ISSN 1068-3623, Journal of Contemporary Mathematical Analysis, 2013, Vol. 48, No. 6, pp. 339–347. © Allerton Press, Inc., 2013.
Original Russian Text © A. Poghosyan, 2013, published in Izvestiya NAN Armenii. Matematika, 2013, No. 6, pp. 82-91.

REAL AND COMPLEX ANALYSIS

On a Convergence of the Rational-Trigonometric-Polynomial Approximations Realized by the Roots of the Laguerre Polynomials

A. Poghosyan*

[1] *Institute of Mathematics of National Academy of Sciences of Armenia, Yerevan, Armenia*

Received February 24, 2012

Abstract—The paper considers a problem of approximation of functions by means of their finite number of Fourier coefficients. Convergence acceleration of approximations by the truncated Fourier series is achieved by application of polynomial and rational correction functions. Rational corrections include unknown parameters whose determination is a crucial problem. We consider an approach connected with the roots of the Laguerre polynomials and study the rates of convergence of such approximations.

MSC2010 numbers: 41A20, 65T40, 41A21, 41A25, 65B99

DOI: 10.3103/S1068362313060101

Keywords: *Fourier series; convergence acceleration; Krylov-Lanczos approximation; rational approximation; Laguerre polynomials.*

1. INTRODUCTION

We consider the problem of approximation of a function f by means of its Fourier coefficients

$$\hat{f}_n = \frac{1}{2}\int_{-1}^{1} f(x)\,\exp\{-i\pi n x\}\,dx, \quad |n| \le N.$$

A natural way is reconstruction of function f by the truncated Fourier series

$$S_N(f,x) = \sum_{n=-N}^{N} \hat{f}_n e^{i\pi n x}.$$

Different methods of convergence acceleration of the truncated Fourier series have been suggested in the literature in the last few decades (see, e.g., [1], [5], [6], and references therein). An approach, which involves a polynomial, representing the discontinuities (jumps) of the underlying function and some of its first derivatives, was suggested in 1906 by Krylov [10] (see also [17], [19]). Later Lanczos [11] developed the same approach in more formalized setting (see also [3], [9], [13]). This approach we will refer as Krylov-Lanczos (KL)-approximation.

In this paper we consider the KL-approximation with additional acceleration of convergence by sequential application of rational (by $e^{i\pi x}$) correction functions along the ideas of the Fourier-Pade approximations (see [2]). The general form of the Fourier-Pade representation has been suggested by Cheney [7]. Similar approximations were studied by Geer [8]. The rational corrections in our approach contain unknown parameters, and different approaches are known for their determination (see [12], [14], [15]). We consider an approach connected with the roots of the Laguerre polynomials and analyze its convergence properties.

*E-mail: arnak@instmath.sci.am

2. KRYLOV-LANCZOS APPROXIMATION

Let $f \in C^q[-1,1]$. By $A_k(f)$ we denote the exact value of the jump in the k-th derivative of f on interval $[-1,1]$:

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \quad k = 0, 1..., q.$$

In this paper we restrict our discussion to the class of functions that are smooth on $[-1,1]$, and assume that the exact values of the jumps are known. Denote by $AC[-1,1]$ the set of absolutely continuous functions on $[-1,1]$. Let $f^{(q-1)} \in AC[-1,1]$ for some $q \geq 1$. The following expansion of the Fourier coefficients is crucial for Krylov-Lanczos approach

$$\hat{f}_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q-1} \frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^{1} f^{(q)}(x) e^{-i\pi n x} dx, \ n \neq 0, \tag{2.1}$$

which leads to the representation of Lanczos [9]:

$$f(x) = \sum_{k=0}^{q-1} A_k(f) B_k(x) + F(x).$$

Here, B_k $(k = 0, \ldots, q-1)$ are 2-periodic Bernoulli polynomials

$$B_0(x) = \frac{x}{2}, \quad B_k(x) = \int B_{k-1}(x) dx, \quad \int_{-1}^{1} B_k(x) dx = 0, \quad x \in [-1,1]$$

with the Fourier coefficients

$$\hat{B}_{k,n} = \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, \quad n \neq 0, \quad \hat{B}_{k,0} = 0,$$

and F is a 2-periodic and smooth function defined on the real line ($F \in C^{q-1}(\mathbb{R})$) with the Fourier coefficients

