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ON AN AUTOCORRECTION PHENOMENON OF THE ECKHOFF INTERPOLATION

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ABSTRACT. The paper considers the Krylov-Lanczos and the Eckhoff interpolations of a function with a discontinuity at a known point. These interpolations are based on certain corrections associated with jumps in the first derivatives. In the Eckhoff interpolation, approximation of the exact jumps is accomplished by the solution of a system of linear equations. We show that in the regions where the 2-periodic extension of the interpolated function is smooth, the Eckhoff interpolation converges faster compared with the Krylov-Lanczos interpolation. This accelerated convergence is known as the autocorrection phenomenon. The paper presents a theoretical explanation of this phenomenon. Numerical experiments confirm theoretical estimates.

Key words and phrases: Trigonometric interpolation, Krylov-Lanczos interpolation, Eckhoff interpolation, Bernoulli polynomials, Convergence acceleration, Autocorrection phenomenon.

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1. INTRODUCTION

We continue investigations started in [4], [22], [23], and [25] where the accuracy of the Krylov-Lanczos (KL) and the Eckhoff approximations and interpolations were explored. The KL approximation or interpolation (see also [1], [5], [6], [12]-[16], and references therein) is a sum of a correction polynomial, representing the discontinuities in the function and some of its first derivatives (jumps), and truncated Fourier series

$$S_N(f) := \sum_{n=-N}^{N} f_n e^{i\pi nx},\ f_n := \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi nx} dx$$

or trigonometric interpolation

$$I_N(f) := \sum_{n=-N}^{N} \check{f}_n e^{i\pi nx},\ \check{f}_n := \frac{1}{2N+1}\sum_{k=-N}^{N} f(x_k) e^{-i\pi n x_k},\ x_k := \frac{2k}{2N+1}$$

of the corrected function. The KL approximation deals with the Fourier coefficients of the approximated function while the KL interpolation treats the discrete Fourier coefficients. It is supposed that the exact values of the jumps are known in the KL methods. Eckhoff et al. [8]-[11] developed a new approach for calculating the polynomial terms in the representation suggested by Krylov and Lanczos, deriving a system of linear equations for calculation of the jumps. The corresponding modifications of the KL approximation and interpolation are known as the Eckhoff approximation and interpolation, respectively. For further developments of these methods see also [2], [3], [17]-[21], [24] and references therein.

In [4] the Eckhoff approximation was explored. In particular, the accuracy of the jumps approximation was studied and L_2 error of the approximation was computed. Based on these estimates, in [25] the asymptotic behavior of the Eckhoff approximation was examined on the subintervals where the approximated function was smooth. It was found that the Eckhoff approximation was more precise (by the rate of convergence) compared with the KL approximation. This convergence acceleration phenomenon, which was quite contrary to the slow convergence that might be expected due to approximate calculation of the jumps, was called the autocorrection phenomenon of the Eckhoff approximation.

In [22] the asymptotic behavior of the KL interpolation was investigated on the subintervals where the interpolated function was smooth. Exact asymptotic constants of the errors were obtained. In [23] the problem of the jumps approximation by the discrete Fourier coefficients was treated. Exact asymptotic errors of the jumps approximation and the corresponding interpolation were derived. Based on these results, in this paper we investigate the Eckhoff interpolation on the subintervals where the interpolated function is smooth. Comparison with the results in [22] reveals the autocorrection phenomenon of the Eckhoff interpolation. We show that the convergence rate of the interpolation even excesses the corresponding convergence rate of the approximation.

The paper is organized as follows:

Section 2 presents the KL and the Eckhoff approximations, describes the Eckhoff method of the jumps approximation and represents the theory of the autocorrection phenomenon. In particular, Subsection 2.1 considers the KL approximation $S_{N,q}(f)$ and investigates the error on the intervals $|x| < 1$. Parameter q indicates the number of derivatives that are involved in the process of approximation. Subsection 2.2 treats the problem of the jumps approximation and explores the Eckhoff approximation $\widetilde{S}_{N,q}(f)$ on the intervals $|x| < 1$. Theorems 2.2 and 2.4 consider even values of q ($q = 2m, m = 1, 2, \cdots$). Theorem 2.2 shows that the rate of convergence of $S_{N,2m}(f)$ is $O(N^{-2m-1})$ as $N \to \infty$. Theorem 2.4 states that the rate of convergence of $\widetilde{S}_{N,2m}(f)$ is $O(N^{-3m-1})$. We see the improvement in convergence rate by the factor $O(N^m)$. This convergence acceleration phenomenon is known as the autocorrection phenomenon of the Eckhoff approximation. Theorems 2.3 and 2.5 reveal this phenomenon for odd values of q ($q = 2m+1$, $m = 0, 1, \cdots$). In this case the rate of convergence of $S_{N,2m+1}(f)$ is $O(N^{-2m-2})$ and the rate of convergence of $\widetilde{S}_{N,2m+1}(f)$ is $O(N^{-3m-2})$. We have an improvement in convergence rate by the factor $O(N^m)$.

Section 3 considers the KL interpolation $I_{N,q}(f)$ and the Eckhoff interpolation $\widetilde{I}_{N,q}(f)$. Theorems 3.1 and 3.2 reveal the asymptotic behavior of the KL interpolation on the interval $|x| < 1$ for even and odd values of q, respectively. For even values of q we have $O(N^{-2m-1})$ and for odd values - $O(N^{-2m-3})$. Subsection 3.2 considers the problem of the jumps approximation via discrete Fourier coefficients and introduce the Eckhoff interpolation.

Section 4 investigates the autocorrection phenomenon of the Eckhoff interpolation. Subsection 4.1 considers even values of parameter q. Theorem 4.3 shows that the rate of convergence of $\widetilde{I}_{N,2m}(f)$ on the interval $|x| < 1$ is $O(N^{-4m-1})$. In comparison with the convergence rate $O(N^{-2m-1})$ of $I_{N,2m}(f)$ we have an improvement in convergence rate by the factor $O(N^{2m})$. We see that for interpolation the autocorrection phenomenon is much larger than for the non-interpolating approximations - improvement by the factor $O(N^{2m})$ instead of $O(N^m)$. Note also that interpolation $\widetilde{I}_{N,2m}(f)$ is even more precise than approximation $\widetilde{S}_{N,2m}(f)$ when $|x| < 1$. Subsection 4.2 explores odd values of parameter q. Theorem 4.7 shows that the rate of convergence of $\widetilde{I}_{N,2m+1}(f)$ on the interval $|x| < 1$ is $O(N^{-4m-3})$ while for $I_{N,2m+1}(f)$ is $O(N^{-2m-3})$. Hence, we have an improvement in convergence rate by the factor $O(N^{2m})$.

Some auxiliary lemmas and useful information are presented in Appendixes.

2. THE KRYLOV-LANCZOS AND THE ECKHOFF APPROXIMATIONS. THE AUTOCORRECTION PHENOMENON

In this section we describe the Krylov-Lanczos and the Eckhoff approximations and present estimates of the asymptotic errors. Comparison of these results reveals the essence of the autocorrection phenomenon. Main results are coming from [4] and [25].

2.1. The Krylov-Lanczos approximation.

Throughout the paper we limit our discussion to a smooth function f on $[-1, 1]$. Suppose $f \in C^q[-1, 1]$ and denote by $A_k(f)$ the exact value of the jump in the k-th derivative of f

$$A_k(f) := f^{(k)}(1) - f^{(k)}(-1), \quad k = 0, \dots, q.$$

The following lemma is crucial for the Krylov-Lanczos approach.

Lemma 2.1. *Let $f \in C^{q-1}[-1, 1]$ and $f^{(q-1)}$ is absolutely continuous on $[-1, 1]$ for some $q \geq 1$. Then the following expansion is valid*

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q-1} \frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^{1} f^{(q)}(x) e^{-i\pi n x} dx, \; n \neq 0.$$

Proof. The proof is trivial due to integration by parts. ∎

Lemma 2.1 implies the representation

$$f(x) = \sum_{k=0}^{q-1} A_k(f) B(x; k) + F(x), \tag{2.1}$$

where $B(x;k)$ are 2-periodic extensions of the Bernoulli polynomials (see Appendix B) and F is a 2-periodic and relatively smooth function on the real line ($F \in C^{q-1}(R)$) with the Fourier coefficients

$$F_n = f_n - \sum_{k=0}^{q-1} A_k(f) B_n(k). \tag{2.2}$$

Approximation of F by the truncated Fourier series leads to the *Krylov-Lanczos (KL) approximation*

$$S_{N,q}(f) := \sum_{n=-N}^{N} \left(f_n - \sum_{k=0}^{q-1} A_k(f) B_n(k) \right) e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f) B(x;k)$$

with the error

$$R_{N,q}(f) := f(x) - S_{N,q}(f).$$

In the next two theorems we present estimates for the accuracy of the KL approximation on the subintervals where the approximated function is smooth.

Theorem 2.2. [25] *Let q be an even number, $q = 2m$, $m = 1, 2, \cdots$. Suppose $f \in C^{2m+1}[-1,1]$ and $f^{(2m+1)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds for $|x| < 1$*

$$R_{N,2m}(f) = A_{2m}(f) \frac{(-1)^{N+m}}{2(\pi N)^{2m+1}} \frac{\sin \pi \left(N + \frac{1}{2}\right) x}{\cos \frac{\pi x}{2}} + o(N^{-2m-1}),\ N \to \infty.$$

Theorem 2.3. [25] *Let q be an odd number, $q = 2m+1$, $m = 0, 1, \cdots$. Suppose $f \in C^{2m+2}[-1,1]$ and $f^{(2m+2)}$ is absolutely continuous on $[-1,1]$. Then the following estimate holds for $|x| < 1$*

$$R_{N,2m+1}(f) = A_{2m+1}(f) \frac{(-1)^{N+m+1}}{2(\pi N)^{2m+2}} \frac{\cos \pi \left(N + \frac{1}{2}\right) x}{\cos \frac{\pi x}{2}} + o(N^{-2m-2}),\ N \to \infty.$$

2.2. **Computation of the jumps. The Eckhoff approximation. The autocorrection phenomenon.** In [8]-[11] Eckhoff suggested to compute approximate jump values $A_k^a(f,N)$ for $A_k(f)$ directly from the Fourier coefficients f_n. As the Fourier coefficients F_n asymptotically ($n \to \infty$) decay faster than the coefficients f_n, therefore they can be discarded for large $|n|$. Hence, from (2.2) we derive the following system of linear equations for determining the approximate jumps

$$f_n = \sum_{k=0}^{q-1} A_k^a(f,N) B_n(k), \quad n = n_1, n_2, \ldots, n_q. \tag{2.3}$$

Thus, for any given N we assume to have chosen q different integer indices

$$n_1 = n_1(N), n_2 = n_2(N), \ldots, n_q = n_q(N)$$

for evaluating system (2.3).

We denote by $\widetilde{S}_{N,q}(f)$ the *Eckhoff approximation* which differs from the KL approximation that uses the approximate jumps $A_k^a(f,N)$ instead of the exact ones

$$\widetilde{S}_{N,q}(f) := \sum_{n=-N}^{N} \left(f_n - \sum_{k=0}^{q-1} A_k^a(f,N) B_n(k) \right) e^{i\pi n x} + \sum_{k=0}^{q-1} A_k^a(f,N) B(x;k).$$

We put

$$\widetilde{R}_{N,q}(f) := f(x) - \widetilde{S}_{N,q}(f).$$

Asymptotic behavior of $\widetilde{R}_{N,q}(f)$ together with the accuracy of the jumps approximation for different choices of the indices n_s were investigated in [4].

In this article we are interested in the following choices of the indices n_s in system (2.3)

$$\begin{aligned} n_s &= N - s + 1,\ s = 1, \cdots, m, \\ n_s &= -(N - s + m + 1),\ s = m + 1, \cdots, 2m \end{aligned} \tag{2.4}$$

for even values of q, $q = 2m$, $m = 1, 2, \cdots$, and

$$\begin{aligned} n_s &= N - s + 1,\ s = 1, \cdots, m + 1, \\ n_s &= -(N - s + m + 2),\ s = m + 2, \cdots, 2m + 1 \end{aligned} \tag{2.5}$$

for odd values of q, $q = 2m + 1$, $m = 0, 1, \cdots$.

The next two theorems address the accuracy of the Eckhoff approximation for these choices of the indices n_s on the subinterval $|x| < 1$ where the approximated function is smooth.

Theorem 2.4. [25] *Let q be an even number, $q = 2m$, $m = 1, 2, \cdots$ and the indices $n_s = n_s(N)$ be chosen as in (2.4). Suppose that $f \in C^{3m+1}[-1, 1]$ and $f^{(3m+1)}$ is absolutely continuous on $[-1, 1]$. Then the following estimate holds for $|x| < 1$ and $N \to \infty$*

$$\begin{aligned} \widetilde{R}_{N,2m}(f) &= A_{2m}(f)\frac{(-1)^{N+m}}{2^{m+1}N^{3m+1}\pi^{2m+1}}\frac{\sin\frac{\pi x}{2}(2N - m + 1)}{(\cos\frac{\pi x}{2})^{m+1}} \\ &\times \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(m+2k)!}{(2k)!} + o(N^{-3m-1}). \end{aligned}$$

Theorem 2.5. [25] *Let q be an odd number, $q = 2m + 1$, $m = 0, 1, \cdots$ and the indices $n_s = n_s(N)$ be chosen as in (2.5). Suppose that $f \in C^{3m+1}[-1, 1]$ and $f^{(3m+1)}$ is absolutely continuous on $[-1, 1]$. Then the following estimate holds for $|x| < 1$ and $N \to \infty$*

$$\begin{aligned} \widetilde{R}_{N,2m+1}(f) &= A_{2m+1}(f)\frac{(-1)^{N+m+1}}{2N^{3m+2}\pi^{2m+2}}\frac{e^{-i\pi(N+1)x}}{(1 + e^{-i\pi x})^{m+1}} \\ &\times \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(m+2k)!(4k+m+2)}{(2k+1)!} + o(N^{-3m-2}). \end{aligned}$$

Theorem 2.2 states that for even values of q ($q = 2m,\ m = 1, 2, \cdots$) approximation $S_{N,2m}(f)$ has the rate of convergence $O(N^{-2m-1})$, while Theorem 2.4 shows that $\widetilde{S}_{N,2m}(f)$ has the rate $O(N^{-3m-1})$. As a result we have an improvement in convergence rate by the factor $O(N^m)$. This convergence acceleration phenomenon is known as the autocorrection phenomenon of the Eckhoff approximation (see [25]). Comparison of Theorems 2.3 and 2.5 reveals this phenomenon for odd values of q ($q = 2m + 1,\ m = 0, 1, \cdots$). In this case the rate of convergence is $O(N^{-3m-2})$ for $\widetilde{S}_{N,2m+1}(f)$ and $O(N^{-2m-2})$ for $S_{N,2m+1}(f)$. We have an improvement in convergence rate by the factor $O(N^m)$. We see that for $m = 0$ ($q = 1$) this phenomenon is absent.

