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FOURIER FORMULAE FOR EQUIDISTANT HERMITE TRIGONOMETRIC INTERPOLATION

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A sequence of Hermite trigonometric interpolation polynomials with equidistant interpolation nodes and uniform multiplicities is investigated. We derive relatively compact formula that gives the interpolants as functions of the coefficients in the DFTs of the derivative values. The coefficients can be calculated by the FFT algorithm. Corresponding quadrature formulae are derived and explored. Convergence acceleration based on the Krylov-Lanczos method for accelerating both the convergence of interpolation and quadrature is considered. Exact constants of the asymptotic errors are obtained and some numerical illustrations are presented.

Keywords: Hermite interpolation; trigonometric interpolation; Krylov-Lanczos interpolation; Bernoulli polynomials; convergence acceleration; Hermite-Krylov-Lanczos interpolation; quadrature formula.

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1. Introduction

For a given smooth function $f \in C^{p-1}[-1,1]$ we consider the sequence $T_{p,N}(f)(x)$, $p \geq 1$, $N \geq 1$, of Hermite trigonometric interpolation polynomials with prescribed values

$$T_{p,N}^{(s)}(f)\left(\frac{2k}{2N+1}\right) = f^{(s)}\left(\frac{2k}{2N+1}\right),\ s = 0, \cdots, p-1;\ |k| \leq N.$$

The case $p = 2$ was first treated by Jackson [15]. The usual trigonometric interpolation is included as the particular case $p = 1$ (see Kress [18]).

Salzer [32] considered the general case of full Hermite trigonometric interpolation with non-equidistant interpolation points. Trigonometric divided differences were used by Lyche [23] to derive a trigonometric analog of the Newton form of the Hermite polynomial. Interpolation methods of Hermite type in translation invariant spaces of trigonometric polynomials for any position of interpolation points and any number of derivatives were constructed by Delvos [7]. This approach might be considered as an extension of the method of Salzer and a generalization of the Kress [19] idea to non-equidistant interpolation points with different multiplicities. Dryanov [8] proved the existence and uniqueness of the Hermite trigonometric interpolation polynomial for the general case, any number of interpolation points and multiplicities. Explicit expressions for the quadrature formulae with maximal trigonometric degree of precision were obtained. Jin [16] established constructively the fundamental Hermite polynomials for the general case. Trigonometric and paratrigonometric Hermite interpolation for any number of interpolation points with different multiplicities were constructed by Du, Han and Jin [9].

Hermite trigonometric interpolation on equidistant nodes were discussed by different authors (see, for example, Kress [19], Nersessian [25], Sahakyan [31] and Berrut, Welscher [5] with references therein). Kress derived an explicit form of the Hermite trigonometric interpolation on equidistant grid with uniform multiplicities and obtained a derivative-free remainder. He investigated also the corresponding quadrature formulae.

A new idea has recently come up in Hermite trigonometric interpolation: considering the separate discrete Fourier transforms (DFTs) of the various derivatives of f and then writing the Hermite interpolant in terms of the thereby obtained coefficients. Berrut and Welcher [5] developed a formula for the Fourier coefficients in terms of those of the two classical trigonometric polynomials interpolating the values and those of the derivative separately. That formula treated the most customary case, i.e., the classical Hermite interpolant that used only the first order derivatives at every point for an even number of equidistant points. As showed by the authors that formula yielded the coefficients with a single FFT.

It is well known that the resulting error of Hermite trigonometric interpolation is strongly dependent on the smoothness of the interpolated function. Interpolation of a 2-periodic and smooth function is highly effective. When the interpolated function has a point of discontinuity, the interpolation leads to the Gibbs phenomenon. The oscillations caused by this phenomenon are typically propagated into regions away from the singularity and degrade the quality of the approximations. Different ways of treating this deficiency have been suggested in the literature for the case $p = 1$. The idea of increasing the convergence rate by subtracting a polynomial that represents the discontinuities in the function and some of its first derivatives ("jumps") was suggested by Krylov [20] in 1906 and later, in 1964, by Lanczos [21], [22]. The key problem lies in determining the singularity amplitudes that has been realized by Eckhoff [10]-[13] where the values of the "jumps" are solutions of the corresponding system of linear equations. The Krylov-Lanczos and the Krylov-Lanczos-Eckhoff methods were developed and generalized by a number of authors, see [1]-[4], [6], [14], [17], [24], [26]-[29] and references therein.

In this paper (see also [30] where some results are presented without proofs (Theorems 2.1, 3.1, and 3.2)) we consider Hermite trigonometric interpolation with equidistant interpolation nodes and uniform multiplicities. Our method of construction of the Hermite trigonometric interpolants may be considered as a continuation of the method of Berrut and Welcher [5]. We derive relatively compact formula for the Hermite trigonometric interpolation that gives the interpolants as functions of the coefficients in the DFTs of the derivative values. We consider the case of an odd number of equidistant points for an arbitrary high number of derivatives of equal order at each of these nodes. Although we are discussing only the case of odd number of points our approach is valid also for even number of nodes. The accelerating convergence of interpolations are achieved by application of the Krylov-Lanczos approach. We also give formulae for the corrections that should be applied in order to soften the effect of the "jumps" at the endpoints when the interpolated function is not periodic. In this paper it will be assumed that the exact values of the "jumps" are known.

We organize this paper as follows. In Section 2 we give the explicit form of the Hermite trigonometric interpolation $T_{p,N}(f)$ and the corresponding quadrature $Q_{p,N}(f)$ that exploit the discrete Fourier coefficients of the interpolated function and its derivatives. In Section 3 the Krylov-Lanczos convergence acceleration method is applied to $T_{p,N}(f)$ and $Q_{p,N}(f)$. We denote the resulting interpolation and quadrature by $T_{q,p,N}(f)$ and $Q_{q,p,N}(f)$, respectively, and call them Hermite-Krylov-Lanczos interpolation and quadrature. This approach uses the values of the "jumps" in the function and some of its first derivatives at the end points of the interval. The parameter q indicates the number of "jumps" that are involved in the process of interpolation. In Subsections 3.2 and 3.3 the exact constants of the asymptotic errors are given. In Subsection 3.4 some numerical results are presented. Section 4 gives a short summary of the results.