$$\hat{F}_n = \hat{f}_n - \sum_{k=0}^{q-1} A_k(f) \hat{B}_{k,n}.$$

An approximation of F by the truncated Fourier series leads to the Krylov-Lanczos (KL) − approximation

$$S_{N,q}(f,x) = \sum_{n=-N}^{N} \hat{F}_n e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f) B_k(x)$$

with the approximation error

$$R_{N,q}(f,x) = f(x) - S_{N,q}(f,x).$$

The next theorem describes the asymptotic behavior of $R_{N,q}(f,x)$ in the interval $(-1,1)$.

Theorem 2.1. *[13]. Let $f^{(q+1)} \in AC[-1,1]$ for some $q \geq 0$. Then the following holds for $|x| < 1$*

$$R_{N,q}(f,x) = A_q(f) \frac{(-1)^N}{2\pi^{q+1} N^{q+1}} \frac{\sin \frac{\pi}{2}(x(2N+1) - q)}{\cos \frac{\pi x}{2}} + o(N^{-q-1}), \quad N \to \infty.$$

3. APPROXIMATION BY RATIONAL FUNCTIONS

Additional convergence acceleration of the KL-approximation can be achieved by application of rational functions (by $e^{i\pi x}$) as corrections of the error. Consider a finite sequence of complex numbers $\theta = \{\theta_k\}_{|k|=1}^{p}$, $p \geq 1$. We denote $\hat{F} = \{\hat{F}_n\}$, and define the generalized finite differences, denoted by $\Delta_n^k(\theta, \hat{F})$, by formula:

$$\Delta_n^0(\theta, \hat{F}) = \hat{F}_n, \quad \Delta_n^k(\theta, \hat{F}) = \Delta_n^{k-1}(\theta, \hat{F}) + \theta_{k\, sgn(n)} \Delta_{(|n|-1)sgn(n)}^{k-1}(\theta, \hat{F}), \quad k \geq 1,$$

where $sign(n) = 1$ if $n \geq 0$ and $sgn(n) = -1$ if $n < 0$. By $\Delta_n^k(\hat{F})$ we denote the classical finite differences that correspond to the generalized differences $\Delta_n^k(\theta, \hat{F})$ for the choice $\theta \equiv 1$. We have

$$R_{N,q}(f, x) = R_N^+(F, x) + R_N^-(F, x),$$

where

$$R_N^+(F, x) = \sum_{n=N+1}^{\infty} \hat{F}_n e^{i\pi n x}, \quad R_N^-(F, x) = \sum_{n=-\infty}^{-N-1} \hat{F}_n e^{i\pi n x}.$$

An application of the Abel transformation implies

$$R_N^+(F, x) = -\frac{\theta_1 \hat{F}_N\, e^{i\pi(N+1)x}}{1 + \theta_1 e^{i\pi x}} + \frac{1}{1 + \theta_1 e^{i\pi x}} \sum_{n=N+1}^{\infty} \Delta_n^1(\theta, \hat{F}) e^{i\pi n x}.$$

Iterating p times we get

$$R_N^+(F, x) = -e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, \hat{F})}{\prod_{s=1}^{k}(1 + \theta_s e^{i\pi x})} + \frac{1}{\prod_{k=1}^{p}(1 + \theta_k e^{i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{F}) e^{i\pi n x},$$

where the first term can be viewed as a correction of the error, while the last term is the actual error. A similar expansion for $R_N^-(F, x)$ leads to the rational-trigonometric-polynomial (RTP)-approximation

$$\begin{aligned} S_{N,q,p}(f, x) = & \sum_{k=0}^{q-1} A_k(f) B_k(x) + \sum_{n=-N}^{N} \hat{F}_n e^{i\pi n x} \\ & - e^{i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_k \Delta_N^{k-1}(\theta, \hat{F})}{\prod_{s=1}^{k}(1 + \theta_s e^{i\pi x})} - e^{-i\pi(N+1)x} \sum_{k=1}^{p} \frac{\theta_{-k} \Delta_{-N}^{k-1}(\theta, \hat{F})}{\prod_{s=1}^{k}(1 + \theta_{-s} e^{-i\pi x})} \end{aligned} \tag{3.1}$$

with the error

$$R_{N,q,p}(f, x) = f(x) - S_{N,q,p}(f, x) = R_{N,q,p}^+(f, x) + R_{N,q,p}^-(f, x),$$

where

$$R_{N,q,p}^{\pm}(f, x) = \frac{1}{\prod_{k=1}^{p}(1 + \theta_{\pm k} e^{\pm i\pi x})} \sum_{n=N+1}^{\infty} \Delta_{\pm n}^p(\theta, \hat{F}) e^{\pm i\pi n x}. \tag{3.2}$$

