1}}\right) \\
&= \frac{i(i\pi)^k e^{i\pi kx}}{(e^{i\pi x}-e^{-i\pi x})^{k+1}} \sum_{n=0}^{k} n! S(k,n)(-1)^n (1-e^{-2i\pi x})^{k-n} \\
&\times \left((1+e^{-i\pi x})^{n+1} + (1-e^{-i\pi x})^{n+1}\right) \\
&= \frac{i(i\pi)^k e^{i\pi kx}}{(e^{i\pi x}-e^{-i\pi x})^{k+1}} \sum_{n=0}^{k} n! S(k,n)(-1)^n \sum_{\ell=0}^{k-n} \binom{k-n}{\ell} \\
&\times (-1)^{\ell} e^{-2i\pi\ell x} \sum_{s=0}^{n+1} \binom{n+1}{s} (1+(-1)^s) e^{-i\pi sx}.
\end{aligned}$$

For even values of k, $k=2m$, we derive

$$\begin{aligned}
\left(\frac{1}{\sin\pi x}\right)^{(2m)} &= \frac{\pi^{2m}e^{2i\pi mx}}{2^{2m}(\sin\pi x)^{2m+1}}\sum_{n=0}^{m}S(2m,2n)(2n)! \\
&\times \sum_{\ell=0}^{2m-2n}\binom{2m-2n}{\ell}(-1)^{\ell}e^{-2i\pi\ell x}\sum_{s=0}^{n}\binom{2n+1}{2s}e^{-2i\pi sx} \\
&- \frac{\pi^{2m}e^{2i\pi mx}}{2^{2m}(\sin\pi x)^{2m+1}}\sum_{n=0}^{m-1}S(2m,2n+1)(2n+1)! \\
&\times \sum_{\ell=0}^{2m-2n-1}\binom{2m-2n-1}{\ell}(-1)^{\ell}e^{-2i\pi\ell x}\sum_{s=0}^{n+1}\binom{2n+2}{2s}e^{-2i\pi sx} \\
&= \frac{\pi^{2m}e^{2i\pi mx}}{2^{2m}(\sin\pi x)^{2m+1}}\sum_{j=0}^{2m}e^{-2i\pi jx} \\
&\times \sum_{\ell=0}^{j}(-1)^{\ell}\sum_{n=0}^{2m-j}(2n)!S(2m,2n)\binom{2m-2n}{\ell}\binom{2n+1}{2j-2\ell} \\
&- \frac{\pi^{2m}e^{2i\pi mx}}{2^{2m}(\sin\pi x)^{2m+1}}\sum_{j=0}^{2m}e^{-2i\pi jx} \\
&\times \sum_{\ell=0}^{j}(-1)^{\ell}\sum_{n=0}^{2m-j}(2n+1)!S(2m,2n+1)\binom{2m-2n-1}{\ell}\binom{2n+2}{2j-2\ell} \\
&= \frac{\pi^{2m}e^{2i\pi mx}}{2^{2m}(\sin\pi x)^{2m+1}}\sum_{j=0}^{2m}e^{-2i\pi jx} \\
&\times \sum_{\ell=0}^{j}(-1)^{\ell}\sum_{n=0}^{4m-2j+1}n!S(2m,n)(-1)^{n}\binom{2m-n}{\ell}\binom{n+1}{2j-2\ell} \\
&= \frac{\pi^{2m}e^{2i\pi mx}}{2^{2m}(\sin\pi x)^{2m+1}}\sum_{j=0}^{2m}\alpha_{2m,j}e^{-2i\pi jx}.
\end{aligned}$$

The proof for odd values of k can be performed in a similar way.

Theorem 3.4. *The following relations for $k\geq 0$ are true*

$$\check{B}_{k,0}=\frac{1}{2(i\pi(2N+1))^{k+1}}\sum_{r\neq 0}\frac{(-1)^{r+1}}{r^{k+1}},$$

$$\check{B}_{k,n}=\frac{(-1)^{n+k+1}}{(2i(2N+1))^{k+1}k!\left(\sin\frac{\pi n}{2N+1}\right)^{k+1}}\sum_{j=0}^{k}\alpha_{k,j}e^{i\frac{\pi n(k-2j)}{2N+1}},\ n\neq 0,$$

where $\alpha_{k,j}$ are defined in Lemma 3.3.

Proof. The first relation immediately follows from the well-known formula (see [22])

$$\check{B}_{k,n}=\sum_{r=-\infty}^{\infty}B_{k,n+r(2N+1)}$$

and the equation (2.2).

For the second relation we write down

$$\begin{aligned}\check{B}_{k,n} &= \sum_{r=-\infty}^{\infty} \frac{(-1)^{n+r+1}}{2(i\pi(n+r(2N+1)))^{k+1}} \\ &= \frac{(-1)^{n+1}}{2(i\pi(2N+1))^{k+1}} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{\left(\frac{n}{2N+1}+r\right)^{k+1}} \\ &= \frac{(-1)^{n+1}}{2(i\pi(2N+1))^{k+1}} \varphi_k\left(\frac{n}{2N+1}\right),\end{aligned}$$

where

$$\varphi_k(x) := \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(x+r)^{k+1}}.$$

It is easy to verify that

$$\varphi_0^{(k)}(x) = (-1)^k k! \varphi_k(x).$$

Taking into account that

$$\varphi_0(x) = \frac{\pi}{\sin \pi x}$$

we get

$$\varphi_k(x) = \frac{(-1)^k}{k!} \varphi_0^{(k)}(x) = \pi \frac{(-1)^k}{k!} \left(\frac{1}{\sin \pi x}\right)^{(k)}$$

and hence

$$\check{B}_{k,n} = \frac{(-1)^{n+k+1}\pi}{2k!(i\pi(2N+1))^{k+1}} \left(\frac{1}{\sin \pi x}\right)^{(k)}_{x=\frac{n}{2N+1}}, \qquad n \neq 0. \tag{3.2}$$

This completes the proof in view of Lemma 3.3.

