RECONSTRUCTION OF CONVEX BODIES FROM PROJECTION CURVATURE RADIUS FUNCTION

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In this article we pose the problem of existence and uniqueness of convex body for which the projection curvature radius function coincides with given function. We find a necessary and sufficient condition that ensures a positive answer to both questions and suggest an algorithm of construction of the body. Also we find a representation of the support function of a convex body by projection curvature radii.

§1. INTRODUCTION

Let $F(\omega)$ be a function defined on the sphere $\mathbf{S}^2$. The existence and uniqueness of convex body $\mathbf{B} \subset \mathbb{R}^3$ for which the mean curvature radius at a point on $\partial\mathbf{B}$ with outer normal direction ω coincides with given $F(\omega)$ was posed by Christoffel (see [2],[7]). Let $R_1(\omega)$ and $R_2(\omega)$ be the principal radii of curvatures of the surface of the body at the point with normal $\omega \in \mathbf{S}^2$. Christoffel problem asked about the existence of $\mathbf{B}$ for which

$$R_1(\omega) + R_2(\omega) = F(\omega). \tag{1.1}$$

The corresponding problem for Gauss curvature $R_1(\omega)R_2(\omega) = F(\omega)$ was posed and solved by Minkovski. W. Blashke reduced the Christoffel problem to a partial differential equation of second order for the support function (see [7]). A. D. Aleksandrov and A. V. Pogorelov generelized these problems, and proved the existence and uniqueness of convex body for which

$$G(R_1(\omega), R_2(\omega)) = F(\omega), \tag{1.2}$$

for a class of symmetric functions G (see [2], [9]).

In this paper we generalize the classic problem in a different direction and pose a similar problem for the projection curvature radii of convex bodies (see [4]). By $\mathcal{B}$ we denote the class of convex bodies $\mathbf{B} \subset \mathbb{R}^3$. We need some notation.

$\mathbf{S}^2$ – the unit sphere in $\mathbb{R}^3$ (the space of spatial directions),

$\mathbf{S}_\omega \subset \mathbf{S}^2$ – the great circle with pole at $\omega \in \mathbf{S}^2$,

$\mathbf{B}(\omega)$ – projection of $\mathbf{B} \in \mathcal{B}$ onto the plane containing the origin in $\mathbb{R}^3$ and ortogonal to ω.

$R(\omega, \varphi)$ – curvature radius of $\partial\mathbf{B}(\omega)$ at the point whose outer normal direction is $\varphi \in \mathbf{S}_\omega$.

Let $F(\omega,\varphi)$ be a nonnegative function defined on $\{(\omega,\varphi):\omega\in\mathbf{S}^2,\varphi\in\mathbf{S}_\omega\}$ (the space of "flags" see [1]).

In this article we pose:

Problem 1. existence and uniqueness (up to a translation) of a convex body for which

$$R(\omega,\varphi)=F(\omega,\varphi)\qquad and \tag{1.3}$$

Problem 2. construction of that convex body.

It is well known (see [10]) that a convex body $\mathbf{B}$ is determined uniquely by its support function $H(\Omega)=\max\{<\Omega,y>: y\in\mathbf{B}\}$ defined for $\Omega\in\mathbf{S}^2$, where $<\cdot,\cdot>$ denotes the standard inner product in $\mathbb{R}^3$. Usually one extends $H(\Omega)$ to a function $H(x)$, $x\in\mathbb{R}^3$ using homogeneity: $H(x)=|x|\,H(\Omega)$, where Ω is the direction of $\overrightarrow{Ox}$ (O is the origin in $\mathbb{R}^3$). Then the definition of convexity of $H(\Omega)$ is written as

$$H(x+y)\le H(x)+H(y)\quad for\,every\quad x,y\in\mathbb{R}^3.$$

Below $\mathcal{C}^k(\mathbf{S}^2)$ denotes the space of k times continuously differentiable functions in $\mathbf{S}^2$. A convex body $\mathbf{B}$ we call k-smooth if $H(\Omega)\in\mathcal{C}^k(\mathbf{S}^2)$.

Given a function $H(\Omega)$ defined for $\Omega\in\mathbf{S}^2$, by $H_\omega(\varphi)$, $\varphi\in\mathbf{S}_\omega$ we denote the restriction of $H(\Omega)$ to the circle $\mathbf{S}_\omega$ for $\omega\in\mathbf{S}^2$.

Below we show that the Problem 1. is equivalent to the problem of existence of a function $H(\Omega)$ defined on $\mathbf{S}^2$ satisfing the differential equation

$$H_\omega(\varphi)+[H_\omega(\varphi)]''_{\varphi\varphi}=F(\omega,\varphi)\quad for\,every\quad \omega\in\mathbf{S}^2\quad and\quad \varphi\in\mathbf{S}_\omega. \tag{1.4}$$

Note, that if restrictions of $H(\Omega)$ satisfies (1.4), then (the extention of) $H(\Omega)$ is convex.