Note that the approximation (3.1) will be completely determined if the values of parameters θ_k are specified. Different methods are known for determination of these parameters (see, [12], [14] − [16]).

In this paper we will focus to the approach discussed in [12], [15] and [16], where the following setting is considered:

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \quad k = 1, \ldots, p. \tag{3.3}$$

By $\gamma_k(\tau)$ we denote the coefficients of the polynomial

$$\prod_{k=1}^{p}(1 + \tau_k x) = \sum_{k=0}^{p} \gamma_k(\tau) x^k. \tag{3.4}$$

The next theorem describes the behavior of $R_{N,q,p}(f, x)$ in the case where the parameter sequence $\theta = \{\theta_k\}$ is specified by (3.3) (see [16]).

Theorem 3.1. *Let $f^{(q+p+1)} \in AC[-1,1]$ for some $q \geq 0$ and $p \geq 1$. Let*

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N}, \quad k = 1, ..., p.$$

Then the following estimate holds for $|x| < 1$

$$R_{N,q,p}(f,x) = A_q(f) \frac{(-1)^{N+p}}{2^{p+1}\pi^{q+1}N^{p+q+1}q!} \frac{\sin\frac{\pi}{2}(x(2N-p+1)-q)}{\cos^{p+1}\frac{\pi x}{2}} \sum_{k=0}^{p}(-1)^k(p-k+q)!\gamma_k(\tau)$$
$$+ o\left(N^{-q-p-1}\right), \quad N \to \infty.$$

Note that Theorem 3.1 remains valid also in the case where the parameters τ_k in (3.3) are undefined. This enables an extra freedom in rational approximations to achieve some additional goals.

In this paper we apply an approach where the parameters τ_k are the roots of the Laguerre polynomials $L_p^q(x)$ (see [4]). The next section contains a theoretical background of such RTP-approximations.

4. RTP-APPROXIMATIONS BY THE ROOTS OF LAGUERRE POLYNOMIALS

Let τ_k be the roots of the Laguerre polynomial $L_p^q(x)$:

$$L_p^q(\tau_k) = 0, \quad k = 1, ..., p.$$

It is well-known that τ_k are distinct and positive, and the Laguerre polynomials have the following representation:

$$L_p^q(x) = \sum_{k=0}^{p}(-1)^k \frac{(p+q)!}{k!(p-k)!(q+k)!}x^k.$$

For our purposes the equation $L_p^q(\tau_s) = 0$ can be written in the form:

$$\sum_{k=0}^{p}\left(-\frac{1}{\tau_s}\right)^k \frac{p!}{k!(p-k)!(q+p-k)!} = 0.$$

A comparison of the last equality with (3.4) shows that

$$\gamma_k(\tau) = \binom{p}{k}\frac{(q+p)!}{(q+p-k)!}. \tag{4.1}$$

Now we are going to estimate the pointwise convergence of such RTP-approximations in the regions away from the endpoints. The next result is an immediate consequence of Theorem 3.1.

Theorem 4.1. *Let $f^{(q+p+1)} \in AC[-1,1]$ for some $q \geq 0$ and $p \geq 1$, and let*

$$\theta_k = \theta_{-k} = 1 - \frac{\tau_k}{N},$$

where τ_k, $k = 1, ..., p$ are the roots of the Laguerre polynomial. Then the following estimate holds for $|x| < 1$

$$R_{N,q,p}(f,x) = o\left(N^{-q-p-1}\right), \quad N \to \infty. \tag{4.2}$$

Proof. In view of (4.1) we have

$$\sum_{k=0}^{p}(-1)^k(p-k+q)!\gamma_k(\tau) = 0.$$

Hence the estimate (4.2) follows from Theorem 3.1.