2. Main Formulas for Hermite Trigonometric Interpolation and Quadrature

Let $f \in C^{p-1}[-1,1]$, $p \geq 1$. Let $\check{f}_m^{(j)}$ denote the discrete Fourier coefficients of the j-th derivative of f

$$\check{f}_m^{(j)} := \frac{1}{2N+1} \sum_{k=-N}^{N} f^{(j)}(x_k) e^{-i\pi m x_k},\ x_k := \frac{2k}{2N+1},\ j = 0, \cdots, p-1,\ |m| \leq N.$$

We set $\check{f}_m := \check{f}_m^{(0)}$.

Following [5], the sequence $T_{p,N}(f)$, $p \geq 1$, $N \geq 1$, of Hermite trigonometric interpolation polynomials will be defined by the formula

$$T_{p,N}(f)(x) := \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \Omega_j(m) \check{f}_m^{(j)},$$

where $\sigma = 0$ for odd values of parameter p and $\sigma = 1$ for even values. The unknown functions $\{\Omega_j\}$ will be determined from the condition that $T_{p,N}(f)$ is exact for the set of functions $\{e^{i\pi rx}\}$, $r = -N(1-\sigma)-\left[\frac{p}{2}\right](2N+1), \cdots, N(1+\sigma)+\left[\frac{p-1}{2}\right](2N+1)$, where $[x]$ denotes the greatest integer less than or equal to x. We set $r = n+s(2N+1)$, $n = -N(1-\sigma), \cdots, N(1+\sigma)$; $s = -\left[\frac{p}{2}\right], \cdots, \left[\frac{p-1}{2}\right]$ and write

$$e^{i\pi(n+s(2N+1))x} = \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} (i\pi(n+s(2N+1)))^j \Omega_j(m) \times \frac{1}{2N+1} \sum_{k=-N}^{N} e^{i\pi(n+s(2N+1))x_k} e^{-i\pi m x_k}.$$

Taking into account that

$$\frac{1}{2N+1} \sum_{k=-N}^{N} e^{i\pi(n+s(2N+1))x_k} e^{-i\pi m x_k} = \delta_{n,m}$$

($\delta_{n,m}$ is the Kronecker symbol) we obtain the system of linear equations

$$e^{i\pi(m+s(2N+1))x} = \sum_{j=0}^{p-1} \alpha_s^j(m)\Omega_j(m), \tag{2.1}$$

with Vandermonde matrix, for determining the functions $\{\Omega_j\}$, where

$$\alpha_s(m) := i\pi(m+s(2N+1)).$$

We proceed as in [3] and construct the explicit solution of (2.1).

Let $P_j(x)$ be the Lagrange fundamental polynomial of degree $p-1$ defined as

$$P_j(x) := \prod_{\substack{\ell=-[\frac{p}{2}] \\ \ell \neq j}}^{[\frac{p-1}{2}]} \frac{x-\alpha_\ell(m)}{\alpha_j(m)-\alpha_\ell(m)} = \sum_{k=0}^{p-1} c_{j,k}(m)x^k, \; j = -\left[\frac{p}{2}\right], \cdots, \left[\frac{p-1}{2}\right],$$

where the $c_{j,k}(m)$ denote the coefficients of the polynomial $P_j(x)$. From the equations

$$P_j(\alpha_i(m)) = \sum_{k=0}^{p-1} c_{j,k}(m)\alpha_i^k(m) = \delta_{i,j}, \quad i,j = -\left[\frac{p}{2}\right], \cdots, \left[\frac{p-1}{2}\right]$$

we see that the transpose of $(c_{j,k})$ is the inverse of the Vandermonde matrix (α_i^k), i.e., the following relations hold

$$\sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m)\alpha_k^s(m) = \delta_{j,s}, \tag{2.2}$$

$$\sum_{j=0}^{p-1} c_{k,j}(m)\alpha_s^j(m) = \delta_{k,s}. \tag{2.3}$$

As was shown in [3]

$$c_{k,j}(m) = \frac{1}{\prod\limits_{\substack{\ell=-[\frac{p}{2}] \\ \ell\neq k}}^{[\frac{p-1}{2}]} (\alpha_k(m) - \alpha_\ell(m))} \sum_{s=j+1}^{p} \gamma_s(m)\alpha_k^{s-j-1}(m), \tag{2.4}$$

where the γ_s are the coefficients of the polynomial

$$\prod_{s=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} (x - \alpha_s(m)) = \sum_{s=0}^{p} \gamma_s(m)\, x^s.$$

Equation (2.1) implies

$$\Omega_j(m) = \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1))x}.$$

This leads to the explicit form of the trigonometric Hermite interpolants

$$T_{p,N}(f)(x) = \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \check{f}_m^{(j)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1))x}, \tag{2.5}$$

where $\sigma = 0$ for the odd values of the parameter p and $\sigma = 1$ for the even values.