For practical realization of the Krylov-Lanczos interpolation the discrete Fourier coefficients $\check{B}_{k,n}$ can be calculated by FFT algorithm but application of Theorem 3.4 provides the same with less complexity and greater accuracy.

4 Computation of the “Jumps". The Eckhoff Interpolation

In this section we investigate the problem of the “jump" approximation via discrete Fourier coefficients. Following Eckhoff we consider the system of linear equations with unknowns $A_k^i(f,N)$ that as we will show below approximate the exact values of the “jumps" $A_k(f)$

$$\check{f}_n = \sum_{k=0}^{q-1} A_k^i(f,N) \check{B}_{k,n}, \qquad n = n_1, n_2, \ldots, n_q. \tag{4.1}$$

Approximation by $I_{N,q}(f)$ where the exact values of the "jumps" are replaced by the approximated ones, calculated from (4.1), we call the *Eckhoff interpolation* and denote by $\widetilde{I}_{N,q}(f)$

$$\widetilde{I}_{N,q}(f) := \sum_{n=-N}^{N}\left(\check{f}_n - \sum_{k=0}^{q-1} A_k^i(f,N)\check{B}_{k,n}\right)e^{i\pi nx} + \sum_{k=0}^{q-1} A_k^i(f,N)B_k(x).$$

The main contribution of this paper is a study of (4.1), calculation of its determinant, obtaining the conditions that provide existence and uniqueness of the solution (see Theorem 4.3). As a result the asymptotic errors of the "jumps" approximation (Theorems 5.2, 5.6) and the corresponding errors of the Eckhoff interpolation (Theorems 5.5, 5.9) are derived. Moreover, we show that (4.1) is equivalent to a system with an upper triangular matrix and therefore $A_k^i(f,N)$ can be calculated by backward substitution (see the equation (4.9)).

First we prove some preliminary lemmas.

Lemma 4.1. *The following identity is for $k = 0,\cdots,q-1$ is valid*

$$\sin^q \pi x\left(\frac{1}{\sin \pi x}\right)^{(k)} = \frac{(i\pi)^k}{(2i)^{q-1}} e^{i\pi(q-1)x} \sum_{j=0}^{q-1} \beta_{k,j} e^{-2i\pi jx},$$

where

$$\beta_{k,j} := \sum_{\ell=0}^{j} (-1)^\ell \sum_{n=0}^{q-1} n!(-1)^n S(k,n)\binom{q-n-1}{\ell}\binom{n+1}{2j-2\ell}.$$

Proof. Starting as in the proof of Lemma 3.3 we get

$$\begin{aligned}
\sin^q \pi x\left(\frac{1}{\sin \pi x}\right)^{(k)} &= \frac{i(i\pi)^k}{(2i)^q}\sum_{n=0}^{k}(-1)^n S(k,n)n! e^{i\pi nx} \\
&\times \ (e^{i\pi x} - e^{-i\pi x})^{q-n-1}((1+e^{-i\pi x})^{n+1} + (1-e^{-i\pi x})^{n+1}) \\
&= \frac{i(i\pi)^k}{(2i)^q} e^{i\pi x(q-1)} \sum_{n=0}^{k}(-1)^n S(k,n)n! \sum_{\ell=0}^{q-n-1}(-1)^\ell e^{-2i\pi \ell x}\binom{q-n-1}{\ell} \\
&\times \ \sum_{s=0}^{n+1}\binom{n+1}{s}(1+(-1)^s)e^{-i\pi sx}.
\end{aligned}$$

Suppose that k is even, $k = 2m$

$$\begin{aligned}
\sin^q \pi x \left(\frac{1}{\sin \pi x}\right)^{(2m)} &= \frac{(i\pi)^{2m}}{(2i)^{q-1}} e^{i\pi(q-1)x} \sum_{n=0}^{m} S(2m,2n)(2n)! \\
&\quad \times \sum_{\ell=0}^{q-2n-1} \binom{q-2n-1}{\ell} (-1)^\ell e^{-2i\pi\ell x} \sum_{s=0}^{n} \binom{2n+1}{2s} e^{-2i\pi s x} \\
&\quad - \frac{(i\pi)^{2m}}{(2i)^{q-1}} e^{i\pi(q-1)x} \sum_{n=0}^{m-1} S(2m,2n+1)(2n+1)! \\
&\times \sum_{\ell=0}^{q-2n-2} \binom{q-2n-2}{\ell} (-1)^\ell e^{-2i\pi\ell x} \sum_{s=0}^{n+1} \binom{2n+2}{2s} e^{-2i\pi s x} \\
&= \frac{(i\pi)^{2m} e^{i\pi(q-1)x}}{(2i)^{q-1}} \sum_{j=0}^{q-1} e^{-2i\pi jx} \\
&\times \sum_{\ell=0}^{j} (-1)^\ell \sum_{n=0}^{q-j-1} (2n)! S(2m,2n) \binom{q-2n-1}{\ell} \binom{2n+1}{2j-2\ell} \\
&\quad - \frac{(i\pi)^{2m} e^{i\pi(q-1)x}}{(2i)^{q-1}} \sum_{j=0}^{q-1} e^{-2i\pi jx} \sum_{\ell=0}^{j} (-1)^\ell \sum_{n=0}^{q-j-1} (2n+1)! \\
&\quad \times S(2m,2n+1) \binom{q-2n-2}{\ell} \binom{2n+2}{2j-2\ell} \\
&= \frac{(i\pi)^{2m} e^{i\pi(q-1)x}}{(2i)^{q-1}} \sum_{j=0}^{q-1} e^{-2i\pi jx} \\
&\quad \times \sum_{\ell=0}^{j} (-1)^\ell \sum_{n=0}^{2q-2j-1} n! S(2m,n)(-1)^n \binom{q-n-1}{\ell} \binom{n+1}{2j-2\ell} \\
&= \frac{(i\pi)^{2m} e^{i\pi(q-1)x}}{(2i)^{q-1}} \sum_{j=0}^{q-1} \beta_{2m,j} e^{-2i\pi jx}.
\end{aligned}$$

The case of odd k can be proved similarly.