Imposing extra smoothness on the underlying functions we can derive more precise estimates. To this end, we first prove some properties of the generalized finite differences.

Lemma 4.1. *Let $f^{(q+p+r+1)} \in AC[-1,1]$ for some $q, r \geq 0$ and $p \geq 1$, and let θ_k be as in Theorem 3.1. Then the following estimate holds as $N \to \infty$ and $|n| \geq N+1$*

$$\Delta_n^w(\Delta_n^p(\theta, \hat{F})) = \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \sum_{k=0}^{p} (sgn(n))^{p-k} \frac{\gamma_k(\tau)}{N^k} \sum_{t=w}^{r+1} \frac{1}{n^{t+p-k}} \tag{4.3}$$

$$\times \sum_{s=w}^{t} (sgn(n))^s \binom{t+p-k+q}{p-k+s} \frac{A_{t+q-s}(f)}{(i\pi)^{t-s}} \alpha_{k,s+p-k}(w) + \frac{o(N^{-p})}{n^{q+r+2}},$$

where

$$\alpha_{k,s}(w) = \sum_{j=0}^{w+p-k} (-1)^j \binom{w+p-k}{j} (k+j)^s.$$

Proof. It is easy to verify that

$$\Delta_n^p(\theta, \hat{F}) = \sum_{k=0}^{p} (-1)^k \frac{\gamma_k(\tau)}{N^k} \Delta_{n-sgn(n)k}^{p-k}(\hat{F}),$$

where the classical finite differences can be calculated as follows

$$\Delta_n^k(\hat{F}) = \sum_{j=0}^{k} \binom{k}{j} \hat{F}_{n-sgn(n)j}.$$

Taking into account that

$$\Delta_n^w(\Delta_n^{p-k}(\hat{F})) = \Delta_n^{w+p-k}(\hat{F}),$$

we get

$$\Delta_n^w(\Delta_n^p(\theta, \hat{F})) = \sum_{k=0}^{p} (-1)^k \frac{\gamma_k(\tau)}{N^k} \sum_{j=0}^{w+p-k} \binom{w+p-k}{j} \hat{F}_{n-sgn(n)(k+j)}. \tag{4.4}$$

In view of (2.1) we have

$$\hat{F}_{n-sgn(n)(k+j)} = \frac{(-1)^{n+k+j+1}}{2} \sum_{s=q}^{q+p-k+r+1} \frac{A_s(f)}{(i\pi(n-(\pm(k+j))))^{s+1}} + o(n^{-q-p+k-r-2}), \ n \to \infty.$$

Next, using (4.4) and denoting $h = q+p-k+r+1$, we can write

$$\Delta_n^w(\Delta_n^p(\theta, \hat{F})) = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{p} \frac{\gamma_k(\tau)}{N^k} \sum_{j=0}^{w+p-k} (-1)^j \binom{w+p-k}{j}$$

$$\times \sum_{s=q}^{h} \frac{A_s(f)}{(i\pi n)^{s+1}} \frac{1}{\left(1 - \frac{\pm(k+j)}{n}\right)^{s+1}} + \frac{o(N^{-p})}{n^{q+r+2}}$$

$$= \frac{(-1)^{n+1}}{2} \sum_{k=0}^{p} \frac{\gamma_k(\tau)}{N^k} \sum_{j=0}^{w+p-k} (-1)^j \binom{w+p-k}{j} \sum_{s=q}^{h} \frac{A_s(f)}{(i\pi n)^{s+1}}$$

$$\times \sum_{t=s}^{\infty} (\pm 1)^{t-s} \binom{t}{s} \frac{(k+j)^{t-s}}{n^{t-s}} + \frac{o(N^{-p})}{n^{q+r+2}}$$

$$= \frac{(-1)^{n+1}}{2(i\pi n)^{q+1}} \sum_{k=0}^{p} \frac{\gamma_k(\tau)}{N^k} \sum_{t=0}^{h-q} \frac{1}{n^t} \sum_{s=0}^{t} (\pm 1)^s \binom{t+q}{s} \frac{A_{t+q-s}(f)}{(i\pi)^{t-s}} \alpha_{k,s}(w) + \frac{o(N^{-p})}{n^{q+r+2}}. \qquad (4.5)$$

Taking into account that $\alpha_{k,s}(w) = 0$ for $s < w + p - k$, due to the well-known identity (see [18])

$$\sum_{k=0}^{p} (-1)^k \binom{p}{k} k^j = 0, \quad j = 0, \ldots, p-1, \qquad (4.6)$$

we take in the right-hand side of (4.5), $s \geq w + p - k$ and consequently $t \geq w + p - k$, and after some algebra we get the desired estimate.