Theorem 2.1. *Let $f \in C^{p-1}[-1,1]$, $p \geq 1$. Then $T_{p,N}(f)$ is a Hermite trigonometric interpolation of f on the equidistant grid $x_k = \frac{2k}{2N+1}$, $|k| \leq N$ with the uniform multiplicities*

$$T_{p,N}^{(s)}(f)(x_k) = f^{(s)}(x_k),\ s = 0, \cdots, p-1.$$

Proof. In view of (2.5), we get

$$T_{p,N}^{(s)}(f)(x_r) = \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \check{f}_m^{(j)} e^{i\pi m x_r} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m)\alpha_k^s(m).$$

Relation (2.2) implies

$$T_{p,N}^{(s)}(f)(x_r) = \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \check{f}_m^{(s)} e^{i\pi m x_r} = f^{(s)}(x_r).$$

□

Definition of the coefficients γ_s implies

$$\gamma_s(m) = (-i\pi(2N+1))^{p-s} \sum_{-[\frac{p}{2}]\leq k_1 < \cdots < k_{p-s} \leq [\frac{p-1}{2}]} \prod_{\ell=k_1}^{k_{p-s}} \left(\frac{m}{2N+1} + \ell\right).$$

Inserting this into (2.4) we get

$$c_{k,j}(m) = \frac{1}{(i\pi(2N+1))^j}\beta_{k,j}\left(\frac{m}{2N+1}\right), \tag{2.6}$$

where

$$\beta_{k,j}(x) := \frac{1}{\prod\limits_{\substack{\ell=-[\frac{p}{2}] \\ \ell\neq k}}^{[\frac{p-1}{2}]}(k-\ell)} \sum_{s=j+1}^{p} (-1)^{p-s}\rho_s(x)(x+k)^{s-j-1} \tag{2.7}$$

and the $\rho_j(x)$ are the coefficients of the polynomial

$$\prod_{s=-[\frac{p}{2}]}^{[\frac{p-1}{2}]}(y+(x+s)) = \sum_{s=0}^{p}\rho_s(x)y^s. \tag{2.8}$$

Equation (2.6) helps to calculate the numbers $c_{0,j}(0)$

$$c_{0,j}(0) = \frac{\beta_{0,j}(0)}{(i\pi(2N+1))^j} = \frac{(-1)^j\rho_{j+1}(0)}{(i\pi(2N+1))^j\prod\limits_{\substack{\ell=-[p/2] \\ \ell\neq 0}}^{[\frac{p-1}{2}]}\ell}. \tag{2.9}$$

Integration of the interpolation $T_{p,N}(f)(x)$ over the interval $(-1,1)$ leads to the quadrature formula

$$Q_{p,N}(f) := \int_{-1}^{1} T_{p,N}(f)(x)dx = 2\sum_{j=0}^{p-1}\check{f}_0^{(j)}c_{0,j}(0).$$

In view of (2.9), we get

$$\begin{aligned} Q_{p,N}(f) &= \frac{2}{\prod\limits_{\substack{\ell=-[p/2] \\ \ell\neq 0}}^{[\frac{p-1}{2}]}\ell}\sum_{j=0}^{p-1}\check{f}_0^{(j)}\frac{(-1)^j\rho_{j+1}(0)}{(i\pi(2N+1))^j} \\ &= \frac{2}{\prod\limits_{\substack{\ell=-[p/2] \\ \ell\neq 0}}^{[\frac{p-1}{2}]}\ell}\sum_{j=0}^{p-1}\frac{(-1)^j\rho_{j+1}(0)}{(i\pi)^j(2N+1)^{j+1}}\sum_{k=-N}^{N}f^{(j)}(x_k). \end{aligned} \tag{2.10}$$

In the next section we review the Krylov-Lanczos method for accelerating the convergence of the classical trigonometric interpolation and apply this method to Hermite trigonometric interpolation (2.5) and quadrature (2.10).

3. Accelerating the Convergence of Trigonometric Hermite Interpolation and Quadrature

For $f \in C^q[-1,1]$ denote by $A_k(f)$ the "jump" of the k-th derivative of f

$$A_k(f) := f^{(k)}(1) - f^{(k)}(-1),\ k = 0, \cdots, q.$$

Throughout the paper it will be assumed that the exact values of the "jumps" are known.

By $f_n^{(j)}$ define the Fourier coefficients of the j-th derivative of f

$$f_n^{(j)} := \frac{1}{2}\int_{-1}^{1} f^{(j)}(x)e^{-i\pi nx}dx,\ j \geq 0.$$

We set $f_n := f_n^{(0)}$.

3.1. *The Krylov-Lanczos correction method*

The following Lemma is crucial for the Krylov-Lanczos method.

Lemma 3.1. *Suppose $f \in C^{q-1}[-1,1]$ for some $q \geq 1$ and $f^{(q-1)}$ is absolutely continuous on $[-1,1]$. Then the following formula holds for $n \neq 0$*

$$f_n = \frac{(-1)^{n+1}}{2}\sum_{k=0}^{q-1}\frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q}\int_{-1}^{1} f^{(q)}(x)e^{-i\pi nx}dx. \tag{3.1}$$

Proof. The proof is trivial by means of integration by parts. □

Equation (3.1) implies the basic expansion

$$f(x) = F(x) + \sum_{k=0}^{q-1} A_k(f)B_k(x) \tag{3.2}$$

of the approximated function where the B_k are 2-periodic extensions of the Bernoulli polynomials with the Fourier coefficients

$$B_{k,n} := \begin{cases} 0, & n = 0, \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n \neq 0, \end{cases}$$

and F is a 2-periodic and smooth function ($F \in C^{q-1}(R)$) on the real line with the discrete Fourier coefficients

$$\check{F}_n = \check{f}_n - \sum_{k=0}^{q-1} A_k(f)\check{B}_{k,n}. \tag{3.3}$$

It is well known that [22]

$$B_0(x) = \frac{x}{2},\ B_k(x) = \int B_{k-1}(x)dx,\ \int_{-1}^{1} B_k(x)dx = 0.$$