Denote

$$V := (v_{ij})_{i,j=0}^{q-1}, \qquad v_{ij} := \sum_{s=0}^{i} (-1)^s \binom{q-j-1}{s} \binom{j+1}{2i-2s}. \tag{4.2}$$

The next lemma provides the LU-factorization of the matrix V.

Lemma 4.2. *The matrix V has the following LU-factorization*

$$V = LU,$$

where

$$L := (\ell_{ij})_{i,j=0}^{q-1}, \qquad U := (u_{ij})_{i,j=0}^{q-1},$$

$$\ell_{ij} := (-1)^i \binom{q-j-1}{q-i-1},\ u_{ij} := (-1)^i \sum_{k=0}^{q-1} \binom{j+1}{2k} \binom{j-k}{i-k}.$$

Proof. We have

$$\begin{aligned}
\sum_{s=0}^{q-1} \ell_{is} u_{sj} &= (-1)^i \sum_{k=0}^{q-1} \binom{j+1}{2k} \sum_{s=0}^{q-1} (-1)^s \binom{q-s-1}{q-i-1} \binom{j-k}{s-k} \\
&= (-1)^{i+q+1} \sum_{k=0}^{q-1} \binom{j+1}{2q-2-2k} \sum_{s=0}^{k} (-1)^s \binom{s}{q-i-1} \binom{j-q+k+1}{k-s} \\
&= (-1)^{i+q+1} \sum_{k=0}^{q-1} (-1)^k \binom{j+1}{2q-2-2k} \sum_{s=0}^{k} (-1)^s \binom{k-s}{q-i-1} \binom{j-q+k+1}{s}.
\end{aligned}$$

According to the identity ([19])

$$\binom{n-p}{n-m} = \sum_{k=0}^{n} (-1)^k \binom{n-k}{m} \binom{p}{k}$$

we derive

$$\begin{aligned}
\sum_{s=0}^{q-1} \ell_{is} u_{sj} &= (-1)^{i+q+1} \sum_{k=0}^{q-1} (-1)^k \binom{j+1}{2q-2-2k} \binom{q-1-j}{k-q+1+i} \\
&= \sum_{k=0}^{i} (-1)^{k+i} \binom{j+1}{2k} \binom{q-1-j}{i-k} \\
&= \sum_{k=0}^{i} (-1)^k \binom{j+1}{2i-2k} \binom{q-1-j}{k} = v_{ij}.
\end{aligned}$$

This ends the proof if we observe that L and U are lower and upper triangular matrices, respectively.

In view of the identity (3.2) we rewrite the system (4.1) in the form

$$\sum_{k=0}^{q-1} A_k^i(f,N) \frac{(-1)^k \pi}{2k!(i\pi(2N+1))^{k+1}} \sin^q \pi\tau_s \left(\frac{1}{\sin \pi x}\right)^{(k)}_{x=\tau_s} = (-1)^{n_s+1} \sin^q \pi\tau_s \check{f}_{n_s},$$
$$\tau_s := \frac{n_s}{2N+1},\ s = 1, \cdots, q.$$

An application of Lemma 4.1 implies the following system of linear equations that is equivalent to the system (4.1)

$$\sum_{k=0}^{q-1} m_{sk} t_k = y_s,\ s = 1, \cdots, q, \tag{4.3}$$

where

$$y_s := (2i)^q (-1)^{n_s+1} \check{f}_{n_s} \sin^q \pi\tau_s e^{-i\pi(q-1)\tau_s},$$

$$t_k := \frac{(-1)^k}{k!(2N+1)^{k+1}} A_k^i(f,N),$$

$$m_{sk} := \sum_{j=0}^{q-1} w_{sj} \sum_{n=0}^{q-1} v_{jn} s_{nk}, \qquad w_{sj} := e^{-2i\pi\tau_s j}, \qquad s_{nk} := n!(-1)^n S(k,n),$$

and v_{ij} is defined by the equation (4.2). The system (4.3) can be rewritten also in the matrix form

$$MT = Y, \tag{4.4}$$

where

$$T := (t_0, \cdots, t_{q-1})^T, Y := (y_1, \cdots, y_q)^T, M := (m_{ij}) = WVS$$

with

$$S := (s_{ij}), W := (w_{ij}).$$

An application of Lemma 4.2 yields the matrix factorization of M

$$M = WLUS. \tag{4.5}$$

Note that W is a Vandermonde matrix and S is an upper triangular matrix.

Factorization (4.5) will help to calculate the determinant of M in the next theorem.

Theorem 4.3. *The system* (4.3)*, and hence* (4.1)*, has a unique solution, provided that the values* $n_1, \cdots, n_q$ *are distinct.*

Proof. The factorization (4.5) implies

$$\det M = \det W \cdot \det L \cdot \det U \cdot \det S.$$

The matrices L, U and S are triangular. Whence

$$\begin{aligned}
\det L &= \prod_{j=0}^{q-1} \ell_{jj} = \prod_{j=0}^{q-1} (-1)^j = (-1)^{\frac{q(q-1)}{2}}, \\
\det U &= \prod_{j=0}^{q-1} u_{jj} = \prod_{j=0}^{q-1} \left((-1)^j \sum_{k=0}^{q-1} \binom{j+1}{2k} \right) \\
&= \prod_{j=0}^{q-1} (-1)^j 2^j = (-2)^{\frac{q(q-1)}{2}}, \\
\det S &= \prod_{j=0}^{q-1} s_{jj} = \prod_{j=0}^{q-1} (-1)^j j! = (-1)^{\frac{q(q-1)}{2}} \prod_{j=0}^{q-1} j!.
\end{aligned}$$

Taking into account that W is a Vandermonde matrix, we get

$$\det W = \prod_{j=1}^{q} \prod_{k=j+1}^{q} \left(e^{-2i\pi\tau_k} - e^{-2i\pi\tau_j}\right).$$

Finally

$$\det M = (-1)^{\frac{q(q-1)}{2}} 2^{\frac{q(q-1)}{2}} \prod_{j=1}^{q} \prod_{k=j+1}^{q} \left(e^{-2i\pi\tau_k} - e^{-2i\pi\tau_j}\right) \prod_{s=0}^{q-1} s!.$$

This completes the proof.