Lemma 4.2. *Let $f^{(q+p+r+1)} \in AC[-1,1]$ for some $q, r \geq 0$ and $p \geq 1$, and let θ_k be as in Theorem 4.1. Then the following estimate holds as $N \to \infty$*

$$\Delta_{\pm N}^{w}(\Delta_n^p(\theta, \hat{F})) = \frac{(-1)^{N+1}}{2(\pm i\pi N)^{q+1} N^p} \sum_{t=\frac{p+w}{2}}^{r+1} \frac{1}{N^t} \sum_{s=w}^{t} A_{t+q-s}(f) \frac{\beta_{p,q}(w,s,t)}{(\pm i\pi)^{t-s}} + o(N^{-q-p-r-2}), \qquad (4.7)$$

where $w \leq p$ and p have the same parity, and

$$\Delta_{\pm N}^{w}(\Delta_n^p(\theta, \hat{F})) = \frac{(-1)^{N+1}}{2(\pm i\pi N)^{q+1} N^p} \sum_{t=\frac{p+w+1}{2}}^{r+1} \frac{1}{N^t} \sum_{s=w}^{t} A_{t+q-s}(f) \frac{\beta_{p,q}(w,s,t)}{(\pm i\pi)^{t-s}} + o(N^{-q-p-r-2}), \qquad (4.8)$$

when $w \leq p+1$ and p have opposite parity, where

$$\beta_{p,q}(w,s,t) = \sum_{k=0}^{p} \gamma_k(\tau) \binom{t+p-k+q}{p-k+s} \alpha_{k,s+p-k}(w),$$

and $\alpha_{k,s}$ are defined in Lemma 4.1.

Proof. Taking $n = \pm N$ in (4.3) we get

$$\Delta_{\pm N}^{w}(\Delta_n^p(\theta, \hat{F})) = \frac{(-1)^{N+1}}{2(\pm i\pi N)^{q+1} N^p} \sum_{t=w}^{r+1} \frac{1}{N^t} \sum_{s=w}^{t} \frac{A_{t+q-s}(f)}{(\pm i\pi)^{t-s}} \beta_{p,q}(w,s,t) + o(N^{-q-p-r-2}). \qquad (4.9)$$

As it was mentioned above, when τ_k are the roots of the Laguerre polynomial $L_p^q(x)$, then the coefficients $\gamma_k(\tau)$ have an explicit form (see (4.1)), and hence $\beta_{p,q}(w,s,t)$ can be written in the following form

$$\beta_{p,q}(w,s,t) = \frac{(p+q)!}{(t+q-s)!} \sum_{k=0}^{p} \binom{p}{k} \frac{(t+p-k+q)!}{(q+p-k)!(p-k+s)!} \qquad (4.10)$$

$$\times \sum_{j=0}^{w+p-k} (-1)^j \binom{w+p-k}{j} (k+j)^{s+p-k}.$$

Below we show that

$$\beta_{p,q}(w,s,t) = 0, \quad t \leq \frac{w+p-1}{2},$$

which proves (4.7) and (4.8).

First observe that, applying Newton's binomial formula, the equality (4.10) we can write in the form

$$\beta_{p,q}(w,s,t) = \frac{(p+q)!}{(t+q-s)!} \sum_{k=0}^{p} \binom{p}{k} \frac{(t+q+p-k)!}{(q+p-k)!(p-k+s)!} \sum_{u=0}^{s+p-k} \binom{s+p-k}{u} k^u$$

$$\times \sum_{j=0}^{w+p-k} (-1)^j \binom{w+p-k}{j} j^{s+p-k-u}.$$

Taking into account that the last sum vanishes for $s+p-k-u<w+p-k$ we get

$$\beta_{p,q}(w,s,t)=(-1)^{w+p}\frac{(p+q)!}{(t+q-s)!}\sum_{\alpha=0}^{s-w}\frac{1}{(s-w-\alpha)!}\sum_{k=0}^{p}(-1)^k k^{s-\alpha-w}\binom{p}{k}$$
$$\times\frac{(t+q+p-k)!}{(q+p-k)!}\frac{(w+p-k)!}{(w+p-k+\alpha)!}S(p-k+\alpha+w,p-k+w),$$