Hence, the discrete Fourier coefficients $\check{B}_{k,n}$ have explicit form, for example, here are three of them:

$$\check{B}_{0,n} = \frac{(-1)^n \ i}{2(2N+1)\sin\frac{\pi n}{2N+1}}, \ n \neq 0, \ \check{B}_{0,0} = 0,$$
$$\check{B}_{1,n} = \frac{(-1)^n \ \cos\frac{\pi n}{2N+1}}{2(2N+1)^2\sin^2\frac{\pi n}{2N+1}}, \ n \neq 0, \ \check{B}_{1,0} = -\frac{1}{12(2N+1)^2},$$
$$\check{B}_{2,n} = \frac{(-1)^{n+1} \ i\left(3+\cos\frac{2\pi n}{2N+1}\right)}{8(2N+1)^3\sin^3\frac{\pi n}{2N+1}}, \ n \neq 0, \ \check{B}_{2,0} = 0.$$

Equation (3.2) yields ($p \leq q$)

$$F^{(j)}(x) = f^{(j)}(x) - \frac{A_{j-1}(f)}{2} - \sum_{k=j}^{q-1} A_k(f)B_{k-j}(x), \ j = 1, \cdots, p-1. \tag{3.4}$$

Therefore

$$\check{F}_n^{(j)} = \check{f}_n^{(j)} - \sum_{k=j}^{q-1} A_k(f)\check{B}_{k-j,n}, \, n \neq 0, \, j = 1, \cdots, p-1 \tag{3.5}$$

and

$$\check{F}_0^{(j)} = \check{f}_0^{(j)} - \frac{A_{j-1}(f)}{2} - \sum_{k=j}^{q-1} A_k(f)\check{B}_{k-j,0}, \, j = 1, \cdots, p-1. \tag{3.6}$$

Approximation of F in (3.2) by $T_{p,N}(f)$, for $q \geq p$, leads to the following Hermite interpolation that we will call *Hermite-Krylov-Lanczos interpolation*

$$\begin{aligned} T_{q,p,N}(f)(x) := &\sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{j=0}^{p-1} \check{F}_m^{(j)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} c_{k,j}(m) e^{i\pi(m+k(2N+1))x} \\ &+ \sum_{k=0}^{q-1} A_k(f)B_k(x), \, q \geq p, \end{aligned} \tag{3.7}$$

where the coefficients $\check{F}_m^{(j)}$ are defined by (3.3), (3.5), and (3.6).

Integration of the interpolation (3.7) over the interval $(-1, 1)$ leads to the following quadrature formula that we will call *Hermite-Krylov-Lanczos quadrature*

$$Q_{q,p,N}(f) := \int_{-1}^{1} T_{q,p,N}(f)(x)dx = \frac{2}{\prod\limits_{\substack{\ell=-[p/2] \\ \ell\neq 0}}^{[\frac{p-1}{2}]} \ell} \sum_{j=0}^{p-1} \check{F}_0^{(j)} \frac{(-1)^j \rho_{j+1}(0)}{(i\pi(2N+1))^j}, \, q \geq p \tag{3.8}$$

or equivalently

$$Q_{q,p,N}(f) = \frac{2}{\prod\limits_{\substack{\ell=-[p/2] \\ \ell\neq 0}}^{[\frac{p-1}{2}]} \ell} \sum_{j=0}^{p-1} \frac{(-1)^j \rho_{j+1}(0)}{(i\pi)^j (2N+1)^{j+1}} \sum_{k=-N}^{N} F^{(j)}(x_k),\ q \geq p, \tag{3.9}$$

where for $F^{(j)}(x)$ we have representation (3.4).

In the next subsections we investigate the convergence of $T_{q,p,N}(f)$ and $Q_{q,p,N}(f)$, deriving the exact constants of the asymptotic errors.

3.2. *Accuracy of the Hermite-Krylov-Lanczos interpolation*

First we prove some lemmas that we need in our analysis.

Lemma 3.2. *Let $f \in C^q[-1,1]$, $q \geq 1$, with absolutely continuous $f^{(q)}$ on $[-1,1]$. Then for $0 \leq j \leq q-1$ the following asymptotic expansion is valid:*

$$F_n^{(j)} = A_q(f) \frac{(-1)^{n+1}}{2(i\pi n)^{q-j+1}} + o(n^{-q+j-1}),\ n \to \infty. \tag{3.10}$$

Proof. Expansion (3.2) implies that $F \in C^q[-1,1]$ and $F^{(q)}$ is absolutely continuous on $[-1,1]$. By means of integration by parts we get

$$F_n^{(j)} = \frac{(-1)^{n+1}}{2} \sum_{k=j}^{q} \frac{A_k(F)}{(i\pi n)^{k-j+1}} + \frac{1}{2(i\pi n)^{q-j+1}} \int_{-1}^{1} F^{(q+1)}(x) e^{-i\pi n x} dx.$$

Taking into account that $A_k(F) = 0$, $k = 0, \cdots, q-1$, $A_q(F) = A_q(f)$, in view of the Riemann-Lebesgue lemma, we obtain the required. $\square$

Lemma 3.3. *Let $f \in C^q[-1,1]$, $q \geq 1$, with absolutely continuous $f^{(q)}$ on $[-1,1]$. Then the following relation is true*

$$F_n^{(j)} = (i\pi n)^j F_n,\ 1 \leq j \leq q. \tag{3.11}$$

Proof. Integration by parts yields $(1 \leq j \leq q)$

$$F_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{j-1} \frac{A_k(F)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^j} \int_{-1}^{1} F^{(j)}(x) e^{-i\pi n x} dx.$$

Taking into account that $A_k(F) = 0$, $k = 0, \cdots, q-1$ we get the required. $\square$

We put

$$R_{q,p,N}(f)(x) := f(x) - T_{q,p,N}(f)(x)$$

and by $||f||$ we denote the standard norm in the space $L_2(-1,1)$

$$||f|| := \left(\int_{-1}^{1} |f(x)|^2 dx \right)^{1/2}.$$

The next theorem reveals the asymptotic behavior of the trigonometric Hermite interpolation.