3. THE KRYLOV-LANCZOS INTERPOLATION. COMPUTATION OF THE JUMPS. THE ECKHOFF INTERPOLATION.

In this section we describe the Krylov-Lanczos interpolation, treat the problem of the jumps approximation by the discrete Fourier coefficients, and define the Eckhoff interpolation. Main results are coming from papers [22] and [23].

3.1. **The Krylov-Lanczos interpolation.** Representation (2.1) allows calculation of the discrete Fourier coefficients of F as well

$$\check{F}_n = \check{f}_n - \sum_{k=0}^{q-1} A_k(f)\check{B}_n(k). \tag{3.1}$$

Approximation of F in (2.1) by $I_N(f)$ leads to the *Krylov-Lanczos (KL) interpolation*

$$I_{N,q}(f) := \sum_{n=-N}^{N} \left(\check{f}_n - \sum_{k=0}^{q-1} A_k(f)\check{B}_n(k) \right) e^{i\pi nx} + \sum_{k=0}^{q-1} A_k(f)B(x;k)$$

with the error

$$r_{N,q}(f) := f(x) - I_{N,q}(f).$$

For explicit calculation of the discrete Fourier coefficients $\check{B}_n(k)$ see Appendix B.

The next two theorems reveal the asymptotic behavior of the KL interpolation on the interval $|x| < 1$. Note that in Theorem 3.2 the required smoothness is higher than in Theorem 3.1, but the convergence rate is correspondingly higher.

Theorem 3.1. [22] *Let $q \ge 1$ be an even number $q = 2m$, $m = 1, 2, \cdots$. If $f \in C^{2m+1}[-1, 1]$ and $f^{(2m+1)}$ is absolutely continuous on $[-1, 1]$ then the following estimate holds as $N \to \infty$ and $|x| < 1$*

$$r_{N,2m}(f) = A_{2m}(f)\frac{(-1)^{N+m}}{2(\pi N)^{2m+1}} \frac{\sin \frac{\pi x}{2}(2N + 1)}{\cos \frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s + 1)^{2m+1}} + o(N^{-2m-1}).$$

Theorem 3.2. [22] *Let $q \ge 1$ be an odd number $q = 2m+1$, $m = 0, 1, \cdots$. If $f \in C^{2m+3}[-1, 1]$ and $f^{(2m+3)}$ is absolutely continuous on $[-1, 1]$ then the following estimate holds as $N \to \infty$ and $|x| < 1$*

$$\begin{aligned} r_{N,2m+1}(f) &= \frac{(-1)^{N+m+1}}{2\pi^{2m+2}N^{2m+3}} \frac{\sin \frac{\pi x}{2}(2N + 1)}{\cos \frac{\pi x}{2}} \\ &\times \left(A_{2m+1}(f)(2m + 2)tg\frac{\pi x}{2} \sum_{s=-\infty}^{\infty} \frac{(-1)^s s}{(2s + 1)^{2m+3}} \right. \\ &+ \left. \frac{A_{2m+2}(f)}{\pi} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s + 1)^{2m+3}} \right) + o(N^{-2m-3}). \end{aligned}$$

3.2. **Approximation of the jumps. The Eckhoff interpolation.** In this subsection we investigate the problem of the jumps approximation via the discrete Fourier coefficients.

Taking into account that in (3.1) the discrete Fourier coefficients $\check{F}_n$ can be discarded compared with $\check{f}_n$ as $n \to \infty$ we get the following system of linear equations with unknowns $A_k^i(f, N)$

$$\check{f}_n = \sum_{k=0}^{q-1} A_k^i(f, N)\check{B}_n(k),\ n = n_1, n_2, \cdots, n_q. \tag{3.2}$$

Approximation by $I_{N,q}(f)$ where the exact values of the jumps are replaced by the approximated ones, calculated from system (3.2) we call the *Eckhoff interpolation* and denote by $\widetilde{I}_{N,q}(f)$

$$\widetilde{I}_{N,q}(f) := \sum_{n=-N}^{N} \left(\check{f}_n - \sum_{k=0}^{q-1} A_k^i(f, N)\check{B}_n(k) \right) e^{i\pi nx} + \sum_{k=0}^{q-1} A_k^i(f, N)B(x;k)$$

with the error

$$\widetilde{r}_{N,q}(f) := f(x) - \widetilde{I}_{N,q}(f).$$

As was shown in [23] the numbers $A_k^i(f,N)$ approximated the exact values $A_k(f)$ as $N \to \infty$ and $cN \le n_s \le N, 0 < c \le 1$. Now, following [23] we will show that system (3.2) is equivalent to a system with upper triangular matrix that will simplify the process of the solution.

According to relation (B.3) we rewrite system (3.2) as follows

$$\check{f}_{n_s} \sin^q \pi\tau_s = \frac{(-1)^{n_s+1}\pi}{2} \sum_{k=0}^{q-1} \frac{(-1)^k A_k^i(f,N)}{(i\pi(2N+1))^{k+1}k!} \left(\frac{1}{\sin \pi x}\right)^{(k)}_{x=\tau_s} \sin^q \pi\tau_s, \tag{3.3}$$

where $\tau_s := \frac{n_s}{2N+1}$. In view of Lemma A.3 we copy out (3.3) in the form

$$\sum_{k=0}^{q-1} \frac{(-1)^k A_k^i(f,N)}{(2N+1)^{k+1}k!} \sum_{j=0}^{q-1} \beta_{j,k}(q) e^{-2i\pi j\tau_s} = (2i)^q (-1)^{n_s+1} e^{-i\pi(q-1)\tau_s} \check{f}_{n_s} \sin^q \pi\tau_s.$$

Taking into account that $\{e^{-2i\pi j\tau_s}\}$ are the elements of Vandermonde matrix we get

$$\begin{aligned}\sum_{k=0}^{q-1} \frac{(-1)^k A_k^i(f,N)}{(2N+1)^{k+1}k!} \beta_{j,k}(q) = (2i)^q \sum_{s=1}^{q} \frac{(-1)^{n_s+1} \check{f}_{n_s} e^{-i\pi(q-1)\tau_s} \sin^q \pi\tau_s}{\prod\limits_{\substack{r=1\\ r\neq s}}^{q} \left(e^{-2i\pi\tau_s} - e^{-2i\pi\tau_r}\right)} \\ \times \sum_{t=j+1}^{q} \gamma_t e^{-2i\pi\tau_s(t-j-1)},\end{aligned} \tag{3.4}$$

where the numbers γ_s are the coefficients of the polynomial

$$\prod_{n=1}^{q} \left(e^{-2i\pi x} - e^{-2i\pi\tau_n}\right) = \sum_{s=0}^{q} \gamma_s(q) e^{-2i\pi s x}.$$

In view of Lemmas A.5 and A.6 we copy out (3.4) in the equivalent form

$$\begin{aligned}\sum_{k=0}^{q-1} u_{j,k} \frac{(-1)^k A_k^i(f,N)}{(2N+1)^{k+1}k!} = (2i)^q \sum_{k=0}^{q-1} \ell_{j,k} \sum_{s=1}^{q} \frac{(-1)^{n_s+1} \check{f}_{n_s} \sin^q \pi\tau_s e^{-i\pi(q-1)\tau_s}}{\prod\limits_{\substack{r=1\\ r\neq s}}^{q} \left(e^{-2i\pi\tau_s} - e^{-2i\pi\tau_r}\right)} \\ \times \sum_{t=k+1}^{q} \gamma_t(q) e^{-2i\pi\tau_s(t-k-1)}.\end{aligned} \tag{3.5}$$

Note that matrix $u_{j,k}$ in (3.5) is an upper triangular matrix, and hence unknowns $A_k^i(f,N)$ can be calculated by backward substitution with $O(q^2)$ operations.

4. THE AUTOCORRECTION PHENOMENON OF THE ECKHOFF INTERPOLATION

In this section we investigate the accuracy of the Eckhoff interpolation on the intervals $|x| < 1$ and reveal the theoretical background of the autocorrection phenomenon.

In the sequel subsections we explore the autocorrection phenomenon for even and odd values of parameter q, separately. Hereafter we will suppose that the indices n_s are chosen in accordance with (2.4) or (2.5).

4.1. **Even values of q.** In this subsection we explore the accuracy of the jumps approximation and the corresponding Eckhoff interpolation for even values of q when the indices n_s are chosen as in (2.4). We will suppose in this subsection that $q = 2m$, $m = 1, 2, \cdots$.

Taking into account the obvious relations

$$\check{B}_{-n}(2k) = -\check{B}_n(2k), \ \ \check{B}_{-n}(2k+1) = \check{B}_n(2k+1) \tag{4.1}$$

we split system (3.2) into two independent subsystems for determining the jumps $A^i_{2k}(f,N)$ and $A^i_{2k+1}(f,N)$, separately

$$\frac{\check{f}_n - \check{f}_{-n}}{2} = \sum_{k=0}^{m-1} A^i_{2k}(f,N)\check{B}_n(2k), \ \ n = N, N-1, \cdots, N-m+1, \tag{4.2}$$

$$\frac{\check{f}_n + \check{f}_{-n}}{2} = \sum_{k=0}^{m-1} A^i_{2k+1}(f,N)\check{B}_n(2k+1), \ \ n = N, N-1, \cdots, N-m+1. \tag{4.3}$$

In the next theorem we present the accuracy of the jumps approximation by (3.2) or equivalently by (4.2)-(4.3).

Theorem 4.1. *Suppose that q is an even number, $q = 2m$, $m = 1, 2, \cdots$ and the indices n_s are chosen as in (2.4). If $f \in C^{4m}[-1,1]$ such that $f^{(4m)}$ is absolutely continuous on $[-1,1]$ then the following estimates hold as $N \to \infty$*

$$A^i_{2k}(f,N) = A_{2k}(f) + A_{2m}(f)\frac{(2k)!\,\nu_{2k}}{(2N)^{2m-2k}} + o(N^{-2m+2k-1}), \tag{4.4}$$

$$A^i_{2k+1}(f,N) = A_{2k+1}(f) + A_{2m+1}(f)\frac{(2k+1)!\,\nu'_{2k+1}}{(2N)^{2m-2k}} + o(N^{-2m+2k}), \tag{4.5}$$

where the numbers ν_{2k} and ν'_{2k+1} are the solutions of systems

$$\sum_{k=s}^{m-1} u_{2s,2k}\nu_{2k} = \mu_{2s}, \ \sum_{k=s}^{m-1} u_{2s,2k+1}\nu'_{2k+1} = \mu'_{2s}, \ s = 0, \cdots, m-1$$

with

$$\mu_s := \sum_{j=0}^{2m-1} \ell_{s,j}\Upsilon_j, \qquad \mu'_s := \sum_{j=0}^{2m-1} \ell_{s,j}\Upsilon'_j,$$

and (see Lemma A.1)

$$\Upsilon_j : = \frac{1}{(2m)!}\sum_{s=0}^{2m} \alpha_{2m,s} \sum_{k=j+1}^{2m} \binom{2m}{k} \omega^*_{2m,s+k-j-1},$$

$$\Upsilon'_j : = \frac{1}{(2m+1)!}\sum_{s=0}^{2m+1} \alpha_{2m+1,s} \sum_{k=j+1}^{2m} \binom{2m}{k} \omega^*_{2m+1,s+k-j-1},$$

$$\omega^*_{r,j} := \sum_{s=0}^{2m-1} \frac{(-1)^{j+s}}{2^{r-s}} \binom{j}{s} \binom{r-s-1}{r-2m}.$$

Proof. Lemma 2.1 leads to the following expansion

$$f(x) = \sum_{k=0}^{4m} A_k(f)B(x;k) + F(x),$$

where $F \in C^{4m}(R)$ and consequently $F_n = o(n^{-4m-1}),\ n \to \infty$. From here we conclude that

$$\check{f}_n = \sum_{k=0}^{4m} A_k(f)\check{B}_n(k) + o(n^{-4m-1}),\ n \to \infty,\ |n| \le N.$$

Using this in (3.2), we derive

$$\begin{aligned}&\sum_{k=0}^{2m-1} (A_k^i(f,N) - A_k(f))\check{B}_{n_s}(k) \\ &\qquad = \sum_{k=2m}^{4m} A_k(f)\check{B}_{n_s}(k) + o(N^{-4m-1}),\ s = 1,\cdots,2m.\end{aligned} \tag{4.6}$$

Proceeding as for the proof of (3.4) we derive

$$\begin{aligned}\sum_{k=0}^{2m-1} \lambda_k \beta_{j,k}(2m) &= (2i)^{2m} \sum_{r=2m}^{4m} A_r(f) \sum_{s=1}^{2m} \frac{(-1)^{n_s+1}\sin^{2m}\pi\tau_s \check{B}_{n_s}(r)e^{-i\pi(2m-1)\tau_s}}{\prod\limits_{\substack{\ell=1\\ \ell\neq s}}^{2m}(e^{-2i\pi\tau_s} - e^{-2i\pi\tau_\ell})} \\ &\times \sum_{t=j+1}^{2m} \gamma_t(2m)e^{-2i\pi\tau_s(t-j-1)} + \sum_{s=1}^{2m} \frac{o(N^{-4m-1})}{\left|\prod\limits_{\substack{\ell=1\\ \ell\neq s}}^{2m}(e^{-2i\pi\tau_s} - e^{-2i\pi\tau_\ell})\right|},\end{aligned} \tag{4.7}$$

where

$$\lambda_k := \frac{(-1)^k}{(2N+1)^{k+1}k!}(A_k^i(f,N) - A(f)).$$

For the last term in (4.7) we write

$$\left|\prod_{\substack{\ell=1\\ \ell\neq s}}^{2m}(e^{-2i\pi\tau_s} - e^{-2i\pi\tau_\ell})\right| = \left|\prod_{\substack{\ell=1\\ \ell\neq s}}^{2m} 2i\sin\pi(\tau_\ell - \tau_s)\right| = O(N^{-2m+1}),\ N \to \infty.$$