where $S(n,k)$ are the Stirling numbers of the second kind (see [18]). Using the following property of Stirling numbers (see [18])

$$S(k+\alpha,k)=\sum_{j=0}^{\alpha}\binom{k+\alpha}{j+\alpha}c_j(\alpha),\quad \alpha\geq 0, \tag{4.11}$$

where $c_j(\alpha)$ are the associated Stirling numbers of the second kind, we can write

$$S(p-k+\alpha+w,p-k+w)=\sum_{j=0}^{\alpha}\binom{p-k+w+\alpha}{j+\alpha}c_j(\alpha).$$

So, for $\beta_{p,q}(w,s,t)$ we obtain

$$\beta_{p,q}(w,s,t)=(-1)^{w+p}\frac{(p+q)!}{(t+q-s)!}\sum_{\alpha=0}^{s-w}\frac{1}{(s-w-\alpha)!}\sum_{j=0}^{\alpha}\frac{c_j(\alpha)}{(j+\alpha)!}\sum_{k=0}^{p}(-1)^k k^{s-\alpha-w}\binom{p}{k}$$
$$\times\frac{(t+q+p-k)!}{(q+p-k)!}\frac{(w+p-k)!}{(w+p-k-j)!}. \tag{4.12}$$

This, in view of identity (4.6), proves that $\beta_{p,q}(w,s,t)=0$ for $t\leq\frac{1}{2}(w+p-1)$. Lemma 4.2 is proved.

The next theorem describes the behavior of $R_{N,q,p}$ for even p.

Theorem 4.2. *Let $f^{(q+p+\frac{p}{2}+1)}\in AC[-1,1]$ for some $q\geq 0$ and $p\geq 1$ (p is even), and let θ_k be as in Theorem 4.1. Then*

$$R_{N,q,p}(f,x)=A_q(f)\frac{(-1)^N}{2^{p+1}\pi^{q+1}N^{q+p+\frac{p}{2}+1}}\frac{\sin\frac{\pi}{2}(x(2N-p+1)-q)}{\cos^{p+1}\frac{\pi x}{2}}\beta_{p,q}\left(0,\frac{p}{2},\frac{p}{2}\right)$$
$$+o(N^{-q-p-\frac{p}{2}-1}),\quad N\to\infty,$$

where $\beta_{p,q}$ are defined in Lemma 4.2.

Proof. An application of the Abel transformation to $R^{+}_{N,q,p}(f,x)$ (see (3.2)) implies

$$R^{\pm}_{N,q,p}(f,x)=-\frac{e^{\pm i\pi(N+1)x}}{\prod_{k=1}^{p}(1+\theta_{\pm k}e^{\pm i\pi x})}\frac{\Delta^0_{\pm N}(\Delta^p_n(\theta,\hat{F}))}{1+e^{\pm i\pi x}}$$
$$-\frac{e^{\pm i\pi(N+1)x}}{\prod_{k=1}^{p}(1+\theta_{\pm k}e^{\pm i\pi x})}\sum_{w=1}^{\frac{p}{2}+1}\frac{\Delta^w_{\pm N}(\Delta^p_n(\theta,\hat{F}))}{(1+e^{\pm i\pi x})^{w+1}}$$
$$+\frac{1}{\prod_{k=1}^{p}(1+\theta_{\pm k}e^{\pm i\pi x})}\frac{1}{(1+e^{\pm i\pi x})^{\frac{p}{2}+2}}\sum_{n=N+1}^{\infty}\Delta^{\frac{p}{2}+2}_{\pm n}(\Delta^p_n(\theta,\hat{F}))e^{\pm i\pi nx}. \tag{4.13}$$