Theorem 3.1. *Let $f \in C^q[-1,1]$, $q \geq 1$ be such that $f^{(q)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds*

$$\lim_{N\to\infty} (2N+1)^{q+\frac{1}{2}} \|R_{q,p,N}(f)\| = |A_q(f)| t(q,p),$$

$$t(q,p) := \frac{1}{\sqrt{2}\,\pi^{q+1}} \left(\int_{-\frac{1-\sigma}{2}}^{\frac{1+\sigma}{2}} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} \left| \sum_{j=0}^{p-1} \beta_{k,j}(x) \sum_s{}^{*} \frac{(-1)^s}{(x+s)^{q-j+1}} \right|^2 dx + \frac{2^{2q+2}}{(2q+1)p^{2q+1}} \right)^{\frac{1}{2}},$$

where

$$\sum_s{}^{*} := \sum_{s=-\infty}^{-[\frac{p}{2}]-1} + \sum_{s=[\frac{p-1}{2}]+1}^{\infty}$$

and the $\beta_{k,j}$ are defined by (2.7).

Proof. It is easy to verify that

$$\begin{aligned} R_{q,p,N}(f)(x) = &\sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} \left(F_{m+k(2N+1)} - \sum_{j=0}^{p-1} c_{k,j}(m) \check{F}_m^{(j)} \right) e^{i\pi(m+k(2N+1))x} \\ &+ \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_k{}^{*} F_{m+k(2N+1)} e^{i\pi(m+k(2N+1))x}. \end{aligned}$$

This yields

$$\begin{aligned} \|R_{q,p,N}(f)\|^2 = 2 &\sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_{k=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} \left| F_{m+k(2N+1)} - \sum_{j=0}^{p-1} c_{k,j}(m) \check{F}_m^{(j)} \right|^2 \\ &+ 2 \sum_{m=-N(1-\sigma)}^{N(1+\sigma)} \sum_k{}^{*} |F_{m+k(2N+1)}|^2. \end{aligned} \tag{3.12}$$

Taking into account that

$$\check{F}_m^{(j)} = \sum_{s=-\infty}^{\infty} F_{m+s(2N+1)}^{(j)},$$

in view of (3.11), we derive

$$F_{m+k(2N+1)} - \sum_{j=0}^{p-1} c_{k,j}(m)\check{F}_m^{(j)} = F_{m+k(2N+1)} - \sum_{s=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} F_{m+s(2N+1)} \sum_{j=0}^{p-1} c_{k,j}(m)\alpha_s^j(m) - \sum_{j=0}^{p-1} c_{k,j}(m) \sum_s{}^{*} F_{m+s(2N+1)}^{(j)}.$$

Relation (2.3) implies

$$F_{m+k(2N+1)} - \sum_{j=0}^{p-1} c_{k,j}(m)\check{F}_m^{(j)} = -\sum_{j=0}^{p-1} c_{k,j}(m) \sum_s{}^{*} F_{m+s(2N+1)}^{(j)}.$$

From expansion (3.10) and equation (2.6), we conclude that

$$F_{m+k(2N+1)} - \sum_{j=0}^{p-1} c_{k,j}\check{F}_m^{(j)} = \frac{(-1)^m A_q(f)}{2(i\pi(2N+1))^{q+1}} \sum_{j=0}^{p-1} \beta_{k,j}\left(z_m\right) \sum_s{}^{*} \frac{(-1)^s}{(z_m+s)^{q-j+1}} + o(N^{-q-1}),\ N\to\infty,\ z_m = \frac{m}{2N+1}.$$

We end the proof by inserting the last estimate into (3.12), letting N tend to infinity, and replacing the Riemann sums by the corresponding integrals. □

Numerical values of $t(q,p)$ for various values of the parameters q and p are shown in Table 1. Note that $p=1$ corresponds to the classical trigonometric interpolation accelerated by the Krylov-Lanczos method (see [29]). We see that, for the fixed values of q, the values of $t(q,p)$ are decreasing when p is increasing. Similarly, when p is fixed then the values of $t(q,p)$ are decreasing when q is increasing. Hence, for the same number of grid points of interpolation we derive greater precision while increasing either one of the parameters p and q. Note also that the parameter q changes the rate of convergence of interpolation while the parameter p influences only the value of the constant $t(q,p)$.

In Figure 1 the numerical values of $t(9,p)$ and $t(10,p)$ are presented. It shows the behavior of $t(q,p)$ when the parameter q is fixed and p is increasing.

In the next subsection we will prove similar results for the corresponding quadrature formula.

3.3. *Accuracy of the Hermite-Krylov-Lanczos quadrature*

Denote

$$r_{q,p,N}(f) := \int_{-1}^{1} f(x)dx - Q_{q,p,N}(f).$$

Table 1. The numerical values of $t(q,p)$ for $1 \le p \le 5$ and $p \le q \le 10$.

$q \setminus p$	1	2	3	4	5
1	0.24	–	–	–	–
2	0.11	0.025	–	–	–
3	0.063	$9.4 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$	–	–
4	0.034	$2.4 \cdot 10^{-3}$	$5.8 \cdot 10^{-4}$	$2.2 \cdot 10^{-4}$	–
5	0.020	$7.7 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$5.0 \cdot 10^{-5}$	$2.3 \cdot 10^{-5}$
6	0.011	$2.2 \cdot 10^{-4}$	$2.6 \cdot 10^{-5}$	$6.2 \cdot 10^{-6}$	$2.2 \cdot 10^{-6}$
7	$7.0 \cdot 10^{-3}$	$6.7 \cdot 10^{-5}$	$5.8 \cdot 10^{-6}$	$1.2 \cdot 10^{-6}$	$3.8 \cdot 10^{-7}$
8	$4.2 \cdot 10^{-3}$	$1.9 \cdot 10^{-5}$	$1.1 \cdot 10^{-6}$	$1.6 \cdot 10^{-7}$	$3.9 \cdot 10^{-8}$
9	$2.5 \cdot 10^{-3}$	$6.0 \cdot 10^{-6}$	$2.3 \cdot 10^{-7}$	$2.8 \cdot 10^{-8}$	$6.0 \cdot 10^{-9}$
10	$1.5 \cdot 10^{-3}$	$1.8 \cdot 10^{-6}$	$4.6 \cdot 10^{-8}$	$4.0 \cdot 10^{-9}$	$6.6 \cdot 10^{-10}$

In the next theorem we reveal the asymptotic behavior of the Hermite-Krylov-Lanczos quadrature.