Definition 1.1. If for given $F(\omega,\varphi)$ there exists $H(\Omega)\in\mathcal{C}^2(\mathbf{S}^2)$ defined on $\mathbf{S}^2$ that satisfies (1.4), then $H(\Omega)$ is called a spherical solution of (1.4).

In (1.4), $H_\omega(\varphi)$ is a flag function, so we recall the basic concepts associated with flags (in integral geometry the concept of a flag was first systematically employed by R. V. Ambartzumian in [1]).

A flag is a pair (ω,φ), where $\omega\in\mathbf{S}^2$ and $\varphi\in\mathbf{S}_\omega$. To each flag (ω,φ) corresponds a dual flag

$$(\omega,\varphi)\ \leftrightarrow\ (\omega,\varphi)^*=(\Omega,\phi), \tag{1.5}$$

where $\Omega\in\mathbf{S}^2$ is the spatial direction same as $\varphi\in\mathbf{S}_\omega$, while $\phi\in\mathbf{S}_\Omega$ is the direction same as ω. Given a flag function $g(\omega,\varphi)$, we denote by g^* the image of g defined by

$$g^*(\Omega,\phi)=g(\omega,\varphi), \tag{1.6}$$

where $(\omega,\varphi)^*=(\Omega,\phi)$.

Definition 1.2. For every $\omega \in \mathbf{S}^2$, (1.4) reduces to a differential equation on the circle $\mathbf{S}_\omega$. Any continuous function $G(\omega, \varphi)$ that is a solution of (1.4) for every $\omega \in \mathbf{S}^2$ we call *a flag solution.*

Definition 1.3. If a flag solution $G(\omega, \varphi)$ satisfies

$$G^*(\Omega, \phi) \;=\; G^*(\Omega) \tag{1.7}$$

(no dependence on the variable ϕ), then $G(\omega, \varphi)$ is called *a consistent flag solution.*

There is an important principle: *each consistent flag solution $G(\omega, \varphi)$ of (1.4) produces a spherical solution of (1.4) via the map*

$$G(\omega, \varphi) \to G(\Omega, \phi) = G(\Omega) = H(\Omega), \tag{1.8}$$

and vice versa: restrictions of any spherical solution of (1.4) onto the great circles is a consistent flag solution.

Hence the problem of finding the spherical solutions reduces to finding the consistent flag solutions.

To solve the latter problem, the present paper applies *the consistency method* first used in [3] and [5] in an integral equations context.

We denote:

$e[\Omega, \phi]$ – the plane containing the origin of $\mathbb{R}^3$, direction $\Omega \in \mathbf{S}^2$ and $\phi \in \mathbf{S}_\Omega$ (ϕ determine rotation of the plane around Ω),

$\mathbf{B}[\Omega, \phi]$ – projection of $\mathbf{B} \in \mathcal{B}$ onto the plane $e[\Omega, \phi]$,

$R^*(\Omega, \phi)$ – curvature radius of $\partial\mathbf{B}[\Omega, \phi]$ at the point whose outer normal direction is Ω.

It is easy to see that

$$R^*(\Omega, \phi) \;=\; R(\omega, \varphi),$$

where (Ω, ϕ) is the flag dual to (ω, φ).

Note, that in the Problem 1. uniquness (up to a translation) follows from the classical uniqueness result on Christoffel problem, since

$$R_1(\Omega) + R_2(\Omega) = \frac{1}{\pi}\int_0^{2\pi} R^*(\Omega, \phi)\, d\phi. \tag{1.9}$$

In case $F(\omega, \varphi) \geq 0$ is nonnegative, the equation (1.4) has the following geometrical interpretation. It follows from [4] that homogeneus fonction $H(x) = |x| H(\Omega)$, where $H(\Omega) \in \mathcal{C}^2(\mathbf{S}^2)$, is convex if and only if

$$H_\omega(\varphi) + [H_\omega(\varphi)]''_{\varphi\varphi} \geq 0 \quad for\,every \quad \omega \in \mathbf{S}^2 \quad and \quad \varphi \in \mathbf{S}_\omega, \tag{1.10}$$

where $H_\omega(\varphi)$ is the restriction of $H(\Omega)$ onto $\mathbf{S}_\omega$.

So in case $F(\omega,\varphi) \geq 0$, it follows from (1.10), that if $H(\Omega)$ is a spherical solution of (1.4) then its homogeneus extention $H(x) = |x|\, H(\Omega)$ is convex.

It is well known from convexity theory that if a function $H(x)$ is convex then there is a unique convex body

$\mathbf{B} \subset \mathbb{R}^3$ with support function $H(x)$.

The support function of each parallel shifts (translation) of that body $\mathbf{B}$ will again be a spherical solution of (1.4). By uniqueness, every two spherical solutions of (1.4) differ by a summand $< a, \Omega >$, where $a \in \mathbb{R}^3$. Thus we proved the following theorem.