According to Lemma 4.1 we have

$$\Delta_n^{\frac{p}{2}+2}(\Delta^p_n(\theta,\hat{F}))=\frac{o(N^{-p})}{n^{q+\frac{p}{2}+2}},\quad N\to\infty,\quad |n|\geq N+1,$$

and hence the last term on the right-hand side of (4.13) is $o(N^{-q-p-\frac{p}{2}-1})$ as $N \to \infty$. The estimates (4.7) and (4.8) show that the second term in (4.13) is $O(N^{-q-p-\frac{p}{2}-2})$ as $N \to \infty$. Therefore,

$$R^{\pm}_{N,q,p}(f,x) = -\frac{e^{\pm i\pi(N+1)x}}{(1+e^{\pm i\pi x})^{p+1}}\Delta^0_{\pm N}(\Delta^p_n(\theta,\hat{F})) + o(N^{-q-p-\frac{p}{2}-1}), \quad N \to \infty. \tag{4.14}$$

Next, the estimate (4.7) implies

$$\begin{aligned}\Delta^0_{\pm N}(\Delta^p_n(\theta,\hat{F})) &= \frac{(-1)^{N+1}}{2(\pm i\pi N)^{q+1}N^p}\sum_{t=\frac{p}{2}}^{\frac{p}{2}+1}\frac{1}{N^t}\sum_{s=0}^{t}A_{t+q-s}(f)\frac{\beta_{p,q}(0,s,t)}{(\pm i\pi)^{t-s}} + o(N^{-q-p-\frac{p}{2}-2})\\ &= \frac{(-1)^{N+1}}{2(\pm i\pi N)^{q+1}N^{p+\frac{p}{2}}}\sum_{s=0}^{\frac{p}{2}}A_{\frac{p}{2}+q-s}(f)\frac{\beta_{p,q}\left(0,s,\frac{p}{2}\right)}{(\pm i\pi)^{\frac{p}{2}-s}} + O(N^{-q-p-\frac{p}{2}-2}).\end{aligned} \tag{4.15}$$

In view of identity (4.6), the equation (4.12) shows that $\beta_{p,q}\left(0,s,\frac{p}{2}\right)=0$ for $s=0,\dots,\frac{p}{2}-1$, and hence, in the right-hand side of (4.15), only the term corresponding to $s=\frac{p}{2}$ is nonzero, which leads to the following estimate

$$\Delta^0_{\pm N}(\Delta^p_n(\theta,\hat{F})) = A_q(f)\frac{(-1)^{N+1}}{2(\pm i\pi N)^{q+1}N^{p+\frac{p}{2}}}\beta_{p,q}\left(0,\frac{p}{2},\frac{p}{2}\right) + O(N^{-q-p-\frac{p}{2}-2}), \quad N \to \infty.$$

Substituting this into (4.14) we get

$$R^{\pm}_{N,q,p}(f,x) = A_q(f)\frac{e^{\pm i\pi(N+1)x}}{(1+e^{\pm i\pi x})^{p+1}}\frac{(-1)^N}{2(\pm i\pi)^{q+1}N^{p+q+\frac{p}{2}+1}}\beta_{p,q}\left(0,\frac{p}{2},\frac{p}{2}\right) + o(N^{-q-p-\frac{p}{2}-1}),$$

yielding the final expansion of the error:

$$R_{N,q,p}(f,x) = A_q(f)\frac{(-1)^N}{\pi^{q+1}N^{p+q+\frac{p}{2}+1}}\beta_{p,q}\left(0,\frac{p}{2},\frac{p}{2}\right) Re\left[\frac{e^{i\pi(N+1)x}}{(1+e^{i\pi x})^{p+1}i^{q+1}}\right] + o(N^{-q-p-\frac{p}{2}-1}).$$

This completes the proof.

Similarly the next theorem can be proved.

Theorem 4.3. *Let $f^{(q+p+\frac{p+1}{2}+1)} \in AC[-1,1]$ for some $q \geq 0$ and $p \geq 1$ (p is odd), and let θ_k be as in Theorem 4.1. Then the following estimate holds for $|x|<1$ and $N \to \infty$*

$$R_{N,q,p}(f,x) = \frac{\varphi_{N,q,p}(x)}{N^{p+q+\frac{p+1}{2}+1}} + o(N^{-p-q-\frac{p+1}{2}-1}),$$