Theorem 3.2. *Let $f \in C^q[-1,1]$, $q \ge 1$ be such that $f^{(q)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds*

$$\lim_{N\to\infty} (2N+1)^{q+1} r_{q,p,N}(f) = A_q(f)\, h(q,p),$$

$$h(q,p) := \frac{1}{(i\pi)^{q+1} \prod\limits_{\substack{\ell=-[\frac{p}{2}] \\ \ell\neq 0}}^{[\frac{p-1}{2}]} \ell} \sum_{j=0}^{p-1} (-1)^j \rho_{j+1}(0) \sum_s{}^{*} \frac{(-1)^s}{s^{q-j+1}},$$

where

$$\sum_s{}^{*} := \sum_{s=-\infty}^{-[\frac{p}{2}]-1} + \sum_{s=[\frac{p-1}{2}]+1}^{\infty}$$

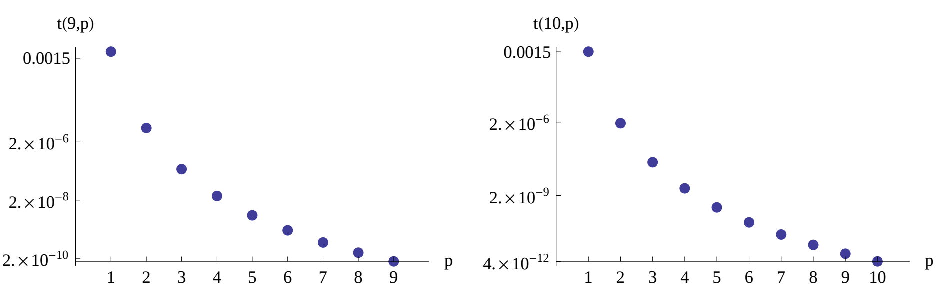

Fig. 1. The values of $t(9,p)$ (left) and $t(10,p)$ (right) for various values of the parameter p ($p \le q$).

and the $\rho_s(x)$ *are defined by (2.8).*

Proof. We have

$$r_{q,p,N}(f) = \int_{-1}^{1} R_{q,p,N}(f)(x)dx = 2F_0 - 2\sum_{j=0}^{p-1} \check{F}_0^{(j)} c_{0,j}(0)$$

$$= 2F_0 - 2\sum_{j=0}^{p-1} c_{0,j}(0) \sum_{s=-\infty}^{\infty} F_{s(2N+1)}^{(j)}.$$

In view of relation (3.11), we derive

$$r_{q,p,N}(f) = 2F_0 - 2\sum_{s=-[\frac{p}{2}]}^{[\frac{p-1}{2}]} F_{s(2N+1)} \sum_{j=0}^{p-1} c_{0,j}(0)\alpha_0^j(0) - 2\sum_{j=0}^{p-1} c_{0,j}(0) \sum_s{}^{*} F_{s(2N+1)}^{(j)}.$$

Identity (2.3) implies

$$r_{q,p,N}(f) = -2\sum_{j=0}^{p-1} c_{0,j}(0) \sum_s{}^{*} F_{s(2N+1)}^{(j)}. \tag{3.13}$$

According to expansion (3.10) and equation (2.9), we obtain

$$r_{q,p,N}(f) = \frac{A_q(f)}{(i\pi(2N+1))^{q+1}} \sum_{j=0}^{p-1} \beta_{0,j}(0) \sum_s{}^{*} \frac{(-1)^s}{s^{q-j+1}} + o(N^{-q-1}),\ N \to \infty.$$

This ends the proof. □

Table 2. Numerical values of $|h(q,p)|$.

$q \setminus p$	1	2	3	4	5
1	0.16	–	–	–	–
2	0	$5.3 \cdot 10^{-2}$	–	–	–
3	$1.9 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$2.5 \cdot 10^{-3}$	–	–
4	0	$6.2 \cdot 10^{-3}$	0	$4.1 \cdot 10^{-4}$	–
5	$2.0 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$8.0 \cdot 10^{-5}$	$8.0 \cdot 10^{-5}$	$1.5 \cdot 10^{-5}$
6	0	$6.5 \cdot 10^{-4}$	0	$1.3 \cdot 10^{-5}$	0
7	$2.1 \cdot 10^{-4}$	$2.1 \cdot 10^{-4}$	$2.3 \cdot 10^{-6}$	$2.3 \cdot 10^{-6}$	$2.2 \cdot 10^{-7}$
8	0	$6.7 \cdot 10^{-5}$	0	$3.6 \cdot 10^{-7}$	0
9	$2.1 \cdot 10^{-5}$	$2.1 \cdot 10^{-5}$	$6.0 \cdot 10^{-8}$	$6.0 \cdot 10^{-8}$	$2.9 \cdot 10^{-9}$
10	0	$6.8 \cdot 10^{-6}$	0	$9.5 \cdot 10^{-9}$	0

Numerical values of $|h(q,p)|$ are presented in Table 2. We see that $|h(q,2p+1)| = |h(q,2p+2)|$ for odd values of q and $1 \le p \le q$. It is easy to verify theoretically that for odd values of q

$$h(q,2p+1) = h(q,2p+2),\ p \ge 1.$$

Moreover, Table 2 shows that $h(q,p)=0$ for even values of q when p is odd. We prove this property in the next theorem.