Hence the last term is $o(N^{-2m-2})$ as $N \to \infty$. Taking into account (B.4), we obtain from (4.7)

$$\begin{aligned}\sum_{k=0}^{2m-1} \lambda_k \beta_{j,k}(2m) &= \sum_{r=2m}^{4m} A_r(f)\frac{(-1)^r}{r!(2N+1)^{r+1}} \sum_{\ell=0}^{r} \alpha_{r,\ell} \sum_{t=j+1}^{2m} \gamma_t \omega_{r,\ell+t-j-1} \\ &+ o(N^{-2m-2}),\ j = 0,\cdots,2m-1,\ N \to \infty,\end{aligned} \tag{4.8}$$

where

$$\omega_{r,j} := \sum_{s=1}^{2m} \frac{e^{-2i\pi j\tau_s}}{(1-e^{-2i\pi\tau_s})^{r-2m+1}\prod\limits_{\substack{\ell=1\\ \ell\neq s}}^{2m}(e^{-2i\pi\tau_s} - e^{-2i\pi\tau_\ell})}.$$

In view of the relations

$$\gamma_s = \binom{2m}{s} + O(N^{-2}),\ N \to \infty$$

and

$$\omega_{r,j} = \omega^*_{r,j} + O(N^{-2}),\ N \to \infty$$

we get from (4.8)

$$\sum_{k=0}^{2m-1} \lambda_k \beta_{j,k}(2m) = \frac{A_{2m}(f)\Upsilon_j}{(2N+1)^{2m+1}} - \frac{A_{2m+1}(f)\Upsilon'_j}{(2N+1)^{2m+2}} + o(N^{-2m-2}).$$

According to Lemma A.6 we derive

$$\sum_{k=s}^{2m-1} u_{s,k}\lambda_k = \frac{A_{2m}(f)\mu_s}{(2N+1)^{2m+1}} - \frac{A_{2m+1}(f)\mu'_s}{(2N+1)^{2m+2}} + o(N^{-2m-2}).$$

From here we conclude that

$$\lambda_k = \frac{A_{2m}(f)\nu_k}{(2N+1)^{2m+1}} - \frac{A_{2m+1}(f)\nu'_k}{(2N+1)^{2m+2}} + o(N^{-2m-2}).$$

Finally, we get

$$\begin{aligned} A^i_k(f,N) &= A_k(f) + A_{2m}(f)\frac{(-1)^k k!\nu_k}{(2N+1)^{2m-k}} \\ &- A_{2m+1}(f)\frac{(-1)^k k!\nu'_k}{(2N+1)^{2m-k+1}} + o(N^{-2m+k-1}),\ N \to \infty, \end{aligned} \tag{4.9}$$

where the numbers ν_k and ν'_k are the solutions of systems

$$\sum_{k=s}^{2m-1} u_{s,k}\nu_k = \mu_s, \qquad \sum_{k=s}^{2m-1} u_{s,k}\nu'_k = \mu'_s. \tag{4.10}$$

For even values of k in (4.9) we have

$$\begin{aligned} A^i_{2k}(f,N) &= A_{2k}(f) + A_{2m}(f)\frac{(2k)!\,\nu_{2k}}{(2N+1)^{2m-2k}} \\ &- A_{2m+1}(f)\frac{(2k)!\,\nu'_{2k}}{(2N+1)^{2m-2k+1}} + o(N^{-2m+2k-1}),\ N \to \infty. \end{aligned} \tag{4.11}$$

From (4.2) we see that $A^i_{2k}(f,N) - A_{2k}(f)$ doesn't depend on $A_{2m+1}(f)$. Therefore $\nu'_{2k} = 0$ and (4.11) coincides with (4.4).

For odd values of k in (4.9) we have as $N \to \infty$

$$\begin{aligned} A^i_{2k+1}(f,N) &= A_{2k+1}(f) - A_{2m}(f)\frac{(2k+1)!\nu_{2k+1}}{(2N+1)^{2m-2k-1}} \\ &+ A_{2m+1}(f)\frac{(2k+1)!\nu'_{2k+1}}{(2N+1)^{2m-2k}} + o(N^{-2m+2k}). \end{aligned} \tag{4.12}$$

From (4.3) we observe that $A^i_{2k+1}(f,N) - A_{2k+1}(f)$ doesn't depend on $A_{2m}(f)$, hence $\nu_{2k+1} = 0$. Now (4.12) coincides with (4.5). ∎

Denote

$$\check{G}_n := \check{f}_n - \sum_{k=0}^{q-1} A^i_k(f,N)\check{B}_n(k), \tag{4.13}$$

and

$$G_n := f_n - \sum_{k=0}^{q-1} A_k^i(f,N)B_n(k). \tag{4.14}$$

For the proof of the main theorem of this subsection we need the following lemma, where the asymptotic behaviors of $\Delta_n^{m+1}(G_n)$, $\Delta_n^{m+1}(\check{G}_n - G_n)$ and $\Delta_{\pm N}^{m}(\check{G}_n)$ are explored (see (B.6) for definition of $\Delta_n^p(f_n)$).

Lemma 4.2. *Suppose that q is an even number, $q = 2m$, $m = 1, 2, \cdots$ and the indices n_s are chosen as in (2.4). If $f \in C^{4m+1}[-1,1]$ such that $f^{(4m+1)}$ is absolutely continuous on $[-1,1]$ then the following estimates hold as $N \to \infty$*

$$\Delta_n^{m+1}(G_n) = \frac{o(N^{-2m})}{n^{2m+2}},\ |n| > N,$$

$$\Delta_n^{m+1}(\check{G}_n - G_n) = o(N^{-4m-2}),\ |n| \le N,$$

and

$$\begin{aligned}\Delta_{\pm N}^{m}(\check{G}_n) &= \pm A_{2m}(f)\frac{(-1)^{N+m}}{2^{2m+1}N^{4m+1}}\sum_{k=0}^{m}\frac{2^{2k}\nu_{2k}(2k+2m)!}{(i\pi)^{2k+1}} \\ &\quad\times \sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2m+2k+1}} + o(N^{-4m-1}),\end{aligned}$$

where $\nu_{2m} = -1/(2m)!$.

Proof. Lemma 2.1 implies the representation

$$f(x) = \sum_{k=0}^{4m+1} A_k(f)B(x;k) + F(x), \tag{4.15}$$

where $F \in C^{4m+1}(R)$. From here we conclude that

$$f_n = \sum_{k=0}^{4m+1} A_k(f)B_n(k) + F_n,\ F_n = o(n^{-4m-2}),\ n \to \infty. \tag{4.16}$$

In view of (4.14), we write

$$G_n = \sum_{k=0}^{2m-1}\left(A_k(f) - A_k^i(f,N)\right)B_n(k) + \sum_{k=2m}^{4m+1}A_k(f)B_n(k) + o(n^{-4m-2}).$$

Hence

$$\begin{aligned}\Delta_n^{m+1}(G_n) &= \sum_{k=0}^{2m-1}\left(A_k(f) - A_k^i(f,N)\right)\Delta_n^{m+1}(B_n(k)) \\ &\quad + \sum_{k=2m}^{4m+1}A_k(f)\Delta_n^{m+1}(B_n(k)) + o(n^{-4m-2}).\end{aligned}$$

This proves the first estimate in view of Theorem 4.1 and Lemma B.1.

For the proof of the second estimate note that representation (4.15) yields

$$\check{f}_n = \sum_{k=0}^{4m+1} A_k(f)\check{B}_n(k) + \check{F}_n. \tag{4.17}$$

Condition $F \in C^{4m+1}(R)$ implies

$$\check{F}_n - F_n = \sum_{s\neq 0} F_{n+s(2N+1)} = o(N^{-4m-2}),\ |n| \leq N,\ N \to \infty. \tag{4.18}$$

From (4.13), (4.14), (4.16), (4.17), and (4.18) we derive

$$\begin{aligned}
\Delta_n^{m+1}(\check{G}_n - G_n) &= \sum_{k=0}^{2m-1} (A_k(f) - A_k^i(f,N))\Delta_n^{m+1}(\check{B}_n(k) - B_n(k)) \\
&+ \sum_{k=2m}^{4m+1} A_k(f)\Delta_n^{m+1}(\check{B}_n(k) - B_n(k)) \\
&+ o(N^{-4m-2}),\ N \to \infty,\ |n| \leq N.
\end{aligned}$$

This concludes the proof of the second estimate by Theorem 4.1 and Lemma B.1.
From (4.13) and (4.17) we get

$$\begin{aligned}
\Delta_{\pm N}^m(\check{G}_n) &= \sum_{k=0}^{m-1}(A_{2k}(f) - A_{2k}^i(f,N))\Delta_{\pm N}^m(\check{B}_n(2k)) \\
&+ \sum_{k=0}^{m-1}(A_{2k+1}(f) - A_{2k+1}^i(f,N))\Delta_{\pm N}^m(\check{B}_n(2k+1)) \\
&+ \sum_{k=m}^{2m-1} A_{2k}(f)\Delta_{\pm N}^m(\check{B}_n(2k)) + \sum_{k=m}^{2m-1} A_{2k+1}(f)\Delta_{\pm N}^m(\check{B}_n(2k+1)) \\
&+ o(N^{-4m-2}).
\end{aligned}$$

This finishes the proof of the third estimate. ∎

In the next theorem we explore the asymptotic behavior of $\widetilde{r}_{N,2m}(f)$ on the interval $|x| < 1$. Comparison with Theorem 3.1 will clarify the essence of the autocorrection phenomenon for the Eckhoff interpolation when parameter q is even.

Theorem 4.3. *Suppose that the conditions of Lemma 4.2 are valid. Then the following estimate holds for* $|x| < 1$

$$\begin{aligned}
\widetilde{r}_{N,2m}(f) &= A_{2m}(f)\frac{(-1)^{N+m+1}}{(2N)^{4m+1}}\frac{\sin\frac{\pi x}{2}(2N+1)}{(\cos\frac{\pi x}{2})^{2m+1}} \\
&\times \sum_{k=0}^{m} \frac{(-1)^k 2^{2k}\nu_{2k}(2k+2m)!}{\pi^{2k+1}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{2m+2k+1}} \\
&+ o(N^{-4m-1}),\ N \to \infty,
\end{aligned}$$

where $\nu_{2m} = -1/(2m)!$.

Proof. It is easy to verify the following transformation

$$\begin{aligned}
\widetilde{r}_{N,2m}(f) &= \sum_{n=-N}^{N}(G_n-\check{G}_n)e^{i\pi nx}+\sum_{n=N+1}^{\infty}G_ne^{i\pi nx}+\sum_{n=-\infty}^{-N-1}G_ne^{i\pi nx}\\
&= \frac{e^{-i\pi Nx}-e^{i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})}\check{G}_N+\frac{e^{i\pi Nx}-e^{-i\pi(N+1)x}}{(1+e^{i\pi x})(1+e^{-i\pi x})}\check{G}_{-N}\\
&+ \frac{1}{(1+e^{i\pi x})(1+e^{-i\pi x})}\sum_{n=-N}^{N}\Delta_n^1(G_n-\check{G}_n)e^{i\pi nx}\\
&+ \frac{1}{(1+e^{i\pi x})(1+e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^1(G_n)e^{i\pi nx}\\
&+ \frac{1}{(1+e^{i\pi x})(1+e^{-i\pi x})}\sum_{n=-\infty}^{-N-1}\Delta_n^1(G_n)e^{i\pi nx}.
\end{aligned}$$

Reiteration of this transformation leads to the subsequent expansion of the error

$$\begin{aligned}
\widetilde{r}_{N,2m}(f) &= (e^{-i\pi Nx}-e^{i\pi(N+1)x})\sum_{k=1}^{m+1}\frac{\Delta_N^{k-1}(\check{G}_n)}{(1+e^{i\pi x})^k(1+e^{-i\pi x})^k}\\
&+(e^{i\pi Nx}-e^{-i\pi(N+1)x})\sum_{k=1}^{m+1}\frac{\Delta_{-N}^{k-1}(\check{G}_n)}{(1+e^{i\pi x})^k(1+e^{-i\pi x})^k}\\
&+\frac{1}{(1+e^{i\pi x})^{m+1}(1+e^{-i\pi x})^{m+1}}\sum_{n=-N}^{N}\Delta_n^{m+1}(G_n-\check{G}_n)e^{i\pi nx}\\
&+\frac{1}{(1+e^{i\pi x})^{m+1}(1+e^{-i\pi x})^{m+1}}\sum_{n=N+1}^{\infty}\Delta_n^{m+1}(G_n)e^{i\pi nx}\\
&+\frac{1}{(1+e^{i\pi x})^{m+1}(1+e^{-i\pi x})^{m+1}}\sum_{n=-\infty}^{-N-1}\Delta_n^{m+1}(G_n)e^{i\pi nx}.
\end{aligned} \tag{4.19}$$

According to Lemma 4.2 the last three terms in (4.19) are $o(N^{-4m-1})$. Taking into account that

$$\Delta_{\pm N}^s(\check{G}_n)=\sum_{k=0}^{2s}\binom{2s}{k}\check{G}_{\pm N+s-k}$$

we conclude that $\Delta_{\pm N}^s(\check{G}_n)=0$ as $s=0,\cdots,m-1$ in view of the relations

$$\check{G}_{n_s}=\check{f}_{n_s}-\sum_{k=0}^{2m-1}A_k^i(f,N)\check{B}_{n_s}(k)=0$$

when the indices n_s are chosen as in (2.4). Substituting all these into (4.19) we derive

$$\begin{aligned}
\widetilde{r}_{N,2m}(f) &= \frac{e^{-i\pi Nx}-e^{i\pi(N+1)x}}{(1+e^{i\pi x})^{m+1}(1+e^{-i\pi x})^{m+1}}\Delta_N^m(\check{G}_n)\\
&+ \frac{e^{i\pi Nx}-e^{-i\pi(N+1)x}}{(1+e^{i\pi x})^{m+1}(1+e^{-i\pi x})^{m+1}}\Delta_{-N}^m(\check{G}_n)+o(N^{-4m-1}),\ N\to\infty.
\end{aligned}$$

This concludes the proof in view of the third estimate of Lemma 4.2. ∎

Denote by $\|\cdot\|_\varepsilon$ the standard norm in the space $L_2(-\varepsilon,\varepsilon)$

$$||f||_\varepsilon := \left(\int_{-\varepsilon}^{\varepsilon} |f(x)|^2 dx\right)^{1/2}.$$

The next follows immediately from here.