We continue investigation of the matrix M. In the next two lemmas the inverses of the matrices L and W are calculated.

Lemma 4.4. *The following relation is true*

$$L^{-1} = L.$$

Proof. We will show that $L^2 = I$, where I is the identity matrix. We have

$$\begin{aligned} \sum_{j=0}^{q-1} \ell_{nj}\ell_{jm} &= \sum_{j=m}^{n} (-1)^n \binom{q-j-1}{q-n-1} (-1)^j \binom{q-m-1}{q-j-1} \\ &= (-1)^n \binom{q-m-1}{q-n-1} \sum_{j=m}^{n} (-1)^j \binom{n-m}{j-m} \\ &= (-1)^{n+m} \binom{q-m-1}{q-n-1} \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{j} \\ &= (-1)^{n+m} \binom{q-m-1}{q-n-1} \delta_{0,n-m} = \delta_{n,m}. \end{aligned}$$

Denote by w_{kj}^{-1} the elements of the matrix W^{-1}. The next lemma gives the explicit form of w_{kj}^{-1}.

Lemma 4.5. *Suppose that the indices n_s, $s = 1, \cdots, q$ are distinct. Then for the elements w_{kj}^{-1} the following representation is true*

$$w_{kj}^{-1} = \frac{1}{\prod\limits_{\substack{m=1 \\ m\neq j}}^{q} \left(e^{-2i\pi\tau_j} - e^{-2i\pi\tau_m}\right)} \sum_{s=k+1}^{q} \gamma_s e^{-2i\pi\tau_j(s-k-1)},$$

$$\tau_s := \frac{n_s}{2N+1}, \qquad j = 1, \cdots, q; \qquad k = 0, \cdots, q-1,$$

where the numbers γ_s are the coefficients of the polynomial

$$\prod_{n=1}^{q} \left(e^{-2i\pi x} - e^{-2i\pi\tau_n}\right) = \sum_{s=0}^{q} \gamma_s e^{-2i\pi s x}.$$

Proof. As in [2] we consider the following trigonometric polynomial

$$Q_j(x) := \prod_{\substack{n=1 \\ n\neq j}}^{q} \frac{e^{-2i\pi x} - e^{-2i\pi\tau_n}}{e^{-2i\pi\tau_j} - e^{-2i\pi\tau_n}} = \sum_{k=0}^{q-1} \rho_{jk} e^{-2i\pi kx}, \qquad j = 1,\cdots,q,$$

where by ρ_{jk} we denote the coefficients of the polynomial $Q_j(x)$. From the equations

$$Q_j(\tau_i) = \sum_{k=1}^{q} \rho_{jk} e^{-2i\pi\tau_i k} = \delta_{ij}, \qquad i,j = 1,\cdots,q$$

we see that the transpose of (ρ_{jk}) is the inverse of the Vandermonde matrix $(e^{-2i\pi\tau_i k})$. Following [2], we write

$$\rho_{jk} = -\frac{1}{\prod\limits_{\substack{m=1 \\ m\neq j}}^{q} \left(e^{-2i\pi\tau_j} - e^{-2i\pi\tau_m}\right)} \sum_{s=0}^{k} \gamma_s e^{-2i\pi\tau_j(s-k-1)}, \qquad k = 0,\cdots,q-1; \quad j = 1,\cdots,q.$$

This ends the proof as $w_{kj}^{-1} = \rho_{jk}$.

Denoting $P = US$, from (4.5) we get the factorization

$$M = WLP, \tag{4.6}$$

where P is an upper triangular matrix. Applying Lemmas 4.4 and 4.5 we can rewrite the system (4.4) in the equivalent form

$$PT = L^{-1}W^{-1}Y = LW^{-1}Y, \tag{4.7}$$

or elementwise

$$\sum_{k=0}^{q-1} p_{jk} t_k = \widetilde{y}_j, \qquad j = 0,\cdots,q-1, \tag{4.8}$$

where

$$\begin{aligned} \widetilde{y}_j \quad &:= \quad \sum_{k=0}^{q-1} \ell_{jk} \sum_{r=1}^{q} w_{kr}^{-1} y_r = (2i)^q \sum_{k=0}^{q-1} (-1)^j \binom{q-k-1}{q-j-1} \\ &\times \quad \sum_{r=1}^{q} \frac{(-1)^{n_r+1} \check{f}_{n_r} e^{-i\pi(q-1)\tau_r} \sin^q \pi\tau_r}{\prod\limits_{\substack{m=1 \\ m\neq r}}^{q} \left(e^{-2i\pi\tau_k} - e^{-2i\pi\tau_m}\right)} \sum_{s=r+1}^{q} \gamma_s e^{-2i\pi\tau_k(s-r-1)} \end{aligned}$$

and p_{jk} are the elements of the matrix P. Now $A_k^i(f,N)$ can be derived by backward substitution from (4.8)

$$\begin{aligned} A_k^i(f,N) \quad &= \quad \frac{k!(-1)^k(2N+1)^{k+1}}{p_{kk}} \widetilde{y}_k \\ &- \sum_{j=k+1}^{q-1} A_j^i(f,N) \frac{(-1)^{j+k} k!}{j!(2N+1)^{j-k}} \frac{p_{kj}}{p_{kk}}, \qquad k = 0,\cdots,q-1. \end{aligned} \tag{4.9}$$

5 Asymptotic Estimates

Hereafter we will suppose that the indices n_s are distinct and

$$\rho N \leq |n_s| \leq N, \qquad s = 1, \ldots, q, \quad 0 < \rho \leq 1. \tag{5.1}$$

In this section we derive asymptotic estimates of the "jumps" approximation and the corresponding asymptotic errors of the Eckhoff interpolation.