Theorem 3.3. *Let parameters q and p be even and odd, respectively. Suppose $f \in C^q[-1,1]$, $q \geq 1$, with absolutely continuous $f^{(q)}$ on $[-1,1]$. Then the following estimate holds*

$$\lim_{N\to\infty} r_{q,p,N}(f) = o(N^{-q-1}),\ N \to \infty.$$

Proof. Suppose that p is odd. According to the definition of the $\rho_s(x)$

$$\sum_{r=0}^{p} \rho_r(0) y^r = \prod_{r=-\frac{p-1}{2}}^{\frac{p-1}{2}} (y+r) = y \prod_{r=1}^{\frac{p-1}{2}} (y+r) \prod_{r=1}^{\frac{p-1}{2}} (y-r) = y \prod_{r=1}^{\frac{p-1}{2}} (y^2-r^2).$$

From here we conclude that $\rho_{2s}(0) = 0$, $s = 0, \cdots, \frac{p-1}{2}$. Now let q be an even number. We have

$$\beta_{0,j}(0) = \sum_{j=0}^{p-1} (-1)^j \rho_{j+1}(0) \sum_{s}{}^{*} \frac{(-1)^s}{s^{q-j+1}} = \sum_{j=0}^{\frac{p-1}{2}} \rho_{2j+1}(0) \sum_{s}{}^{*} \frac{(-1)^s}{s^{q-2j+1}} = 0$$

as the last sum vanishes. This completes the proof in view of Theorem 3.2. □

The next theorem proves more accurate estimate. First we need the following lemma.

Lemma 3.4. *Let $f \in C^{q+1}[-1,1]$, $q \geq 1$, with absolutely continuous $f^{(q+1)}$ on $[-1,1]$. Then for $0 \leq j \leq q-1$ and $n \neq 0$ the following asymptotic expansion is valid*

$$F_n^{(j)} = A_q(f) \frac{(-1)^{n+1}}{2(i\pi n)^{q-j+1}} + A_{q+1}(f) \frac{(-1)^{n+1}}{2(i\pi n)^{q-j+2}} + o(n^{-q+j-2}),\ n \to \infty. \tag{3.14}$$

Proof. The proof coincides with the one for the expansion (3.10). □

Theorem 3.4. *Let q and p be even and odd numbers, respectively. Suppose $f \in C^{q+1}[-1,1]$, $q \geq 1$, with absolutely continuous $f^{(q+1)}$ on $[-1,1]$. Then the following estimate holds*

$$\lim_{N\to\infty} (2N+1)^{q+2} r_{q,p,N}(f) = A_{q+1}(f)\, H(q,p),$$

$$H(q,p) := \frac{(-1)^{\frac{q+p+1}{2}}}{\pi^{q+2}} \frac{1}{(\frac{p-1}{2}!)^2} \sum_{j=0}^{p-1} (-1)^j \rho_{j+1}(0) \sum_{s}{}^{*} \frac{(-1)^s}{s^{q-j+2}},$$

where

$$\sum_{s}{}^{*} := \sum_{s=-\infty}^{-\frac{p-1}{2}-1} + \sum_{s=\frac{p-1}{2}+1}^{\infty}.$$

Proof. In view of expansion (3.14), the proof follows from (3.13). $\square$

Numerical values of $|H(q,p)|$ are presented in Table 3.

Table 3. Numerical values of $|H(q,p)|$.

$q \setminus p$	1	3	5	7	9
2	$1.9 \cdot 10^{-2}$	–	–	–	–
4	$2.1 \cdot 10^{-3}$	$8.0 \cdot 10^{-5}$	–	–	–
6	$2.1 \cdot 10^{-4}$	$2.3 \cdot 10^{-6}$	$2.2 \cdot 10^{-7}$	–	–
8	$2.1 \cdot 10^{-5}$	$6.0 \cdot 10^{-8}$	$2.9 \cdot 10^{-9}$	$4.1 \cdot 10^{-10}$	–
10	$2.2 \cdot 10^{-6}$	$1.6 \cdot 10^{-9}$	$3.6 \cdot 10^{-11}$	$3.1 \cdot 10^{-12}$	$5.3 \cdot 10^{-13}$
12	$2.2 \cdot 10^{-7}$	$4.0 \cdot 10^{-11}$	$4.3 \cdot 10^{-13}$	$2.2 \cdot 10^{-14}$	$2.6 \cdot 10^{-15}$
14	$2.2 \cdot 10^{-8}$	$1.0 \cdot 10^{-12}$	$4.9 \cdot 10^{-15}$	$1.5 \cdot 10^{-16}$	$1.2 \cdot 10^{-17}$

3.4. *Numerical results*

Consider the following simple function

$$f(x) = \sin(x-1). \tag{3.15}$$

In Table 4 the uniform errors $\max_{|x|\le 1} |R_{q,p,N}(f)(x)|$ are presented for various values of p, q and $N = 1$ while interpolating the function (3.15) by the Hermite-Krylov-Lanczos interpolation. We see that the errors vary from the value 0.15 ($p = q = 1$) to the value $3 \cdot 10^{-15}$ ($q = 10$ and $p = 6$). The graphs of the corresponding absolute errors are presented in Figure 2. Note that we are interpolating the function (3.15) using only $2N + 1 = 3$ grid points.