Theorem 4.4. *Suppose that the conditions of Theorem 4.3 are valid. Then the following estimate holds for* $0<\varepsilon<1$

$$\lim_{N\to\infty} N^{4m+1}\|\widetilde{r}_{N,2m}(f)\|_\varepsilon = \frac{|A_{2m}(f)|}{2^{4m+\frac{3}{2}}}\left|\sum_{k=0}^{m}\frac{(-1)^k 2^{2k}\nu_{2k}(2k+2m)!}{\pi^{2k+1}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2m+2k+1}}\right| \times \left(\int_{-\varepsilon}^{\varepsilon}\frac{dx}{\cos^{4m+2}\frac{\pi x}{2}}\right)^{1/2},$$

where $\nu_{2m}=-1/(2m)!$.

Theorems 3.1, 4.3 and 4.4 investigate the KL and the Eckhoff interpolations for even values of the parameter q ($q=2m,\ m=1,2,\cdots$) and reveal the asymptotic behavior of the corresponding interpolations on the interval $|x|<1$. Theorem 3.1 shows that the rate of convergence of $I_{N,2m}(f)$ is $O(N^{-2m-1})$ as $N\to\infty$. Theorems 4.3 and 4.4 state that the rate of convergence of $\widetilde{I}_{N,2m}(f)$ is $O(N^{-4m-1})$ as $N\to\infty$. Therefore we have an improvement in convergence rate by the factor $O(N^{2m})$. This convergence acceleration phenomenon we will call the autocorrection phenomenon of the Eckhoff interpolation similarly to the autocorrection phenomenon of the Eckhoff approximation described in [25]. We see that the autocorrection phenomenon is much larger for the interpolation rather than for the approximation - we have an improvement by the factor $O(N^{2m})$ instead of $O(N^m)$ as in the Eckhoff approximation. Note also (compare Theorems 2.4, 4.3) that interpolation $\widetilde{I}_{N,2m}(f)$ is even more precise than approximation $\widetilde{S}_{N,2m}(f)$ when $|x|<1$ - we have an improvement by the factor $O(N^m)$.

The numerical results described below will accomplish the theoretical investigations.

Consider the following simple function

$$f(x)=\sin(x-1). \tag{4.20}$$

In Table 4.1 we present the L_2-errors of the approximations $S_{N,2m}(f)$ and $\widetilde{S}_{N,2m}(f)$ on the interval $[-0.7,0.7]$. The approximation of the jumps are derived from (4.2)-(4.3).

	N=16	N=32	N=64	N=128
$\|\|R_{N,2}(f)\|\|_{0.7}$	$3.6\cdot10^{-6}$	$4.8\cdot10^{-7}$	$6.1\cdot10^{-8}$	$7.7\cdot10^{-9}$
$\|\|\widetilde{R}_{N,2}(f)\|\|_{0.7}$	$3.5\cdot10^{-7}$	$2.3\cdot10^{-8}$	$1.5\cdot10^{-9}$	$9.3\cdot10^{-11}$
$\|\|R_{N,4}(f)\|\|_{0.7}$	$1.3\cdot10^{-9}$	$4.6\cdot10^{-11}$	$1.5\cdot10^{-12}$	$4.7\cdot10^{-14}$
$\|\|\widetilde{R}_{N,4}(f)\|\|_{0.7}$	$3.4\cdot10^{-11}$	$2.6\cdot10^{-13}$	$2.0\cdot10^{-15}$	$1.5\cdot10^{-17}$
$\|\|R_{N,6}(f)\|\|_{0.7}$	$4.9\cdot10^{-13}$	$4.4\cdot10^{-15}$	$3.6\cdot10^{-17}$	$2.9\cdot10^{-19}$
$\|\|\widetilde{R}_{N,6}(f)\|\|_{0.7}$	$6.2\cdot10^{-15}$	$5.4\cdot10^{-18}$	$4.6\cdot10^{-21}$	$4.3\cdot10^{-24}$

Table 4.1: L_2-errors while approximating the function (4.20) by $S_{N,2m}(f)$ and $\widetilde{S}_{N,2m}(f)$ on the interval $[-0.7,0.7]$ when the indices (2.4) are considered.

Here one can see that according to the autocorrection phenomenon the approximation by $\widetilde{S}_{N,2m}(f)$ is more precise than the approximation by $S_{N,2m}(f)$.

From Table 4.1 we get

$$\frac{\|R_{16,2}(f)\|_{0.7}}{\|R_{32,2}(f)\|_{0.7}} = 7.5,\ \frac{\|R_{32,2}(f)\|_{0.7}}{\|R_{64,2}(f)\|_{0.7}} = 7.86,\ \frac{\|R_{64,2}(f)\|_{0.7}}{\|R_{128,2}(f)\|_{0.7}} = 7.92.$$

These results coincide with the statement of Theorem 2.2, where $\|R_{N,2}\|_{0.7} = O(N^{-3})$ as $N \to \infty$ which implies asymptotically

$$\frac{\|R_{2^z,2}(f)\|_{0.7}}{\|R_{2^{z+1},2}(f)\|_{0.7}} = 8.$$

In view of Theorem 2.4 we have that $\|\widetilde{R}_{N,2}\|_{0.7} = O(N^{-4})$ as $N \to \infty$, which implies asymptotically

$$\frac{\|\widetilde{R}_{2^z,2}(f)\|_{0.7}}{\|\widetilde{R}_{2^{z+1},2}(f)\|_{0.7}} = 16.$$

This theoretical estimate coincides with the results in Table 4.1

$$\frac{\|\widetilde{R}_{16,2}(f)\|_{0.7}}{\|\widetilde{R}_{32,2}(f)\|_{0.7}} = 15.2,\ \frac{\|\widetilde{R}_{32,2}(f)\|_{0.7}}{\|\widetilde{R}_{64,2}(f)\|_{0.7}} = 15.3,\ \frac{\|\widetilde{R}_{64,2}(f)\|_{0.7}}{\|\widetilde{R}_{128,2}(f)\|_{0.7}} = 16.1.$$

Consequently, the theoretical and the numerical estimates coincide – the magnitude of the autocorrection phenomenon of the Eckhoff approximation for $q = 2$ is 1 power of N. Similarly, we can calculate from Table 4.1 that for $q = 4$ and $q = 6$ the magnitude of the autocorrection phenomenon is 2 and 3 power of N, respectively.

In Table 4.2 we show the corresponding results for the interpolations. From here we get

$$\frac{\|r_{16,2}(f)\|_{0.7}}{\|r_{32,2}(f)\|_{0.7}} = 7.52,\ \frac{\|r_{32,2}(f)\|_{0.7}}{\|r_{64,2}(f)\|_{0.7}} = 7.75,\ \frac{\|r_{64,2}(f)\|_{0.7}}{\|r_{128,2}(f)\|_{0.7}} = 8.$$

These results coincide with the statement of Theorem 3.1, where $\|r_{N,2}\|_{0.7} = O(N^{-3})$ as $N \to \infty$, which implies asymptotically

$$\frac{\|r_{2^z,2}(f)\|_{0.7}}{\|r_{2^{z+1},2}(f)\|_{0.7}} = 8.$$

Similarly

$$\frac{\|\widetilde{r}_{16,2}(f)\|_{0.7}}{\|\widetilde{r}_{32,2}(f)\|_{0.7}} = 28.57,\ \frac{\|\widetilde{r}_{32,2}(f)\|_{0.7}}{\|\widetilde{r}_{64,2}(f)\|_{0.7}} = 31.11,\ \frac{\|\widetilde{r}_{64,2}(f)\|_{0.7}}{\|\widetilde{r}_{128,2}(f)\|_{0.7}} = 31.58.$$

These estimates coincide with the statement of Theorem 4.3 or 4.4, where $\|r_{N,2}\|_{0.7} = O(N^{-5})$ as $N \to \infty$, which implies asymptotically

$$\frac{\|\widetilde{r}_{2^z,2}(f)\|_{0.7}}{\|\widetilde{r}_{2^{z+1},2}(f)\|_{0.7}} = 32.$$

We see that the theoretical and the numerical estimates coincide – the magnitude of the autocorrection phenomenon of the Eckhoff interpolation for $q = 2$ is 2 power of N. Similarly we can calculate from Table 4.2 that for $q = 4$ and $q = 6$ the magnitude of the autocorrection phenomenon is 4 and 6 power of N, respectively. Comparison of Tables 4.1 and 4.2 shows that the magnitude of the autocorrection phenomenon of interpolation is bigger than of the approximation which corresponds to the estimates of Theorems 2.4 and 4.3. Moreover, we see that the interpolation is more precise than the approximation when $|x| < 1$.

In Figures 1 and 2 we visually show the autocorrection phenomenon of the Eckhoff approximation and interpolation, respectively, while approximating the function (4.20) on the interval $[-0.7, 0.7]$ when $q = 4$ and $N = 32$. Comparison of these figures shows also that the Eckhoff

	N=16	N=32	N=64	N=128
$\|r_{N,2}(f)\|_{0.7}$	$7.0 \cdot 10^{-6}$	$9.3 \cdot 10^{-7}$	$1.2 \cdot 10^{-7}$	$1.5 \cdot 10^{-8}$
$\|\widetilde{r}_{N,2}(f)\|_{0.7}$	$1.6 \cdot 10^{-7}$	$5.6 \cdot 10^{-9}$	$1.8 \cdot 10^{-10}$	$5.7 \cdot 10^{-12}$
$\|r_{N,4}(f)\|_{0.7}$	$2.6 \cdot 10^{-9}$	$9.1 \cdot 10^{-11}$	$2.9 \cdot 10^{-12}$	$9.4 \cdot 10^{-14}$
$\|\widetilde{r}_{N,4}(f)\|_{0.7}$	$1.1 \cdot 10^{-11}$	$3.1 \cdot 10^{-14}$	$6.4 \cdot 10^{-17}$	$1.4 \cdot 10^{-15}$
$\|r_{N,6}(f)\|_{0.7}$	$9.4 \cdot 10^{-13}$	$8.7 \cdot 10^{-15}$	$7.2 \cdot 10^{-17}$	$5.8 \cdot 10^{-19}$
$\|\widetilde{r}_{N,6}(f)\|_{0.7}$	$2.0 \cdot 10^{-15}$	$4.6 \cdot 10^{-19}$	$6.5 \cdot 10^{-23}$	$9.2 \cdot 10^{-27}$

Table 4.2: L_2-errors while approximating the function (4.20) by $I_{N,2m}(f)$ and $\widetilde{I}_{N,2m}(f)$ on the interval $[-0.7, 0.7]$ when the indices (2.4) are considered.

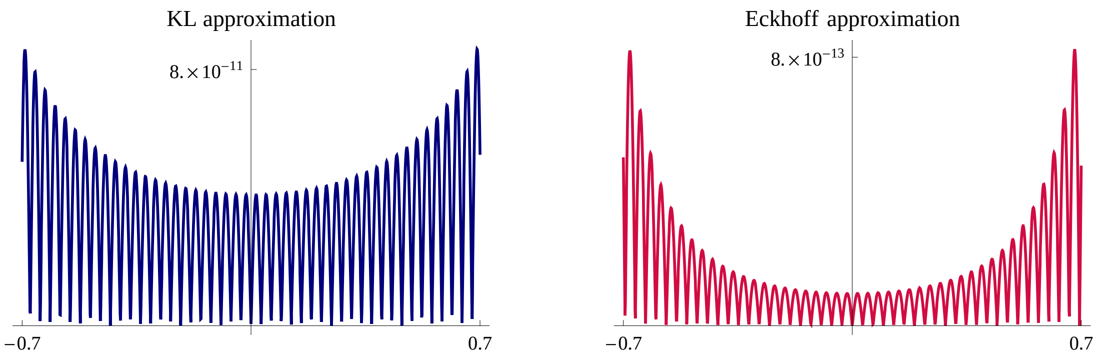

Figure 1: Graphics of $|R_{N,q}(f)|$ (left) and $|\widetilde{R}_{N,q}(f)|$ (right) while approximating the function (4.20) on the interval $[-0.7, 0.7]$ for $q = 4$, $N = 32$ when the indices (2.4) are considered.

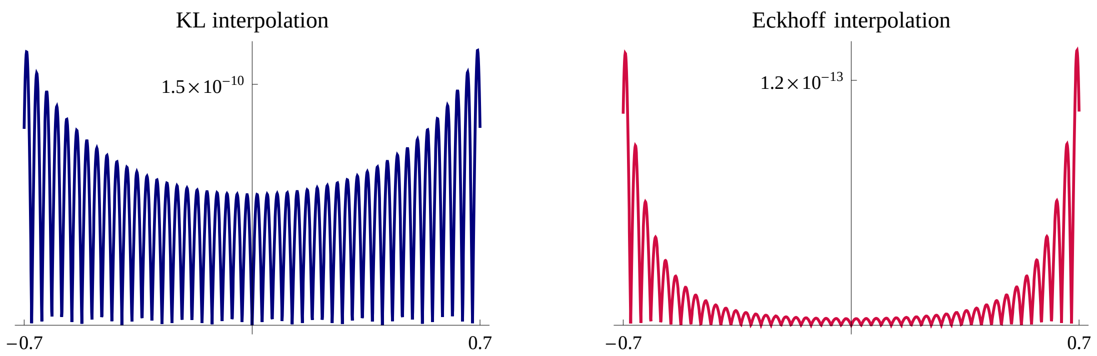

Figure 2: Graphics of $|r_{N,q}(f)|$ (left) and $|\widetilde{r}_{N,q}(f)|$ (right) while approximating the function (4.20) on the interval $[-0.7, 0.7]$ for $q = 4$, $N = 32$ when the indices (2.4) are considered.

interpolation is more precise than the Eckhoff approximation, while the KL interpolation is less accurate compared with the KL approximation.