Denote

$$\omega_{r,m} := \sum_{s=1}^{q} \frac{e^{-2i\pi m\tau_s}}{(1-e^{-2i\pi\tau_s})^{r-q+1} \prod\limits_{\substack{n=1 \\ n\neq s}}^{q} (e^{-2i\pi\tau_s} - e^{-2i\pi\tau_n})}, \qquad r \geq q, \quad m \geq 0, \tag{5.2}$$

where $\tau_s := n_s/(2N+1)$, $\tau_i \neq \tau_j$, $i \neq j$. Evidently

$$\omega_{r,m} = \frac{1}{2\pi i} \int_\Gamma \frac{z^m}{(1-z)^{r-q+1} \prod_{n=1}^{q} (z - e^{-2i\pi\tau_n})} \mathrm{d}z, \tag{5.3}$$

where Γ is a closed curve containing the points $z = e^{-2i\pi\tau_s}$, $s = 1, \cdots, q$, $N \geq 1$ but not containing the point $z = 1$. In particular, we put

$$\begin{aligned}
&\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\
&\Gamma_1 : z = Re^{i\varphi}, \frac{1}{3}\pi\rho \leq \varphi \leq \tfrac{1}{3}\pi(6-\rho), R > 1, \\
&\Gamma_2 : z = re^{i\varphi}, \frac{1}{3}\pi\rho \leq \varphi \leq \tfrac{1}{3}\pi(6-\rho), r < 1, \\
&\Gamma_3 : z = te^{i\frac{5}{6}\pi\rho}, r \leq t \leq R, \\
&\Gamma_4 : z = te^{-i\frac{5}{6}\pi\rho}, r \leq t \leq R.
\end{aligned} \tag{5.4}$$

Lemma 5.1. *Suppose that the indices $n_s = n_s(N)$ are distinct and* (5.1) *is true. Then*

$$\omega_{r,m} = O(1), \qquad N \to \infty.$$

Proof. The proof follows from (5.1) as

$$\frac{2}{3}\pi\rho \leq 2\pi|\tau_s| \leq \pi.$$

The next theorem explores the accuracy of the "jumps" approximation.

Theorem 5.2. *Suppose that the indices n_s, $s=1,\cdots,q$ are distinct and*

$$\lim_{N\to\infty}\frac{n_s}{2N+1}=\lim_{N\to\infty}\tau_s=c_s\neq 0,\quad s=1,\cdots,q.$$

Let ε ($\varepsilon\geq 1$) be the greatest multiplicity of a number in the sequence $\{e^{-2i\pi c_s}\}$, $s=1,\ldots,q$. Then, for $f\in C^{q+\varepsilon-1}[-1,1]$ such that $f^{(q+\varepsilon-1)}$ is absolutely continuous on $[-1,1]$, the following estimate as $N\to\infty$ holds

$$A_j^i(f,N)=A_j(f)+A_q(f)\frac{(-1)^j j!\nu_j}{(2N+1)^{q-j}}+o(N^{-q+j}),\qquad j=0,\ldots,q-1,$$

where the numbers ν_j are defined by the recurrence relation

$$\nu_j:=\frac{\mu_j}{p_{jj}}-\sum_{k=j+1}^{q-1}\nu_k\frac{p_{jk}}{p_{jj}},\qquad j=0,\cdots,q-1$$

and

$$\mu_s:=\frac{(-1)^q}{q!}\sum_{j=0}^{q-1}\ell_{sj}\sum_{m=0}^{q}\alpha_{q,m}\sum_{k=j+1}^{q}\gamma_k^*\omega_{q,m+k-j-1}^*.$$

The numbers $\omega_{r,m}^*$ are defined by the formula (see (5.4))

$$\omega_{r,m}^*:=\frac{1}{2\pi i}\int_\Gamma\frac{z^m}{(1-z)^{r-q+1}\prod_{n=1}^{q}(z-e^{-2i\pi c_n})}\mathrm{d}z$$

and γ_k^* are the coefficients of the polynomial

$$\prod_{n=1}^{q}\left(e^{-2i\pi x}-e^{-2i\pi c_n}\right)=\sum_{s=0}^{q}\gamma_s^* e^{-2i\pi sx}.$$

Proof. In view of Lemma 2.1, we have the representation

$$f(x)=F(x)+\sum_{k=0}^{q+\varepsilon-1}A_k(f)B_k(x),$$

where F is a 2-periodic and smooth function $F\in C^{q+\varepsilon-1}(R)$. This representation implies

$$\check{f}_n=\check{F}_n+\sum_{k=0}^{q+\varepsilon-1}A_k(f)\check{B}_{k,n}.$$

Taking into account that

$$\check{F}_n=\sum_{s=-\infty}^{\infty}F_{n+s(2N+1)}$$

we get

$$\check{F}_n=o(n^{-q-\varepsilon}),\qquad n\to\infty,\quad |n|\leq N$$

and therefore

$$\check{f}_n = \sum_{k=0}^{q+\varepsilon-1} A_k(f)\check{B}_{k,n} + o(n^{-q-\varepsilon}),\ n\to\infty, \qquad |n|\le N.$$

Using this in (4.1), we derive

$$\sum_{k=0}^{q-1}(A_k^i(f,N)-A_k(f))\check{B}_{k,n_s} = \sum_{k=q}^{q+\varepsilon-1} A_k(f)\check{B}_{k,n_s} + o(n_s^{-q-\varepsilon}), \qquad s=1,\cdots,q.$$