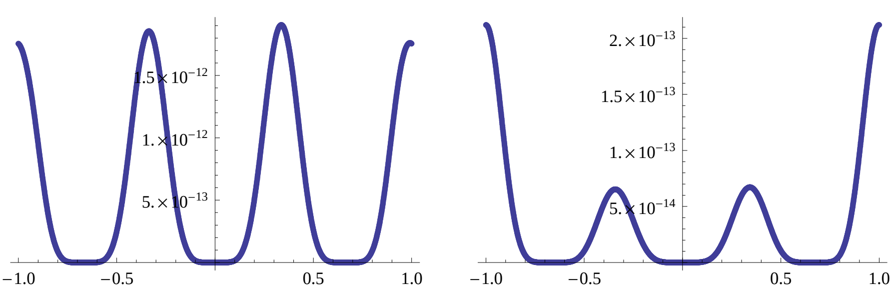

Fig. 2. The absolute errors while approximating (3.15) by $T_{8,6,1}(f)$ (left) and $T_{9,6,1}(f)$ (right).

Table 4. The uniform errors while approximating the function (3.15) by the Hermite-Krylov-Lanczos interpolation for various values of the parameters p and q. Here, $N = 1$.

$q \setminus p$	1	2	3	4	5
1	0.15	–	–	–	–
2	$6.4 \cdot 10^{-3}$	$2.4 \cdot 10^{-3}$	–	–	–
3	$2.4 \cdot 10^{-3}$	$6.7 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$	–	–
4	$1.8 \cdot 10^{-4}$	$2.5 \cdot 10^{-5}$	$4.9 \cdot 10^{-6}$	$2.3 \cdot 10^{-6}$	–
5	$5.2 \cdot 10^{-5}$	$5.9 \cdot 10^{-6}$	$9.9 \cdot 10^{-7}$	$4.4 \cdot 10^{-7}$	$1.9 \cdot 10^{-7}$
6	$4.7 \cdot 10^{-6}$	$2.9 \cdot 10^{-7}$	$2.4 \cdot 10^{-8}$	$7.7 \cdot 10^{-9}$	$2.4 \cdot 10^{-9}$
7	$1.2 \cdot 10^{-6}$	$5.7 \cdot 10^{-8}$	$3.8 \cdot 10^{-9}$	$1.1 \cdot 10^{-9}$	$3.3 \cdot 10^{-10}$
8	$1.2 \cdot 10^{-7}$	$3.2 \cdot 10^{-9}$	$1.1 \cdot 10^{-10}$	$2.4 \cdot 10^{-11}$	$4.7 \cdot 10^{-12}$
9	$3.1 \cdot 10^{-8}$	$5.8 \cdot 10^{-10}$	$1.5 \cdot 10^{-11}$	$2.9 \cdot 10^{-12}$	$5.4 \cdot 10^{-13}$
10	$3.1 \cdot 10^{-9}$	$3.6 \cdot 10^{-11}$	$4.5 \cdot 10^{-13}$	$6.9 \cdot 10^{-14}$	$8.5 \cdot 10^{-15}$

Table 5 shows the numerical values of $|r_{q,p,1}(f)|$ for even values of q and for odd values of p when $f = \sin(x-1)$. Recall that for these values of p and q Theorem 3.4 is valid. In this case also using the same number of nodes $(2N+1 = 3)$ we increase the precision of the quadrature by increasing the values of parameters p and q.

Table 5. Numerical values of $|r_{q,p,1}(f)|$ for even values of q and for odd values of p when $f = \sin(x-1)$.

$q \setminus p$	1	3	5	7	9
2	$3.4 \cdot 10^{-4}$	–	–	–	–
4	$4.0 \cdot 10^{-6}$	$1.6 \cdot 10^{-7}$	–	–	–
6	$4.6 \cdot 10^{-8}$	$4.9 \cdot 10^{-10}$	$4.8 \cdot 10^{-11}$	–	–
8	$5.2 \cdot 10^{-10}$	$1.4 \cdot 10^{-12}$	$7.0 \cdot 10^{-14}$	$9.8 \cdot 10^{-15}$	–
10	$5.8 \cdot 10^{-12}$	$4.1 \cdot 10^{-15}$	$9.6 \cdot 10^{-17}$	$8.3 \cdot 10^{-18}$	$1.4 \cdot 10^{-18}$
12	$6.6 \cdot 10^{-14}$	$1.2 \cdot 10^{-17}$	$1.3 \cdot 10^{-19}$	$6.6 \cdot 10^{-21}$	$7.8 \cdot 10^{-22}$
14	$7.4 \cdot 10^{-16}$	$3.3 \cdot 10^{-20}$	$1.6 \cdot 10^{-22}$	$5.1 \cdot 10^{-24}$	$4.1 \cdot 10^{-25}$

4. Conclusion

We have introduced the Hermite trigonometric interpolation $T_{p,N}(f)$ that exploited the discrete Fourier coefficients of the interpolated function and its derivatives. Those coefficients could be calculated by the FFT algorithm. At least when one might choose N as a power of two, that formula yielded the coefficients in $O(N \log N)$ operations. We presented also the corresponding quadrature formula.

Application of the Krylov-Lanczos correction method led to the Hermite-Krylov-Lanczos interpolation $T_{q,p,N}(f)$ and quadrature $Q_{q,p,N}(f)$ where the parameter q

indicated the number of "jumps" that were involved in the process of interpolation or quadrature. The accelerated convergence was achieved through the use of the Bernoulli polynomials along Krylov's idea.

In Section 3 we proved (Theorem 3.1) that the rate of convergence of the Hermite-Krylov-Lanczos interpolation was $O(N^{-q-\frac{1}{2}})$ in the L_2 norm as $N \to \infty$ while the rate of convergence of the corresponding quadrature was $O(N^{-q-1})$ (see Theorem 3.2). Theorems 3.3 and 3.4 improved that estimate for the even values of q when the values of p were odd. In that case the rate of convergence was $o(N^{-q-1})$ and $O(N^{-q-2})$, respectively.

Numerical and theoretical results showed that with the same number of interpolation nodes the precision could be dramatically improved by increasing either p or q.

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