4.2. **Odd values of q.** In this subsection we explore the accuracy of the jumps approximation and the corresponding Eckhoff interpolation for odd values of q when the indices n_s are chosen as in (2.5). Suppose that $q = 2m + 1$, $m = 0, 1, \cdots$.

Relations (4.1) imply the following modification of (3.2)

$$\frac{\check{f}_n - \check{f}_{-n}}{2} = \sum_{k=0}^{m} A_{2k}^i(f,N)\check{B}_n(2k), \ n = N, N-1, \cdots, N-m+1, \tag{4.21}$$

$$\frac{\check{f}_n + \check{f}_{-n}}{2} = \sum_{k=0}^{m-1} A_{2k+1}^i(f,N)\check{B}_n(2k+1), \ n = N, N-1, \cdots, N-m+1, \tag{4.22}$$

$$\check{f}_{N-m} = \sum_{k=0}^{2m} A_k^i(f,N)\check{B}_{N-m}(k). \tag{4.23}$$

The next is the analog of Theorem 4.1 for odd values of q.

Theorem 4.5. *Suppose that q is an odd number, $q = 2m+1$, $m = 0, 1, \cdots$ and the indices n_s are chosen as in (2.5). If $f \in C^{4m+2}[-1,1]$ such that $f^{(4m+2)}$ is absolutely continuous on $[-1,1]$ then the following estimates hold as $N \to \infty$*

$$A_{2k}^i(f,N) = A_{2k}(f) + (2k)!\frac{A_{2m+1}(f)\tilde{\nu}_{2k}^{'} + A_{2m+2}(f)\tilde{\nu}_{2k}^{''}}{(2N)^{2m-2k+2}} + o(N^{-2m+2k-2}), \tag{4.24}$$

$$A_{2k+1}^i(f,N) = A_{2k+1}(f) - A_{2m+1}(f)\frac{(2k+1)!\,\tilde{\nu}_{2k+1}}{(2N)^{2m-2k}} + O(N^{-2m+2k-1}), \tag{4.25}$$

where the numbers ν_{2k+1}, $\nu_{2k}^{'}$ and $\nu_{2k}^{''}$ are the solutions of systems

$$\sum_{k=s}^{m-1} u_{2s,2k+1}\tilde{\nu}_{2k+1} = \tilde{\mu}_{2s}, \ s = 0, \cdots, m-1,$$

$$\sum_{k=s}^{2m} u_{s,k}\tilde{\nu}_k^{'} = \tilde{\mu}_s^{'}, \ s = 0, \cdots, 2m, \ \sum_{k=s}^{m} u_{2s,2k}\tilde{\nu}_{2k}^{''} = \tilde{\mu}_{2s}^{''}, \ s = 0, \cdots, m$$

with

$$\tilde{\mu}_s := \sum_{j=0}^{2m} \ell_{s,j}\tilde{\Upsilon}_j, \ \tilde{\mu}_s^{'} := \sum_{j=0}^{2m} \ell_{s,j}\tilde{\Upsilon}_j^{'}, \ \tilde{\mu}_s^{''} := \sum_{j=0}^{2m} \ell_{s,j}\tilde{\Upsilon}_j^{''},$$

and

$$\begin{aligned}
\tilde{\Upsilon}_j &:= -\frac{1}{(2m+1)!}\sum_{s=0}^{2m+1} \alpha_{2m+1,s} \sum_{k=j+1}^{2m+1} \binom{2m+1}{k} \tilde{\omega}_{2m+1,s+k-j-1}^{*}, \\
\tilde{\Upsilon}_j^{'} &:= \frac{i\pi(2m+1)}{(2m+1)!}\sum_{s=0}^{2m+1} \alpha_{2m+1,s} \sum_{k=j+1}^{2m} \left[\binom{2m+1}{k} \tilde{\omega}_{2m+1,s+k-j-1}^{'} \right. \\
&\quad - \left. \binom{2m}{k} \tilde{\omega}_{2m+1,s+k-j-1}^{*}\right], \\
\tilde{\Upsilon}_j^{''} &:= \frac{1}{(2m+2)!}\sum_{s=0}^{2m+2} \alpha_{2m+2,s} \sum_{k=j+1}^{2m+1} \binom{2m+1}{k} \tilde{\omega}_{2m+2,s+k-j-1}^{*},
\end{aligned}$$

$$\tilde{\omega}_{r,j}^{*} = \sum_{s=0}^{2m} \frac{(-1)^{j+s}}{2^{r-s}} \binom{j}{s} \binom{r-s-1}{r-2m-1},$$

$$\tilde{\omega}'_{r,j} = \sum_{s=0}^{2m+1} \frac{(-1)^{j+s}}{2^{r-s+1}} \binom{j}{s} \binom{r-s}{r-2m-1}.$$

Proof. Starting as in the proof of Theorem 4.1 we obtain

$$\begin{aligned} \sum_{k=0}^{2m} \lambda_k \beta_{j,k}(2m+1) &= \sum_{r=2m+1}^{4m+2} A_r(f) \frac{(-1)^r}{r!(2N+1)^{r+1}} \sum_{\ell=0}^{r} \alpha_{r,\ell} \sum_{t=j+1}^{2m+1} \gamma_t(2m+1) \tilde{\omega}_{r,\ell+t-j-1} \\ &+ o(N^{-2m-3}),\ j=0,\cdots,2m-1,\ N\to\infty, \end{aligned} \tag{4.26}$$

where

$$\tilde{\omega}_{r,j} := \sum_{s=1}^{2m+1} \frac{e^{-2i\pi j \tau_s}}{(1-e^{-2i\pi\tau_s})^{r-2m} \prod\limits_{\substack{\ell=1\\ \ell\neq s}}^{2m+1} (e^{-2i\pi\tau_s} - e^{-2i\pi\tau_\ell})}.$$

Taking into account that

$$\gamma_s(2m+1) = \binom{2m+1}{s} + i\pi \frac{2m+1}{2N+1} \binom{2m}{s} + O(N^{-2}),\ N\to\infty$$

and

$$\tilde{\omega}_{r,j} = \tilde{\omega}^*_{r,j} - \frac{i\pi(2m+1)}{2N+1} \tilde{\omega}'_{r,j} + O(N^{-2}),\ N\to\infty$$

we get

$$\begin{aligned} \sum_{k=0}^{2m} \lambda_k \beta_{j,k}(2m+1) &= A_{2m+1}(f) \frac{\tilde{\Upsilon}_j}{(2N+1)^{2m+2}} + A_{2m+1}(f) \frac{\tilde{\Upsilon}'_j}{(2N+1)^{2m+3}} \\ &+ A_{2m+2}(f) \frac{\tilde{\Upsilon}''_j}{(2N+1)^{2m+3}} + o(N^{-2m-3}),\ N\to\infty. \end{aligned}$$

According to the LU-factorization of matrix $(\beta_{k,j})$ we derive

$$\begin{aligned} \sum_{k=0}^{2m} u_{s,k} \lambda_k &= A_{2m+1}(f) \frac{\tilde{\mu}_s}{(2N+1)^{2m+2}} + A_{2m+1}(f) \frac{\tilde{\mu}'_s}{(2N+1)^{2m+3}} \\ &+ A_{2m+2}(f) \frac{\tilde{\mu}''_s}{(2N+1)^{2m+3}} + o(N^{-2m-3}),\ N\to\infty. \end{aligned}$$

This implies

$$\begin{aligned} A_k^i(f,N) &= A_k(f) + A_{2m+1}(f) \frac{\tilde{\nu}_k (-1)^k k!}{(2N+1)^{2m+1-k}} + A_{2m+1}(f) \frac{\tilde{\nu}'_k (-1)^k k!}{(2N+1)^{2m+2-k}} \\ &+ A_{2m+2}(f) \frac{\tilde{\nu}''_k (-1)^k k!}{(2N+1)^{2m+2-k}} + o(N^{-2m-2+k}),\ N\to\infty, \end{aligned} \tag{4.27}$$

where the numbers $\tilde{\nu}_k$, $\tilde{\nu}'_k$, and $\tilde{\nu}''_k$ are the solutions of systems

$$\sum_{k=s}^{2m} u_{s,k} \tilde{\nu}_k = \tilde{\mu}_s,\ \sum_{k=s}^{2m} u_{s,k} \tilde{\nu}'_k = \tilde{\mu}'_s,\ \sum_{k=s}^{2m} u_{s,k} \tilde{\nu}''_k = \tilde{\mu}''_s.$$

Equation (4.27) coincides with (4.25) for odd values of k. For even values we rewrite as

$$\begin{aligned} A^i_{2k}(f,N) &= A_{2k}(f) + A_{2m+1}(f)\frac{\tilde{\nu}_{2k}(2k)!}{(2N+1)^{2m-2k+1}} + A_{2m+1}(f)\frac{\tilde{\nu}'_{2k}(2k)!}{(2N+1)^{2m-2k+2}} \\ &+ A_{2m+2}(f)\frac{\tilde{\nu}''_{2k}(2k)!}{(2N+1)^{2m-2k+2}} + o(N^{-2m+2k-2}),\ N\to\infty. \end{aligned}$$

For finishing the proof we need to show that $\tilde{\nu}_{2k} = 0,\ k = 0, \cdots, m-1$. The system of linear equations

$$\sum_{k=s}^{2m} u_{s,k}\tilde{\nu}_k = \tilde{\mu}_s$$

is equivalent to the following system

$$\sum_{k=0}^{2m} \beta_{j,k}\tilde{\nu}_k = \tilde{\Upsilon}_j,\ j = 0, \cdots, 2m.$$

This we copy out in the form

$$\sum_{k=0}^{m} \beta_{m-1-\ell,2k}\tilde{\nu}_{2k} + \sum_{k=0}^{m-1} \beta_{,m-1-\ell,2k+1}\tilde{\nu}_{2k+1} = \tilde{\Upsilon}_{m-1-\ell},\ \ell = 0, \cdots, m-1,$$

$$\sum_{k=0}^{m} \beta_{\ell+m+1,2k}\tilde{\nu}_{2k} + \sum_{k=0}^{m-1} \beta_{\ell+m+1,2k+1}\tilde{\nu}_{2k+1} = \tilde{\Upsilon}_{\ell+m+1},\ \ell = 0, \cdots, m-1,$$

$$\sum_{k=0}^{m} \beta_{m,2k}\tilde{\nu}_{2k} + \sum_{k=0}^{m-1} \beta_{m,2k+1}\tilde{\nu}_{2k+1} = \tilde{\Upsilon}_m.$$

Application of Lemma A.4 leads to the following system of linear equations for determining the numbers ν_{2k}

$$\sum_{k=0}^{m} \beta_{m-1-\ell,2k}\tilde{\nu}_{2k} = \frac{1}{2}(\tilde{\Upsilon}_{m-1-\ell} + \tilde{\Upsilon}_{\ell+m+1}),\ \ell = 0, \cdots, m-1,$$

$$\sum_{k=0}^{m} \beta_{m,2k}\tilde{\nu}_{2k} = \tilde{\Upsilon}_m.$$

It remains to show that $\tilde{\Upsilon}_m = 0$ and $\tilde{\Upsilon}_{m-1-\ell} = -\tilde{\Upsilon}_{\ell+m+1}$. We have (see Lemma A.9)

$$\tilde{\Upsilon}_m = -\frac{1}{(2m+1)!2^{2m+1}} \sum_{r=0}^{2m+1} \alpha_{2m+1,r}\delta_r.$$

In view of Lemmas A.9 and A.2, we get

$$\begin{aligned} \tilde{\Upsilon}_m &= -\frac{1}{(2m+1)!2^{2m+1}} \sum_{r=0}^{m} \alpha_{2m+1,r}\delta_r - \frac{1}{(2m+1)!2^{2m+1}} \sum_{r=m+1}^{2m+1} \alpha_{2m+1,r}\delta_r \\ &= -\frac{1}{2(2m+1)!} \sum_{\ell=0}^{m} (\alpha_{2m+1,m-\ell} - \alpha_{2m+1,\ell+m+1}) = 0. \end{aligned}$$

Now we return to the identity $\tilde{\Upsilon}_{m-1-\ell} = -\tilde{\Upsilon}_{\ell+m+1}$. We will prove that $\tilde{\Upsilon}_j = -\tilde{\Upsilon}_{2m-j}$, $j = 0, \cdots, m-1$. Application of Lemma A.2 implies

$$\begin{aligned}\tilde{\Upsilon}_j + \tilde{\Upsilon}_{2m-j} &= (-1)^j \sum_{r=0}^{2m+1} (-1)^r \alpha_{2m+1,r} \\ &\times \left(\sum_{\ell=j+1}^{2m+1} \binom{2m+1}{\ell} (-1)^\ell \sum_{k=0}^{2m} 2^k (-1)^k \binom{\ell+2m-r-j}{k} \right. \\ &+ \left. \sum_{\ell=0}^{j} \binom{2m+1}{\ell} (-1)^\ell \sum_{k=0}^{2m} 2^k (-1)^k \binom{r+j-\ell}{k} \right).\end{aligned}$$

According to Lemma A.8 we get

$$\begin{aligned}\tilde{\Upsilon}_j + \tilde{\Upsilon}_{2m-j} &= (-1)^j \sum_{r=0}^{2m+1} (-1)^r \alpha_{2m+1,r} \sum_{\ell=0}^{j} \binom{2m+1}{\ell} (-1)^\ell \\ &\times \left(-\sum_{k=0}^{2m} 2^k (-1)^k \binom{\ell+2m-r-j}{k} + \sum_{k=0}^{2m} 2^k (-1)^k \binom{r+j-\ell}{k} \right).\end{aligned}$$

This ends the proof in view of Lemma A.7. ∎

In the next lemma we explore the asymptotic behaviors of $\Delta_n^{m+2}(G_n)$, $\Delta_n^{m+2}(\check{G}_n - G_n)$, $\Delta_{\pm N}^{m+1}(\check{G}_n)$ and $\Delta_{\pm N}^{m}(\check{G}_N)$.