An application of Theorem 3.4 and the factorization (4.6) implies

$$\begin{aligned}\sum_{k=0}^{q-1}(WLP)_{sk}z_k &= \sum_{r=q}^{q+\varepsilon-1} A_r(f)\frac{(-1)^r}{r!(2N+1)^{r+1}}\frac{1}{(1-e^{-2i\pi\tau_s})^{r+1-q}}\\ &\quad\times\sum_{m=0}^{r}\alpha_{r,m}e^{-2i\pi m\tau_s} + o(n_s^{-q-\varepsilon}), \quad s=1,\cdots,q,\ N\to\infty,\end{aligned}$$

where

$$z_k := \frac{(-1)^k}{k!(2N+1)^{k+1}}(A_k^i(f,N)-A_k(f)). \tag{5.5}$$

Applying Lemma 4.5 we write down

$$\begin{aligned}\sum_{k=0}^{q-1}(LP)_{jk}z_k &= \sum_{r=q}^{q+\varepsilon-1} A_r(f)\frac{(-1)^r}{r!(2N+1)^{r+1}}\sum_{m=0}^{r}\alpha_{r,m}\sum_{t=j+1}^{q}\gamma_t\omega_{r,m+t-j-1}\\ &\quad+\sum_{s=1}^{q}\frac{o(n_s^{-q-\varepsilon})}{\prod\limits_{\substack{m=1\\ m\ne s}}^{q}(e^{-2i\pi\tau_s}-e^{-2i\pi\tau_m})}, \qquad j=0,\cdots,q-1,\ N\to\infty.\end{aligned} \tag{5.6}$$

For the last term we note that

$$\left|\prod_{\substack{m=1\\ m\ne s}}^{q}(e^{-2i\pi\tau_s}-e^{-2i\pi\tau_m})\right| = \left|\prod_{\substack{m=1\\ m\ne s}}^{q}2i\sin\pi(\tau_m-\tau_s)\right| = O(N^{-\varepsilon+1}), \qquad N\to\infty.$$

Hence the last term in (5.6) is $o(N^{-q-1})$ as $N\to\infty$. By Lemma 5.2 the terms with $r>q$ are also $o(N^{-q-1})$ as $\gamma_t=O(1)$ and we need to consider only the term $r=q$. Taking into account that $\gamma_s=\gamma_s^*+o(1)$ and $\omega_{q,m+t-j-1}=\omega^*_{q,m+t-j-1}+o(1)$ as $N\to\infty$, we get

$$\sum_{k=0}^{q-1}(LP)_{jk}z_k = A_q(f)\frac{(-1)^q}{q!(2N+1)^{q+1}}\sum_{m=0}^{q}\alpha_{q,m}\sum_{k=j+1}^{q}\gamma_k^*\omega^*_{q,m+k-j-1}+o(N^{-q-1}).$$

Lemma 4.4 implies

$$\sum_{k=s}^{q-1}p_{sk}z_k = A_q(f)\frac{\mu_s}{(2N+1)^{q+1}}+o(N^{-q-1}), \qquad N\to\infty.$$

From here we derive

$$z_k = A_q(f)\frac{\nu_k}{(2N+1)^{q+1}} + o(N^{-q-1}), \qquad N\to\infty, k=0,\cdots,q-1.$$

This finalizes the proof.

This theorem allows to estimate the accuracy of the Eckhoff interpolation.

We put

$$\check{G}_n := \check{f}_n - \sum_{k=0}^{q-1} A_k^i(f,N)\check{B}_{k,n}, \tag{5.7}$$

$$G_n := f_n - \sum_{k=0}^{q-1} A_k^i(f,N)B_{k,n}. \tag{5.8}$$

In the next two lemmas we explore the asymptotic behavior of $\check{G}_n$ and G_n.

Lemma 5.3. *Suppose that the conditions of Theorem* 5.2 *are valid. Then the following asymptotic estimate as* $N\to\infty,\ |n|>N$ *is true*

$$G_n = A_q(f)\frac{(-1)^n}{2(2N+1)^{q+1}}\sum_{k=0}^{q}\frac{k!(-1)^k\nu_k}{\left(i\pi\frac{n}{2N+1}\right)^{k+1}} + \frac{o(N^{-q})(-1)^n}{n},$$

where the numbers ν_k, $k=0,\cdots,q-1$ *are defined in Theorem* 5.2 *and* $\nu_q=(-1)^{q+1}/q!$.

Proof. Lemma 2.1 implies

$$f_n = \sum_{k=0}^{q-1}A_k(f)B_{k,n} + A_q(f)B_{q,n} + o(n^{-q-1}), \qquad n\to\infty.$$

From here and (5.8) we obtain

$$G_n = \sum_{k=0}^{q-1}\left(A_k(f)-A_k^i(f,N)\right)B_{k,n} + A_q(f)B_{q,n} + o(n^{-q-1}), \qquad n\to\infty. \tag{5.9}$$

Theorem 5.2 yields

$$\begin{aligned} G_n &= A_q(f)\frac{(-1)^n}{2(2N+1)^{q+1}}\sum_{k=0}^{q-1}\frac{k!(-1)^k\nu_k}{\left(i\pi\frac{n}{2N+1}\right)^{k+1}} + A_q(f)B_{q,n} \\ &+ o(n^{-q-1}) + o(N^{-q-1})(-1)^n\sum_{k=0}^{q-1}\frac{1}{\left(i\pi\frac{n}{N}\right)^{k+1}}. \end{aligned}$$

This completes the proof.

Lemma 5.4. *Suppose that the conditions of Theorem* 5.2 *are valid. Then the following asymptotic estimate as* $N\to\infty$, $|n|\le N$ *is true*

$$\check{G}_n - G_n = A_q(f)\frac{(-1)^n}{2(2N+1)^{q+1}}\sum_{k=0}^{q}\frac{k!(-1)^k\nu_k}{(i\pi)^{k+1}}\sum_{s\neq 0}\frac{(-1)^s}{\left(\frac{n}{2N+1}+s\right)^{k+1}} + o(N^{-q-1}),$$

where the numbers ν_k, $k=0,\cdots,q-1$ *are defined in Theorem* 5.2 *and* $\nu_q=(-1)^{q+1}/q!$.