where

$$\begin{aligned}\varphi_{N,q,p}(x) &= A_q(f)\frac{(-1)^N}{\pi^{q+1}}\frac{\sin\frac{\pi}{2}(x(2N-p+1)-q)}{2^{p+1}\cos^{p+1}\frac{\pi x}{2}}\beta_{p,q}\left(0,\frac{p+1}{2},\frac{p+1}{2}\right)\\ &- A_{q+1}(f)\frac{(-1)^N}{\pi^{q+2}}\frac{\cos\frac{\pi}{2}(x(2N-p+1)-q)}{2^{p+1}\cos^{p+1}\frac{\pi x}{2}}\beta_{p,q}\left(0,\frac{p-1}{2},\frac{p+1}{2}\right)\\ &+ A_q(f)\frac{(-1)^N}{\pi^{q+1}}\frac{\sin\frac{\pi}{2}(x(2N-p)-q)}{2^{p+2}\cos^{p+2}\frac{\pi x}{2}}\beta_{p,q}\left(1,\frac{p+1}{2},\frac{p+1}{2}\right),\end{aligned}$$

and $\beta_{p,q}$ are defined in Lemma 4.2.

REFERENCES

1. B. Adcock, *Modified Fourier Expansions: Theory, Construction and Applications* (PHD thesis, Trinity Hall, University of Cambridge, 2010).

2. G. A. Baker and P. Graves-Morris, *Padè Approximants* (Encyclopedia of mathematics and its applications, Vol. 59, Cambridge Univ. Press, Cambridge, 1966).
3. G. Baszenski, F.-J. Delvos, and M. Tasche, "A united approach to accelerating trigonometric expansions", Comput. Math. Appl. **30**(3 − 6), 33–49, 1995.
4. H. Bateman, *Higher Transcendental Functions, vol. II* (McGraw-Hill Book Company, 1953).
5. D. Batenkov and Y. Yomdin, "Algebraic Fourier reconstruction of piecewise smooth functions", Mathematics of Computation, **81**, 277–318, 2012.
6. J. P. Boyd, "Acceleration of algebraically-converging Fourier series when the coefficients have series in powers of $1/n$", J. Comp. Phys. **228**(5), 1404 − 1411, 2009.
7. E.W. Cheney, *Introduction to Approximation Theory* (McGraw-Hill, New York, 1966).
8. J. Geer, "Rational trigonometric approximations using Fourier series partial sums", Journal of Scientific Computing, **10**(3), 325-356, 1995.
9. W. B. Jones and G. Hardy, "Accelerating Convergence of Trigonometric Approximations", Math. Comp., **24**, 47-60, 1970.
10. A. Krylov, *On Approximate Calculations, Lectures Delivered in 1906* (Tipolitography of Birkenfeld, St. Petersburg, 1907).
11. C. Lanczos, *Discourse on Fourier Series* (Oliver and Boyd, Edinburgh, 1966).
12. A. Nersessian, and A. Poghosyan, "On a rational linear approximation of Fourier series for smooth functions", Journal of Scientific Computing, **26**(1), 111–125, 2006.
13. A. Poghosyan, "On an autocorrection phenomenon of the Krylov-Gottlieb-Eckhoff method", IMA Journal of Numerical Analysis, **31**(2), 512–527, doi:10.1093/imanum/drp043, 2011.
14. A. Poghosyan, "Fast convergence of the Fourier-Pade approximation for smooth functions", Abstracts of International Conference Harmonic Analysis and Approximations V, 10-17 September, Tsaghkadzor, Armenia, 2011, http://math.sci.am/conference/sept2011/abstracts.html.
15. A. Poghosyan, T. Barkhudaryan, and A. Nurbekyan, "Convergence accleration of Fourier series by the roots of the Laguerre polynomial", Proceedings of the Third Russian-Armenian workshop on mathematical physics, complex analysis and related topics, Tsaghkadzor, Armenia, 137–141, 2010, http://math.sci.am/conference/oct2010/abstractsbook.html.
16. A. Poghosyan, "On a convergence of the L_2-optimal rational approximation", Reports of NAS RA **112**(4), 341–349, 2012.
17. I. I. Privalov, *Fourier Series*, (Gostekhizdat, Moscow-Leningrad, 1931).
18. J. Riordan, *Combinatorial Identities* (Wiley, New York, 1979).
19. G. P. Tolstov, *Fourier Series, (translated from the Russian (1950) by R. A. Silverman)* (Prentice-Hall, New Jersey, 1962).