Lemma 4.6. *Suppose that q is an odd number, $q = 2m+1$, $m = 0, 1, \cdots$ and indices n_s are chosen as in (2.5). If $f \in C^{4m+3}[-1,1]$ such that $f^{(4m+3)}$ is absolutely continuous on $[-1,1]$ then the asymptotic expansions are valid as $N \to \infty$*

$$\Delta_n^{m+2}(G_n) = \frac{o(N^{-2m})}{n^{2m+4}},\ |n| > N,$$

$$\Delta_n^{m+2}(\check{G}_n - G_n) = o(N^{-4m-4}),\ |n| \le N,$$

$$\Delta_{\pm N}^{m+1}(\check{G}_n) = o(N^{-4m-4}),$$

and

$$\begin{aligned}\Delta_{\pm N}^{m}(\check{G}_N) &= \pm \frac{(-1)^{N+m}}{2^{2m+3} N^{4m+3}} \sum_{k=0}^{m+1} 2^{2k} (2k+2m)! \frac{A_{2m+1}\tilde{\nu}_{2k}' + A_{2m+2}\tilde{\nu}_{2k}''}{(i\pi)^{2k+1}} \\ &\times \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{2m+2k+1}} + A_{2m+1}(f) \frac{(-1)^{N+m}}{2^{2m+2} N^{4m+3}} \\ &\times \sum_{k=0}^{m} \frac{2^{2k+1} \tilde{\nu}_{2k+1} (2k+2m+2)!}{(i\pi)^{2k+2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s s}{(2s+1)^{2m+2k+3}} + o(N^{-4m-3}),\end{aligned}$$

where $\tilde{\nu}_{2m+1} = 1/(2m+1)!$, $\tilde{\nu}_{2m+2}' = 0$ and $\tilde{\nu}_{2m+2}'' = -1/(2m+2)!$.

Proof. For the first estimate we write

$$\begin{aligned}\Delta_n^{m+2}(G_n) &= \sum_{k=0}^{2m}\left(A_k(f)-A_k^i(f,N)\right)\Delta_n^{m+2}(B_n(k))\\ &+ \sum_{k=2m+1}^{4m+3} A_k(f)\Delta_n^{m+2}(B_n(k))+o(n^{-4m-4})\end{aligned}$$

and apply Lemma B.1 with Theorem 4.5.

For the second estimate we write

$$\begin{aligned}\Delta_n^{m+2}(\check{G}_n-G_n) &= \sum_{k=0}^{2m}\left(A_k(f)-A_k^i(f,N)\right)\Delta_n^{m+2}(\check{B}_n(k)-B_n(k))\\ &+ \sum_{k=2m+1}^{4m+3} A_k(f)\Delta_n^{m+2}(\check{B}_n(k)-B_n(k))+o(N^{-4m-4})\end{aligned}$$

and apply Lemma B.1 with Theorem 4.5.

Similarly for the third estimate we have

$$\begin{aligned}\Delta_{\pm N}^{m+1}(\check{G}_n) &= \sum_{k=0}^{2m}\left(A_k(f)-A_k^i(f,N)\right)\Delta_{\pm N}^{m+1}(\check{B}_n(k))\\ &+ \sum_{k=2m+1}^{4m+3} A_k(f)\Delta_{\pm N}^{m+1}(\check{B}_n(k))+o(N^{-4m-4}).\end{aligned}$$

Application of Lemma B.1 with Theorem 4.5 will end the proof of the third estimate.

Finally, for the fourth estimate we write

$$\begin{aligned}\Delta_{\pm N}^{m}(\check{G}_n) &= \sum_{k=0}^{m}\left(A_{2k}(f)-A_{2k}^i(f,N)\right)\Delta_{\pm N}^{m}(\check{B}_n(2k))\\ &+ \sum_{k=0}^{m-1}\left(A_{2k+1}(f)-A_{2k+1}^i(f,N)\right)\Delta_{\pm N}^{m}(\check{B}_n(2k+1))\\ &+ \sum_{k=m+1}^{2m+1} A_{2k}(f)\Delta_{\pm N}^{m}(\check{B}_n(2k))+\sum_{k=m}^{2m} A_{2k+1}(f)\Delta_{\pm N}^{m}(\check{B}_n(2k+1))\\ &+ o(N^{-4m-3}),\ N\to\infty.\end{aligned}$$

This concludes the proof in view of Lemma B.1 with Theorem 4.5. ∎

The next theorem explores the behavior of the Eckhoff interpolation for the odd values of q for $|x|<1$.

Theorem 4.7. *Suppose that the conditions of Lemma 4.6 are valid. Then the following estimate holds for* $|x| < 1$

$$
\begin{aligned}
\widetilde{r}_{N,2m+1}(f) &= \frac{(-1)^{N+m}}{2^{4m+3}N^{4m+3}} \frac{ie^{-i\frac{\pi x}{2}}\sin(N+\frac{1}{2})\pi x}{\cos^{2m+2}\frac{\pi x}{2}} \\
\times & \left(A_{2m+1}(f)\sum_{k=0}^{m} 2^{2k+1}\frac{\tilde{\nu}_{2k+1}(2k+2m+2)!}{(i\pi)^{2k+2}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s s}{(2s+1)^{2m+2k+3}} \right. \\
- & \left. \sum_{k=0}^{m+1} 2^{2k-1}(2k+2m)!\frac{A_{2m+1}(f)\tilde{\nu}_{2k}' + A_{2m+2}(f)\tilde{\nu}_{2k}''}{(i\pi)^{2k+1}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2m+2k+1}} \right) \\
+ & \ o(N^{-4m-3}),\ N\to\infty,
\end{aligned}
$$

where $\tilde{\nu}_{2m+1} = 1/(2m+1)!$, $\tilde{\nu}_{2m+2}' = 0$ *and* $\tilde{\nu}_{2m+2}'' = -1/(2m+2)!$.

Proof. Similar to (4.19) we get the following transformation

$$
\begin{aligned}
\widetilde{r}_{N,2m+1}(f) &= (e^{-i\pi Nx} - e^{i\pi(N+1)x})\sum_{k=1}^{m+2}\frac{\Delta_N^{k-1}(\check{G}_n)}{(1+e^{i\pi x})^k(1+e^{-i\pi x})^k} \\
&+ (e^{i\pi Nx} - e^{-i\pi(N+1)x})\sum_{k=1}^{m+2}\frac{\Delta_{-N}^{k-1}(\check{G}_n)}{(1+e^{i\pi x})^k(1+e^{-i\pi x})^k} \\
&+ \frac{1}{(1+e^{i\pi x})^{m+2}(1+e^{-i\pi x})^{m+2}}\sum_{n=-N}^{N}\Delta_n^{m+2}(G_n-\check{G}_n)e^{i\pi nx} \\
&+ \frac{1}{(1+e^{i\pi x})^{m+2}(1+e^{-i\pi x})^{m+2}}\sum_{|n|=N+1}^{\infty}\Delta_n^{m+2}(G_n)e^{i\pi nx}.
\end{aligned}
\tag{4.28}
$$

According to Lemma 4.6 the last three terms are $o(N^{-4m-3})$ as $N\to\infty$. Taking into account that

$$
\Delta_{\pm N}^{s}(\check{G}_n) = \sum_{k=0}^{2s}\binom{2s}{k}\check{G}_{\pm N+s-k}
\tag{4.29}
$$

we conclude that $\Delta_{\pm N}^{s}(\check{G}_n) = 0$ as $s = 0,\cdots,m-1$ and $\Delta_N^{m}(\check{G}_n) = 0$. Substituting all these into (4.28) we derive

$$
\begin{aligned}
\widetilde{r}_{N,2m+1}(f) &= \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})^{m+1}(1+e^{-i\pi x})^{m+1}}\Delta_{-N}^{m}(\check{G}_n) \\
&+ \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})^{m+2}(1+e^{-i\pi x})^{m+2}}\Delta_{N}^{m+1}(\check{G}_n) \\
&+ \frac{e^{i\pi Nx} - e^{-i\pi(N+1)x}}{(1+e^{i\pi x})^{m+1}(1+e^{-i\pi x})^{m+1}}\Delta_{-N}^{m+1}(\check{G}_n) \\
&+ o(N^{-4m-3}),\ N\to\infty.
\end{aligned}
$$

This concludes the proof in view of Lemma 4.6. ∎

The next is immediate consequence of the previous one.

Theorem 4.8. *Suppose that the conditions of Theorem 4.7 are valid. Then the following estimate holds for every* $0 < \varepsilon < 1$

$$\lim_{N\to\infty} N^{4m+3}\|\widetilde{r}_{N,2m+1}(f)\|_{\varepsilon} = \frac{1}{\sqrt{2}\,2^{4m+3}}\left(\int_{-\varepsilon}^{\varepsilon}\frac{dx}{\cos^{4m+4}\frac{\pi x}{2}}\right)^{\frac{1}{2}}$$

$$\times \left| A_{2m+1}(f)\sum_{k=0}^{m}2^{2k+1}\frac{\tilde{\nu}_{2k+1}(2k+2m+2)!}{(i\pi)^{2k+2}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s s}{(2s+1)^{2m+2k+3}}\right.$$

$$\left. - \sum_{k=0}^{m+1}2^{2k-1}(2k+2m)!\frac{A_{2m+1}(f)\tilde{\nu}'_{2k}+A_{2m+2}(f)\tilde{\nu}''_{2k}}{(i\pi)^{2k+1}}\sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^{2m+2k+1}}\right|,$$

where $\tilde{\nu}_{2m+1} = 1/(2m+1)!$, $\tilde{\nu}'_{2m+2} = 0$ *and* $\tilde{\nu}''_{2m+2} = -1/(2m+2)!$.

Theorem 3.2 shows that the rate of convergence of $I_{N,2m+1}(f)$ is $O(N^{-2m-3})$ as $N \to \infty$. Theorem 4.7 states that the rate of convergence of $\widetilde{I}_{N,2m+1}(f)$ is $O(N^{-4m-3})$ as $N \to \infty$. We have an improvement in convergence rate by the factor $O(N^{2m})$. We see that for odd values of the parameter q the autocorrection phenomenon is much larger for the interpolations rather than for the approximations - we have an improvement by the factor $O(N^{2m})$ instead of $O(N^m)$ as in the Eckhoff approximation. Compared with $\widetilde{S}_{N,2m+1}(f)$ when $|x| < 1$ interpolation $\widetilde{I}_{N,2m+1}(f)$ is more precise - we have an improvement by the factor $O(N^m)$.

In Table 4.3 we present the L_2-errors of the approximations $S_{N,2m+1}(f)$ and $\widetilde{S}_{N,2m+1}(f)$ on the interval $[-0.7, 0.7]$. The approximation of the jumps are derived from (4.21)-(4.23).

We see that for $q = 1$ the autocorrection phenomenon for the Eckhoff approximation is absent. Calculations show that for $q = 3$ we have

$$\frac{\|R_{16,3}(f)\|_{0.7}}{\|R_{32,3}(f)\|_{0.7}} = 15.27,\ \frac{\|R_{32,3}(f)\|_{0.7}}{\|R_{64,3}(f)\|_{0.7}} = 15.32,\ \frac{\|R_{64,3}(f)\|_{0.7}}{\|R_{128,3}(f)\|_{0.7}} = 15.67.$$

These results coincide with the statement of Theorem 2.3, where $\|R_{N,3}\|_{0.7} = O(N^{-4})$ as $N \to \infty$. In view of Theorem 2.5 we have that $\|\widetilde{R}_{N,2}\|_{0.7} = O(N^{-5})$ as $N \to \infty$, which implies asymptotically

$$\frac{\|\widetilde{R}_{2^z,3}(f)\|_{0.7}}{\|\widetilde{R}_{2^{z+1},3}(f)\|_{0.7}} = 32.$$

This theoretical estimate coincides with the results in Table 4.3

$$\frac{\|\widetilde{R}_{16,3}(f)\|_{0.7}}{\|\widetilde{R}_{32,3}(f)\|_{0.7}} = 30.77,\ \frac{\|\widetilde{R}_{32,3}(f)\|_{0.7}}{\|\widetilde{R}_{64,3}(f)\|_{0.7}} = 32.5,\ \frac{\|\widetilde{R}_{64,3}(f)\|_{0.7}}{\|\widetilde{R}_{128,3}(f)\|_{0.7}} = 31.37.$$

Consequently, the theoretical and the numerical estimates coincide – the magnitude of the autocorrection phenomenon for $q = 3$ is 1 power of N. Similarly, we can calculate from Table 4.3 that for $q = 4$ and $q = 6$ the magnitude of the autocorrection phenomenon is 2 and 3 power of N, respectively.

In Table 4.4 we show the corresponding results for the interpolations. Again, for $q = 1$ the autocorrection phenomenon is absent. For $q = 3$ we get

$$\frac{\|r_{16,3}(f)\|_{0.7}}{\|r_{32,3}(f)\|_{0.7}} = 28,\ \frac{\|r_{32,3}(f)\|_{0.7}}{\|r_{64,3}(f)\|_{0.7}} = 31.25,\ \frac{\|r_{64,3}(f)\|_{0.7}}{\|r_{128,3}(f)\|_{0.7}} = 29.09.$$

These results coincide with the statement of Theorem 3.2, where $\|r_{N,3}\|_{0.7} = O(N^{-5})$ which implies asymptotically

$$\frac{\|r_{2^z,3}(f)\|_{0.7}}{\|r_{2^{z+1},3}(f)\|_{0.7}} = 32.$$

	N=16	N=32	N=64	N=128
$\|R_{N,1}(f)\|_{0.7}$	$3.0 \cdot 10^{-4}$	$8.0 \cdot 10^{-5}$	$2.0 \cdot 10^{-5}$	$4.9 \cdot 10^{-6}$
$\|\widetilde{R}_{N,1}(f)\|_{0.7}$	$4.0 \cdot 10^{-4}$	$1.0 \cdot 10^{-4}$	$3.0 \cdot 10^{-5}$	$6.9 \cdot 10^{-6}$
$\|R_{N,3}(f)\|_{0.7}$	$1.1 \cdot 10^{-7}$	$7.2 \cdot 10^{-9}$	$4.7 \cdot 10^{-10}$	$3.0 \cdot 10^{-11}$
$\|\widetilde{R}_{N,3}(f)\|_{0.7}$	$1.6 \cdot 10^{-8}$	$5.2 \cdot 10^{-10}$	$1.6 \cdot 10^{-11}$	$5.1 \cdot 10^{-13}$
$\|R_{N,5}(f)\|_{0.7}$	$3.9 \cdot 10^{-11}$	$6.8 \cdot 10^{-13}$	$1.1 \cdot 10^{-14}$	$1.8 \cdot 10^{-16}$
$\|\widetilde{R}_{N,5}(f)\|_{0.7}$	$1.6 \cdot 10^{-12}$	$6.0 \cdot 10^{-15}$	$2.3 \cdot 10^{-17}$	$8.5 \cdot 10^{-20}$
$\|R_{N,7}(f)\|_{0.7}$	$1.4 \cdot 10^{-14}$	$6.5 \cdot 10^{-17}$	$2.8 \cdot 10^{-19}$	$1.1 \cdot 10^{-21}$
$\|\widetilde{R}_{N,7}(f)\|_{0.7}$	$3.0 \cdot 10^{-16}$	$1.2 \cdot 10^{-19}$	$5.3 \cdot 10^{-23}$	$2.4 \cdot 10^{-26}$

Table 4.3: L_2-errors while approximating the function (4.20) by $S_{N,2m+1}(f)$ and $\widetilde{S}_{N,2m+1}(f)$ on the interval $[-0.7, 0.7]$ when the indices (2.5) are considered.