Proof. The proof follows from the formula

$$\check{G}_n-G_n=\sum_{s=-\infty}^{\infty}G_{n+s(2N+1)}-G_n=\sum_{s\neq 0}G_{n+s(2N+1)} \tag{5.10}$$

in view of Lemma 5.3.

Denote

$$\widetilde{r}_{N,q}(f):=f(x)-\widetilde{I}_{N,q}(f).$$

The next theorem reveals the asymptotic behavior of the Eckhoff interpolation.

Theorem 5.5. *Suppose that the conditions of Theorem* 5.2 *are valid. Then*

$$\begin{aligned}
&\lim_{N\to\infty}(2N+1)^{2q+1}\|\widetilde{r}_{N,q}(f)\|^2=a_2(q)|A_q(f)|,\\
&a_2(q):=\left(\frac{1}{2}\int_{-1/2}^{1/2}\left|\sum_{k=0}^{q}\frac{k!(-1)^k\nu_k}{(i\pi)^{k+1}}\sum_{s\neq 0}\frac{(-1)^s}{(x+s)^{k+1}}\right|^2\mathrm{d}x\right.\\
&\qquad\left.+\frac{1}{2}\int_{|x|>1/2}\left|\sum_{k=0}^{q}\frac{k!(-1)^k\nu_k}{(i\pi x)^{k+1}}\right|^2\mathrm{d}x\right)^{\frac{1}{2}},
\end{aligned}$$

where the numbers ν_k, $k=0,\cdots,q-1$ *are defined in Theorem* 5.2 *and* $\nu_q=(-1)^{q+1}/q!$.

Proof. The proof follows from the formula

$$\|\widetilde{r}_{N,q}(f)\|^2=2\sum_{n=-N}^{N}|G_n-\check{G}_n|^2+2\sum_{|n|>N}|G_n|^2 \tag{5.11}$$

and Lemmas 5.3, 5.4.

Omitting any condition on the indices n_s besides (5.1) we are still able to calculate the convergence rate of A_j^i and the corresponding interpolation. The coming theorems of this section illustrate these facts.

Theorem 5.6. *Suppose that* $f\in C^{2q-1}[-1,1]$ *and* $f^{(2q-1)}$ *is absolutely continuous on* $[-1,1]$ *for some* $q\geq 1$. *Then, if the indices* n_s *are distinct and* (5.1) *is valid, the following estimate holds*

$$A_j^i(f,N)=A_j(f)+A_q(f)O(N^{-q+j}),\qquad j=0,\ldots,q-1,\qquad N\to\infty.$$

Proof. The proof mimics that of Theorem 5.2. Starting as there, replacing ε by q, we get

$$\begin{aligned}\sum_{k=0}^{q-1}(LP)_{jk}z_k &= \sum_{r=q}^{2q-1}A_r(f)\frac{(-1)^r}{r!(2N+1)^{r+1}}\sum_{m=0}^{r}\alpha_{r,m}\sum_{t=j+1}^{q}\gamma_t\omega_{r,m+t-j-1}\\ &\quad+\sum_{s=1}^{q}\frac{o(n_s^{-2q})}{\prod\limits_{\substack{n=1\\n\neq s}}^{q}(e^{-2i\pi\tau_s}-e^{-2i\pi\tau_n})}.\end{aligned}\tag{5.12}$$

For the last term we observe that

$$\left|\prod_{\substack{n=1\\n\neq s}}^{q}(e^{-2i\pi\tau_s}-e^{-2i\pi\tau_n})\right|=O(N^{-q+1}),\qquad N\to\infty.$$

Hence the last term in (5.12) is $o(N^{-q-1})$ as $N\to\infty$. In view of Lemma 5.2 we get

$$\sum_{k=0}^{q-1}(LP)_{jk}z_k=A_q(f)O(N^{-q-1}),\qquad N\to\infty,$$

where z_k is defined by (5.5). Lemma 4.4 implies

$$\sum_{k=s}^{q-1}p_{sk}z_k=A_q(f)O(N^{-q-1}),\qquad N\to\infty.$$

From here we derive

$$z_k=A_q(f)O(N^{-q-1}),\qquad N\to\infty,\quad k=0,\cdots,q-1.$$

This finishes the proof.

Now, we will estimate the accuracy of the Eckhoff interpolation similarly. First we given some auxiliary lemmas.

Lemma 5.7. *Suppose that the conditions of Theorem* 5.6 *are valid. Then the following estimate holds*

$$G_n=A_q(f)O(N^{-q})\frac{(-1)^n}{n},\qquad n\to\infty,\quad |n|>N.$$

Proof. The proof immediately follows from the equation (5.9) and Theorem 5.6.

Lemma 5.8. *Suppose that the conditions of Theorem* 5.6 *are valid. Then*

$$\check{G}_n-G_n=A_q(f)O(N^{-q-1}),\qquad N\to\infty,\quad |n|\le N.$$

Proof. The proof follows from (5.10) and Lemma 5.7.

Finally we get the required estimate.

Theorem 5.9. *Suppose that the conditions of Theorem* 5.6 *are valid. Then*

$$\|\widetilde{r}_{N,q}(f)\|=|A_q(f)|O(N^{-q-\frac{1}{2}}),\qquad N\to\infty.$$

Proof. The equation (5.11) completes the proof, due to Lemmas 5.7 and 5.8.

6 Numerical Results

Hereafter we consider a special choice of the indices n_s. If q is an even number $q = 2m$, $m = 1, 2, \cdots$, then we put

$$\begin{aligned} n_s &= N - s + 1, \qquad s = 1, \cdots, m, \\ n_s &= -(N - s + m + 1), \qquad s = m+1, \cdots, 2m. \end{aligned} \tag{6.1}$$

If q is odd, $q = 2m+1$, $m = 0, 1, \cdots$ then we put

$$\begin{aligned} n_s &= N - s + 1, \quad s = 1, \cdots, m+1, \\ n_s &= -(N - s + m + 2), \quad s = m+2, \cdots, 2m+1. \end{aligned} \tag{6.2}$$

In Table 1 we show the numerical values of the constants $a_1(q)$, $a_2(q)$, and the ratio $a_2(q)/a_1(q)$ for different values of q. The indices n_s are chosen as in (6.1) and (6.2) and the constants $a_1(q)$ and $a_2(q)$ are calculated according to Theorems 3.1 and 5.5. The ratio $a_2(q)/a_1(q)$ shows the deficiency of the Eckhoff interpolation compared to the Krylov-Lanczos interpolation.

Table 1. The numerical values of $a_1(q)$, $a_2(q)$, and the ratio $a_2(q)/a_1(q)$ for different values of q for the above choices of n_s.

q	1	2	3	4	5	6	7	8	9	10
a_1	0.237	0.107	0.063	0.035	0.020	0.012	0.007	0.004	0.003	0.002
a_2	0.237	0.173	0.141	0.122	0.108	0.098	0.090	0.084	0.078	0.074
a_2/a_1	1	1.6	2.3	3.5	5.4	8.3	12.9	20.1	31.2	48.6

By $\|T\|_\infty$ and $\kappa(T)$ denote the norm and the condition number of a matrix T respectively

$$\|T\|_\infty := \max_{1\le i\le q} \sum_{j=1}^{q} |t_{ij}|, \ \kappa(T) := \|T\|_\infty \|T^{-1}\|_\infty,$$

where T^{-1} is the inverse of T and t_{ij} are the elements of T.

In Table 2 the condition numbers of the matrix $\check{B}_{k,n_s}$ are presented for the above choices of the indices n_s and for different values of N and q. We see that this matrix is ill-conditioned. Therefore, the factorization (4.6) will help for solving the system (4.1) effectively. More concrete by taking into account that W is a Vandermonde matrix, a practical solution can be achieved using the well-known Björk-Pereyra algorithm, [3]. This $O(q^2)$ algorithm has a number of beneficial properties. In particular, under certain mild hypotheses the magnitude of the numerical errors depends only on the machine precision used, and is independent of the condition number of the matrix, [9].

Table 2. The condition numbers of the matrix $\check{B}_{k,n_s}$, $k=0,\cdots,q-1$ for different values of q and N

q	2	3	4	5	6	7
$N=32$	2690	4×10^{6}	3×10^{9}	2×10^{12}	1×10^{15}	5×10^{17}
$N=64$	10594	6×10^{7}	2×10^{11}	5×10^{14}	1×10^{18}	2×10^{21}
$N=128$	42048	9×10^{8}	1×10^{13}	1×10^{17}	1×10^{21}	8×10^{24}
$N=256$	167539	1×10^{10}	8×10^{14}	3×10^{19}	1×10^{24}	3×10^{28}
$N=512$	668849	2×10^{11}	5×10^{16}	8×10^{21}	1×10^{27}	1×10^{32}
$N=1024$	3×10^{6}	4×10^{12}	3×10^{18}	2×10^{24}	1×10^{30}	5×10^{35}
$N=2048$	1×10^{7}	6×10^{13}	2×10^{20}	5×10^{26}	1×10^{33}	2×10^{39}
$N=4096$	4×10^{7}	9×10^{14}	1×10^{22}	1×10^{29}	1×10^{36}	9×10^{42}

Consider the following simple function

$$f(x)=\sin(x-1) \tag{6.3}$$

and put

$$\sigma_{q,N}(f) := \left(\frac{1}{q}\sum_{k=0}^{q-1}\left|A_k(f)-A_k^i(f,N)\right|^2\right)^{\frac{1}{2}}.$$

In Table 3 the values of $\sigma_{q,N}(f)$ are calculated for different choices of q and N.

Table 3. Nmerical values of $\sigma_{q,N}(f)$ for different values of q and N when is interpolated

q	2	3	4	5	6	7
$N=32$	0.0002	0.0009	0.0003	0.001	0.0003	0.001
$N=64$	0.00005	0.0002	0.00007	0.0003	0.00008	0.0004
$N=128$	0.00001	0.00006	0.00002	0.00008	0.00002	0.00009
$N=256$	3×10^{-6}	0.00001	5×10^{-6}	0.00001	5×10^{-6}	0.00002
$N=512$	9×10^{-7}	4×10^{-6}	1×10^{-6}	5×10^{-6}	1×10^{-6}	6×10^{-6}
$N=1024$	2×10^{-7}	9×10^{-7}	3×10^{-7}	1×10^{-6}	3×10^{-7}	1×10^{-6}
$N=2048$	5×10^{-8}	2×10^{-7}	7×10^{-8}	3×10^{-7}	9×10^{-8}	4×10^{-7}
$N=4096$	1×10^{-8}	6×10^{-8}	2×10^{-8}	8×10^{-8}	2×10^{-8}	9×10^{-8}

For the function (6.3) denote

$$a_{1,q,N}(f) := \frac{(2N+1)^{q+\frac{1}{2}}}{|A_q(f)|}\|r_{q,N}(f)\|,\ a_{2,q,N}(f) := \frac{(2N+1)^{q+\frac{1}{2}}}{|A_q(f)|}\|\widetilde{r}_{q,N}(f)\|.$$

Table 4 shows the numerical values of the constants $a_{1,q,N}(f)$ and $a_{2,q,N}(f)$ for $N = 32$ and different values of q. A comparison with Table 1 shows that the theoretical estimates coincide with experimental results even for moderate values of N.

Table 4. Numerical values of $a_{1,q,N}(f)$ and $a_{2,q,N}(f)$ for different values of q and $N = 32$ when () is interpolated.

q	1	2	3	4	5	6
$a_{1,q,N}$	0.2372	0.1074	0.0626	0.0344	0.0201	0.0117
$a_{2,q,N}$	0.2375	0.1737	0.1422	0.1228	0.1099	0.1004
$a_{2,q,N}/a_{1,q,N}$	1.001	1.62	2.27	3.564	5.48	8.58

Acknowledgement. I am thankful to Levon Nurbekian for useful discussions while working on this paper.

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