Similarly

$$\frac{\|\widetilde{r}_{16,3}(f)\|_{0.7}}{\|\widetilde{r}_{32,3}(f)\|_{0.7}} = 100, \ \frac{\|\widetilde{r}_{32,3}(f)\|_{0.7}}{\|\widetilde{r}_{64,3}(f)\|_{0.7}} = 128.57, \ \frac{\|\widetilde{r}_{64,3}(f)\|_{0.7}}{\|\widetilde{r}_{128,3}(f)\|_{0.7}} = 116.67.$$

These estimates coincide with the statements of Theorems 4.7 and 4.8, where $\|r_{N,3}\|_{0.7} = O(N^{-7})$ as $N \to \infty$, which implies asymptotically

$$\frac{\|\widetilde{r}_{2^z,3}(f)\|_{0.7}}{\|\widetilde{r}_{2^{z+1},3}(f)\|_{0.7}} = 128.$$

Similar calculations we can carry out also for $q = 5$ and $q = 7$. Therefore, Theorems 4.7 and 4.8 show that the magnitude of the autocorrection phenomenon of the interpolation for $q = 3$ is 2 power of N, for $q = 5$ is 4 power of N and for $q = 7$ is 6 power of N.

	N=16	N=32	N=64	N=128
$\|r_{N,1}(f)\|_{0.7}$	$4.0 \cdot 10^{-5}$	$5.3 \cdot 10^{-6}$	$6.6 \cdot 10^{-7}$	$8.4 \cdot 10^{-8}$
$\|\widetilde{r}_{N,1}(f)\|_{0.7}$	$5.0 \cdot 10^{-5}$	$6.9 \cdot 10^{-6}$	$8.7 \cdot 10^{-7}$	$1.1 \cdot 10^{-7}$
$\|r_{N,3}(f)\|_{0.7}$	$2.8 \cdot 10^{-8}$	$1.0 \cdot 10^{-9}$	$3.2 \cdot 10^{-11}$	$1.1 \cdot 10^{-12}$
$\|\widetilde{r}_{N,3}(f)\|_{0.7}$	$1.8 \cdot 10^{-9}$	$1.8 \cdot 10^{-11}$	$1.4 \cdot 10^{-13}$	$1.2 \cdot 10^{-15}$
$\|r_{N,5}(f)\|_{0.7}$	$1.5 \cdot 10^{-11}$	$1.4 \cdot 10^{-13}$	$1.2 \cdot 10^{-15}$	$9.7 \cdot 10^{-18}$
$\|\widetilde{r}_{N,5}(f)\|_{0.7}$	$2.2 \cdot 10^{-13}$	$1.7 \cdot 10^{-16}$	$9.1 \cdot 10^{-20}$	$5.0 \cdot 10^{-23}$
$\|r_{N,7}(f)\|_{0.7}$	$6.9 \cdot 10^{-15}$	$1.8 \cdot 10^{-17}$	$3.8 \cdot 10^{-20}$	$7.9 \cdot 10^{-23}$
$\|\widetilde{r}_{N,7}(f)\|_{0.7}$	$5.1 \cdot 10^{-17}$	$3.5 \cdot 10^{-21}$	$1.3 \cdot 10^{-25}$	$4.8 \cdot 10^{-30}$

Table 4.4: L_2-errors while approximating the function (4.20) by $I_{N,2m+1}(f)$ and $\widetilde{I}_{N,2m+1}(f)$ on the interval $[-0.7, 0.7]$ when the indices (2.5) are considered.

In Figures 3 and 4 we visually show the autocorrection phenomenon of the Eckhoff approximation and interpolation, respectively, while approximating the function (4.20) on the interval $[-0.7, 0.7]$ when $q = 5$ and $N = 32$. We see that for the odd values of q the KL interpolation and approximation are more precise than the Eckhoff interpolation and approximation, respectively.

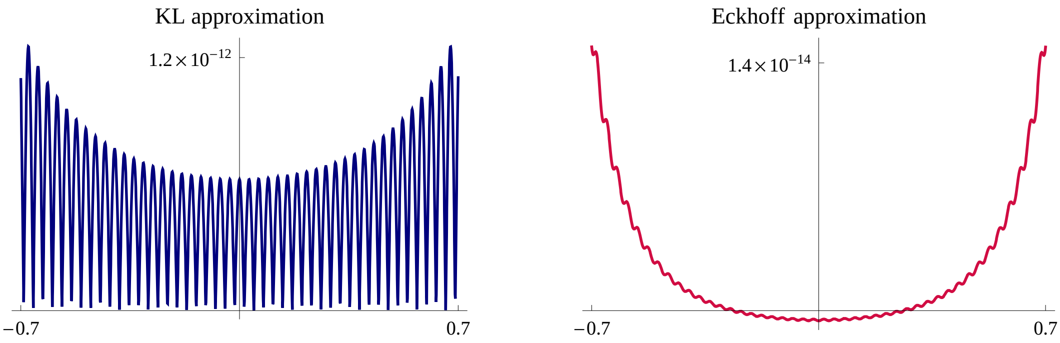

Figure 3: Graphics of $|R_{N,q}(f)|$ (left) and $|\widetilde{R}_{N,q}(f)|$ (right) while approximating the function (4.20) on the interval $[-0.7, 0.7]$ for $q = 5$, $N = 32$ when the indices (2.5) are considered.

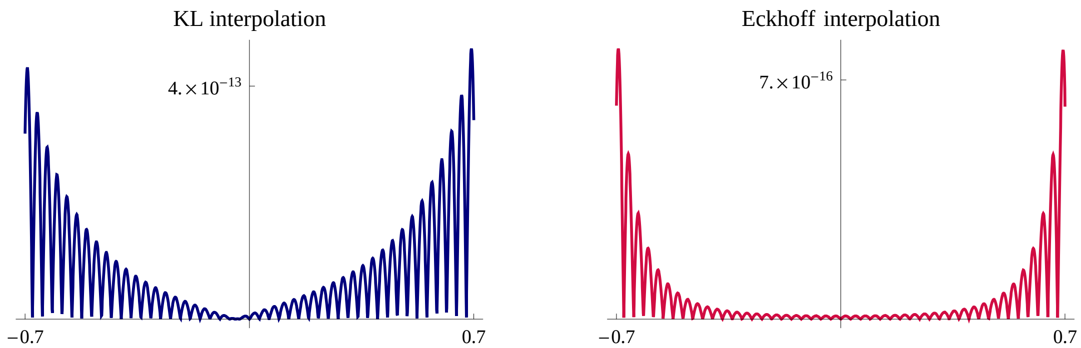

Figure 4: Graphics of $|r_{N,q}(f)|$ (left) and $|\widetilde{r}_{N,q}(f)|$ (right) while approximating the function (4.20) on the interval $[-0.7, 0.7]$ for $q = 5$, $N = 32$ when the indices (2.5) are considered.

A. AUXILIARY IDENTITIES AND LEMMAS

The next lemma gives the explicit form of $\left(\frac{1}{\sin \pi x}\right)^{(k)}$. We need this result for calculation of the discrete Fourier coefficients of the Bernoulli polynomials explicitly (see Appendix B and, in particular, (B.4)). Below the numbers $S(k, n)$ are the Stirling numbers of the second kind ([27]).

Lemma A.1. [23] *The following identity holds*

$$\left(\frac{1}{\sin \pi x}\right)^{(k)} = \frac{\pi^k}{2^k (\sin \pi x)^{k+1}} \sum_{j=0}^{k} \alpha_{k,j} e^{i\pi(k-2j)x}, \ k \geq 0, \tag{A.1}$$

where

$$\alpha_{k,j} := \sum_{\ell=0}^{j} (-1)^{\ell} \sum_{n=0}^{k} n!(-1)^n S(k, n) \binom{k-n}{\ell} \binom{n+1}{2j-2\ell}. \tag{A.2}$$

The next lemma addresses the properties of the numbers $\alpha_{k,j}$.

Lemma A.2. *The following relations are valid*

$$\begin{aligned} \alpha_{2s,s-\ell-1} &= \alpha_{2s,\ell+s+1}, \ \ell = 0, \cdots, s-1, \ s \geq 1, \\ \alpha_{2s+1,s-\ell} &= \alpha_{2s+1,\ell+s+1}, \ \ell = 0, \cdots, s, \ s \geq 0. \end{aligned}$$

Proof. The proof immediately follows from the definition of $\alpha_{k,j}$ as the left hand side of (A.1) takes only the real values so the same is true for the right hand side. ∎

Lemma A.3. [23] *The following identity is valid for* $k = 0, \cdots, q-1$

$$\sin^q \pi x \left(\frac{1}{\sin \pi x}\right)^{(k)} = \frac{(i\pi)^k}{(2i)^{q-1}} e^{i\pi(q-1)x} \sum_{j=0}^{q-1} \beta_{j,k}(q) e^{-2i\pi j x}, \tag{A.3}$$

where

$$\beta_{j,k}(q) := \sum_{\ell=0}^{j} (-1)^\ell \sum_{n=0}^{q-1} n!(-1)^n S(k,n) \binom{q-n-1}{\ell} \binom{n+1}{2j-2\ell}. \tag{A.4}$$

The next lemma addresses the properties of the numbers $\beta_{j,k}$.

Lemma A.4. *The following relations are valid for* $\ell = 0, \cdots, m-1$

$$\begin{aligned} \beta_{m-\ell-1,2s}(2m) &= -\beta_{\ell+m,2s}(2m), \ s = 0, \cdots, m-1, \\ \beta_{m-\ell-1,2s+1}(2m) &= \beta_{m+\ell,2s+1}(2m), \ s = 0, \cdots, m-1, \end{aligned}$$

and

$$\begin{aligned} \beta_{m-\ell-1,2s}(2m+1) &= \beta_{\ell+m+1,2s}(2m+1), \ s = 0, \cdots, m, \\ \beta_{m-\ell-1,2s+1}(2m+1) &= -\beta_{m+\ell+1,2s+1}(2m+1), \ s = 0, \cdots, m-1, \\ \beta_{m,2s+1}(2m+1) &= 0, \ s = 0, \cdots, m-1. \end{aligned}$$

Proof. The proof follows from definition of $\beta_{k,j}$ as the left hand side of (A.3) takes only the real values so the same is true for the right hand side. ∎

We define matrix V as $V = (\beta_{j,k})$ and in the next lemma present the LU-factorization of it.

Lemma A.5. [23] *Matrix* V *has the following LU-factorization*

$$V = LU,$$

where

$$L := (\ell_{j,k})_{j,k=0}^{q-1}, \ U := (u_{j,k})_{j,k=0}^{q-1},$$

$$\ell_{j,k} := (-1)^j \binom{q-k-1}{q-j-1}, \ u_{j,k} := \sum_{n=0}^{q-1} \sum_{\ell=0}^{q-1} (-1)^{j+n} \binom{n+1}{2\ell} \binom{n-\ell}{j-\ell} n! S(k,n).$$

Another interesting property of matrix L.

Lemma A.6. [23] L *is an involutive matrix*

$$L^{-1} = L. \tag{A.5}$$

Now we will prove some combinatorial identities.

Lemma A.7. *The following identity is valid*

$$\sum_{k=0}^{N} 2^k \binom{N-n}{k} (-1)^k = (-1)^N \sum_{k=0}^{N} 2^k \binom{n}{k} (-1)^k, \ n \in \mathbf{Z}. \tag{A.6}$$

Proof. We carry out the proof by the help of mathematical induction. For $N = 0$ the identity is obvious. Supposing that it is true for N, we write for $N + 1$

$$\begin{aligned}
&\sum_{k=0}^{N+1} 2^k \binom{N-n+1}{k} (-1)^k = \sum_{k=0}^{N+1} 2^k \binom{N-n}{k} (-1)^k + \sum_{k=1}^{N+1} 2^k \binom{N-n}{k-1} (-1)^k \\
= &\sum_{k=0}^{N} 2^k \binom{N-n}{k} (-1)^k + 2^{N+1} \binom{N-n}{N+1} (-1)^{N+1} - 2\sum_{k=0}^{N} 2^k \binom{N-n}{k} (-1)^k \\
= &-\sum_{k=0}^{N} 2^k \binom{N-n}{k} (-1)^k + 2^{N+1} \binom{N-n}{N+1} (-1)^{N+1} \\
= &(-1)^{N+1} \sum_{k=0}^{N} 2^k \binom{n}{k} (-1)^k + 2^{N+1} \binom{N-n}{N+1} (-1)^{N+1} \\
= &(-1)^{N+1} \sum_{k=0}^{N+1} 2^k \binom{n}{k} (-1)^k.
\end{aligned}$$

We used the fact that for $0 \le n \le N$ we have

$$\binom{N-n}{N+1} = \binom{n}{N+1} = 0,$$

for $n > N$

$$2^{N+1} \binom{N-n}{N+1} (-1)^{N+1} = 2^{N+1} \binom{n}{N+1}$$

and for $n < 0$

$$(-1)^{N+1} 2^{N+1} \binom{n}{N+1} = 2^{N+1} \binom{N-n}{N+1}.$$

For $n > N$ and $n < 0$ we applied also the identity

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

These end the proof. ∎

Lemma A.8. *Suppose*

$$\eta_n := \sum_{\ell=0}^{2m+1} (-1)^\ell \binom{2m+1}{\ell} \sum_{k=0}^{2m} 2^k \binom{n-\ell}{k} (-1)^k, \ 0 \le n \le 3m+1. \tag{A.7}$$

Then

$$\eta_n = 0.$$

Proof. According to the well-known identity (see [26])

$$\sum_{k=0} (-1)^k \binom{n-k}{m} \binom{p}{k} = \binom{n-p}{n-m}, \tag{A.8}$$

where the summation is executed by the all possible values of k we get for $2m+1 \le n \le 3m+1$

$$\eta_n = \sum_{k=0}^{2m} 2^k (-1)^k \binom{n-2m-1}{n-k} = 0$$

as

$$\binom{n-2m-1}{n-k} = 0,\ k = 0, \cdots, 2m.$$

For $0 \le n \le 2m$ we have from Lemma A.7

$$\begin{aligned}\eta_n &= \sum_{\ell=0}^{2m+1} (-1)^\ell \binom{2m+1}{\ell} \sum_{k=0}^{2m} 2^k \binom{2m-n+\ell}{k} (-1)^k \\ &= -\sum_{k=0}^{2m} 2^k (-1)^k \sum_{\ell=0}^{2m+1} (-1)^\ell \binom{2m+1}{\ell} \binom{4m+1-n-\ell}{k}.\end{aligned}$$

Again by the identity (A.8) we derive

$$\eta_n = -\sum_{k=0}^{2m} 2^k (-1)^k \binom{2m-n}{4m+1-n-k} = 0$$

as

$$\binom{2m-n}{4m+1-n-k} = 0,\ n, k = 0, \cdots, 2m.$$

This ends the proof. ∎

Lemma A.9. *Let*

$$\delta_r := (-1)^r \sum_{\ell=0}^{m} (-1)^\ell \binom{2m+1}{\ell+m+1} \sum_{k=0}^{2m} 2^k \binom{\ell+r}{k} (-1)^k. \tag{A.9}$$

Then

$$\delta_r = \begin{cases} 2^{2m}, & 0 \le r \le m, \\ -2^{2m}, & m+1 \le r \le 2m+1. \end{cases}$$

Proof. We write for $0 \le r \le m$

$$\begin{aligned}\delta_r &= (-1)^r \sum_{\ell=0}^{m} (-1)^\ell \binom{2m+1}{\ell+m+1} \sum_{k=0}^{\ell+r} 2^k \binom{\ell+r}{k} (-1)^k \\ &= \sum_{\ell=0}^{m} \binom{2m+1}{\ell+m+1} = 2^{2m}.\end{aligned}$$

We have for $m+1 \le r \le 2m+1$

$$\begin{aligned}\delta_{m+1} &= -\delta_m + (-1)^{m+1} \sum_{\ell=0}^{m} (-1)^\ell \binom{2m+1}{\ell+m+1} \sum_{k=0}^{2m-1} 2^{k+1} \binom{\ell+m}{k} (-1)^{k+1} \\ &= -\delta_m + 2\sum_{\ell=0}^{m-1} \binom{2m+1}{\ell+m+1} + 2\sum_{k=0}^{2m-1} 2^k \binom{2m}{k} (-1)^k \\ &= -\delta_m = -2^{2m}.\end{aligned}$$

We derive for $m+1 \le r \le 2m$ similarly

$$\delta_{r+1} = \delta_r - 2^{2m+1}(-1)^{r+1} \sum_{\ell=0}^{m} (-1)^\ell \binom{2m+1}{\ell+m+1} \binom{r+\ell}{2m} = \delta_r = -2^{2m}.$$

We used the fact that (see (A.8))

$$\sum_{\ell=0}^{m}(-1)^{\ell}\binom{2m+1}{\ell+m+1}\binom{r+\ell}{2m}=0,\ m+1\le r\le 2m.$$

∎

B. BERNOULLI POLYNOMIALS

The 2-periodic extensions of the Bernoulli polynomials are defined recurrently ([8])

$$B(x;0)=\frac{x}{2},\ B(x;k)=\int B(x;k-1)dx,\ x\in[-1,1], \tag{B.1}$$

where the constant of integration is defined by the relation

$$\int_{-1}^{1}B(x;k)dx=0.$$

It is easy to verify that the Fourier coefficients have the form

$$B_n(k):=\begin{cases}0,\ n=0\\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}},\ n=\pm1,\pm2,\dots\end{cases} \tag{B.2}$$

Relations (B.1) allow the calculation of $\check{B}_n(k)$ explicitly. Here are three of them

$$\check{B}_n(0)=\frac{(-1)^n i}{2(2N+1)\sin\frac{\pi n}{2N+1}},\ n\neq0,\ \check{B}_0(0)=0,$$

$$\check{B}_n(1)=\frac{(-1)^n\cos\frac{\pi n}{2N+1}}{2(2N+1)^2\sin^2\frac{\pi n}{2N+1}},\ n\neq0,\ \check{B}_0(1)=-\frac{1}{12(2N+1)^2},$$

$$\check{B}_n(2)=\frac{(-1)^{n+1}i\left(3+\cos\frac{2\pi n}{2N+1}\right)}{8(2N+1)^3\sin^3\frac{\pi n}{2N+1}},\ n\neq0,\ \check{B}_0(2)=0.$$

It is possible to get the closed form of $\check{B}_n(k)$. Note that for $n\neq0$

$$\check{B}_n(k)=\sum_{r=-\infty}^{\infty}B_{n+r(2N+1)}(k)=\sum_{r=-\infty}^{\infty}\frac{(-1)^{n+r+1}}{2(i\pi(n+r(2N+1)))^{k+1}}$$

$$=\frac{(-1)^{n+1}}{2(i\pi(2N+1))^{k+1}}\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{\left(\frac{n}{2N+1}+r\right)^{k+1}}.$$

Hence

$$\check{B}_n(k)=\frac{(-1)^{n+k+1}\pi}{2k!(i\pi(2N+1))^{k+1}}\left(\frac{1}{\sin\pi x}\right)^{(k)}_{x=\frac{n}{2N+1}}. \tag{B.3}$$

In view of Lemma A.1 Equation (B.3) yields

$$\check{B}_n(k)=\frac{(-1)^{n+k+1}}{(2i(2N+1))^{k+1}k!\left(\sin\frac{\pi n}{2N+1}\right)^{k+1}}\sum_{j=0}^{k}\alpha_{k,j}e^{i\frac{\pi n(k-2j)}{2N+1}},\ n\neq0,\ k\ge0. \tag{B.4}$$

For $n=0$ and $k\ge0$ we have

$$\check{B}_0(k)=\sum_{r=-\infty}^{\infty}B_{r(2N+1)}(k)=\frac{1}{2(i\pi(2N+1))^{k+1}}\sum_{r\neq0}\frac{(-1)^{r+1}}{r^{k+1}}. \tag{B.5}$$

Denote

$$\Delta_n^p(f_n) := \sum_{k=0}^{2p} \binom{2p}{k} f_{n+p-k},\ p \geq 0,\ \forall n. \tag{B.6}$$

We will frequently use the following result.

Lemma B.1. [22] *The following estimates hold for $p \geq 0$ and $k \geq 0$*

$$\Delta_n^p(B_n(k)) = \frac{(-1)^{n+p+1}(k+2p)!}{2(i\pi n)^{k+1} n^{2p} k!} + O(n^{-k-2p-2}),\ n \to \infty,$$

$$\begin{aligned}\Delta_n^p(\check{B}_n(k) - B_n(k)) &= \frac{(-1)^{n+p+1}(k+2p)!}{2(i\pi N)^{k+1} N^{2p} k!} \sum_{s\neq 0} \frac{(-1)^s}{(2s+\frac{n}{N})^{k+2p+1}} \\ &+ O(N^{-k-2p-2}),\ |n| \leq N,\ N \to \infty,\end{aligned}$$

$$\begin{aligned}\Delta_{\pm N}^p(\check{B}_n(2k)) &= \pm\frac{(-1)^{N+p+1}(2k+2p)!}{2(i\pi N)^{2k+1} N^{2p}(2k)!} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{2p+2k+1}} \\ &+ O(N^{-2k-2p-2}),\ N \to \infty,\end{aligned}$$

$$\begin{aligned}\Delta_{\pm N}^p(\check{B}_n(2k+1)) &= \frac{(-1)^{N+p}(2k+2p+2)!}{2(i\pi N)^{2k+2} N^{2p+1}(2k+1)!} \sum_{s=-\infty}^{\infty} \frac{(-1)^s s}{(2s+1)^{2p+2k+3}} \\ &+ O(N^{-2k-2p-4}),\ N \to \infty.\end{aligned}$$

REFERENCES

[1] B. ADCOCK, Convergence acceleration of Fourier-like series in one or more dimensions, *Technical report NA2008/11*, DAMTP, University of Cambridge, 2008.

[2] B. ADCOCK, Multivariate modified Fourier series and application to boundary value problems, *Numer. Math.*, **115**(2010). No. 4, pp. 511-552.

[3] B. ADCOCK, Convergence acceleration of modified Fourier series in one or more dimensions, *Math. Comp.*, **80**(2011), No. 273, pp. 225-261.

[4] A. BARKHUDARYAN, R. BARKHUDARYAN, and A. POGHOSYAN, Asymptotic behavior of Eckhoff's method for Fourier series convergence acceleration, *Analysis in Theory and Applications*, **23**(2007), No. 3, pp. 228–242.

[5] G. BASZENSKI, F.-J. DELVOS, and M. TASCHE, A united approach to accelerating trigonometric expansions, *Computers Math. Applic.*, **30**(1995), No. 3-6, pp. 33–49.

[6] D. BATENKOV, and Y. YOMDIN, Algebraic Fourier reconstruction of piecewise smooth functions, *Math. Comp.*, 81(2012), No. 277, pp. 277–318.

[7] J. P. BOYD, Acceleration of algebraically-converging Fourier series when the coefficients have series in powers of 1/n, *Journal of Computational Physics*, **228**(2009), No. 5, pp. 1404–1411.

[8] K. S. ECKHOFF, Accurate and efficient reconstruction of discontinuous functions from truncated series expansions, *Math. Comp.*, **61**(1993), No. 204, pp. 745–763.

[9] K. S. ECKHOFF, Accurate reconstructions of functions of finite regularity from truncated Fourier series expansions, *Math. Comp.*, **64**(1995), No. 210, pp. 671–690.

[10] K. S. ECKHOFF, On a high order numerical method for functions with singularities, *Math. Comp.*, **67**(1998), No. 223, pp. 1063–1087.

[11] K. S. ECKHOFF, and C.E. WASBERG, On the numerical approximation of derivatives by a modified Fourier collocation method, *Technical Report N 99*, Dept. of Mathematics, University of Bergen, Norway, 1995.

[12] W. B. JONES, and G. HARDY, Accelerating convergence of trigonometric approximations, *Math. Comp.*, **24**(1970), No. 111, pp. 547–560.

[13] A. KRYLOV, *On Approximate Calculations. Lectures Delivered in 1906 (in Russian)*, Tipolitography of Birkenfeld, St. Petersburg, 1907.

[14] C. LANCZOS, Evaluation of noisy data, *J. Soc. Indust. Appl. Math., Ser. B Numer. Anal.*, **1**(1964), pp. 76–85.

[15] C. LANCZOS, *Discourse on Fourier Series*, Oliver and Boyd, Edinburgh, 1966.

[16] J. N. LYNESS, Computational techniques based on the Lanczos representation, *Math. Comp.*, **28**(1974), pp. 81–123.

[17] A. NERSESSIAN, and A. POGHOSYAN, L_2-estimates for convergence rate of polynomial-periodic approximations by translates, *J. Contemp. Math. Anal.*, **36**(2002), No. 3, pp. 56–74.

[18] A. NERSESSIAN, and A. POGHOSYAN, *Asymptotic errors of accelerated two-dimensional trigonometric approximations*, Proceedings of the International ISAAC Conference "Complex Analysis, Differential Equations and Related Topics" Yerevan 2004, 70–78 (eds. G. A. Barsegian, H. G. W. Begehr, H. Ghazaryan, A. Nersessian).

[19] A. NERSESSIAN, and A. POGHOSYAN, *Asymptotic error of a polynomial-periodic interpolation*, Proceedings of the International ISAAC Conference "Complex Analysis, Differential Equations and Related Topics" Yerevan 2004, 88–98 (eds. G. A. Barsegian, H. G. W. Begehr, H. Ghazaryan, A. Nersessian).

[20] A. NERSESSIAN, and A. POGHOSYAN, On a rational linear approximation of Fourier series for smooth functions, *Journal of Scientific Computing*, **26**(2006), No. 1, pp. 111–125.

[21] A. NERSESSIAN, and A. POGHOSYAN, Convergence acceleration of two-dimensional Fourier interpolation, *Armenian Journal of Mathematics*, **1**(2008), No. 1, pp. 50–63.

[22] A. POGHOSYAN, Asymptotic behavior of the Krylov-Lanczos interpolation, *Analysis and Applications*, **7**(2009), No. 2, pp. 199–211.

[23] A. POGHOSYAN, Asymptotic behavior of the Eckhoff's method for convergence acceleration of trigonometric interpolation, *Analysis in Theory and Applications*, **26**(2010), No. 3, pp. 236–260, doi:10.1007/s10496-010-0236-3.

[24] A. POGHOSYAN, Fourier formulae for equidistant Hermite trigonometric interpolation, *Asian-European Journal of Mathematics*, **4**(2011), No. 1, pp. 127–144, doi: 10.1142/S1793557111000113.

[25] A. POGHOSYAN, On an autocorrection phenomenon of the Krylov-Gottlieb-Eckhoff method, *IMA Journal of Numerical Analysis*, **31**(2011), No. 2, pp. 512–527, doi:10.1093/imanum/drp043.

[26] J. RIORDAN, *Combinatorial Identities*, Wiley, New York, 1979.

[27] J. RIORDAN, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1980.