Theorem 1.1. Let $F(\omega,\varphi) \geq 0$ be a nonnegative function defined on $\{(\omega,\varphi) : \omega \in \mathbf{S}^2, \varphi \in \mathbf{S}_\omega\}$. If the equation (1.4) has a spherical solution $H(\Omega)$ then there exists a convex body $\mathbf{B}$ with projection curvature radius function $F(\omega,\varphi)$, whose support function is $H(\Omega)$. Every spherical solutions of (1.4) has the form $H(\Omega) + < a, \Omega >$, where $a \in \mathbb{R}^3$, each being the support function of a translation of the convex body $\mathbf{B}$ by $\overrightarrow{Oa}$.

The converse statement is also true. It follows from the theory for 2-dimension (see [8]), that the support function $H(\Omega)$ of a convex body $\mathbf{B}$ satisfies (1.4) for $F(\omega,\varphi) = R(\omega,\varphi)$, where $R(\omega,\varphi)$ is the projection curvature radius function of $\mathbf{B}$.

Before going to the main result, we make some remarks. The purpose of the present paper is to find a necessary and sufficient condition that ensures a positive answer to both Problems 1,2 and suggest an algorithm of construction of the body $\mathbf{B}$ by finding a representation of the support function in terms of projection curvature radius function. This happens to be a spherical solution of the equation (1.4). In this paper the support function of a convex body $\mathbf{B}$ is considered with respect to a special choice of the origin O^*. It turns out that each 1-smooth convex body $\mathbf{B}$ has the special point O^* we will call the centroid of $\mathbf{B}$ (see Theorem 6.1). The centroid coincides with the centre of symmetry for centrally symmetrical convex bodies.

For convex bodies $\mathbf{B}$ with positive Gaussian curvature one can define the centroid as follows: within $\mathbf{B}$ there exists a unique point O^* such that (see Lemma 6.1)

$$\int_{\mathbf{S}_\Omega} < \overrightarrow{O^* P_\Omega(\tau)}, \Omega > \, d\tau = 0 \quad \text{for every} \quad \Omega \in \mathbf{S}^2,$$

where $P_\Omega(\tau)$ is the point on $\partial\mathbf{B}$ whose outer normal has the direction $\tau \in \mathbf{S}_\Omega$, $d\tau$ is the usual angular measure on $\mathbf{S}_\Omega$). The set of points $\{P_\Omega(\tau), \tau \in \mathbf{S}_O m\}$ we will call the belt of $\mathbf{B}$ with normal Ω.

Throughout the paper (in particular, in Theorem 1.2 that follows) we use usual spherical coordinates ν, τ for points $\omega \in \mathbf{S}^2$ based on a choice of a North Pole $\mathcal{N} \in \mathbf{S}^2$ and a reference point $\tau = 0$ on the equator $\mathbf{S}_\mathcal{N}$. We put $\nu = \frac{\pi}{2} - (\widehat{\omega, \mathcal{N}})$ so that the points $(0, \tau)$ lie on the equator $\mathbf{S}_\mathcal{N}$. The point with coordinates ν, τ we will denote by $(\nu, \tau)_\mathcal{N}$. On each $\mathbf{S}_\omega$ we choose E=the direction East for the reference point and the anticlockwise direction as positive.

Now we describe the main result.

Theorem 1.2. The support function of any 3*-smooth convex body* $\mathbf{B}$ *with respect to the centroid* O^* *has the representation*

$$H(\Omega) = \frac{1}{4\pi} \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} R((0,\tau)_\Omega, \varphi) \cos\varphi \, d\varphi \right] d\tau +$$

$$+\frac{1}{8\pi^2} \int_0^{2\pi} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R((0,\tau)_\Omega, \varphi) \left((\pi + 2\varphi)\cos\varphi - 2\sin^3\varphi\right) d\varphi \right] d\tau -$$

$$-\frac{1}{2\pi^2} \int_0^{\frac{\pi}{2}} \frac{sin\nu}{\cos^2\nu} d\nu \int_0^{2\pi} d\tau \int_0^{2\pi} R((\nu,\tau)_\Omega, \varphi) \sin^3\varphi \, d\varphi \tag{1.11}$$

where $R(\omega, \varphi)$ *is the projection curvature radius function of* $\mathbf{B}$*, on* $\mathbf{S}_\omega$ *we measure* φ *from the East direction with respect to* Ω*. (1.11) is a spherical solution of the equation (1.4) for* $F(\omega, \varphi) = R(\omega, \varphi)$*.*

Remark, that the order of integration in the last integral of (1.11) is important.

Obviously Theorem 1.2 suggests a practical algorithm of reconstruction of convex bodies from projection curvature radius function $R(\omega, \varphi)$ by calculation of the support function $H(\Omega)$.

We turn to Problem 1. Let $R(\omega, \varphi)$ be the projection curvature radius function of a convex body $\mathbf{B}$. Then $F(\omega, \varphi) \equiv R(\omega, \varphi)$ necessarily satisfies the following conditions:

1.

$$\int_0^{2\pi} F(\omega, \varphi) \sin\varphi \, d\varphi = \int_0^{2\pi} F(\omega, \varphi) \cos\varphi \, d\varphi = 0, \tag{1.12}$$

for every $\omega \in \mathbf{S}^2$ and any reference point on $\mathbf{S}_\omega$ (follows from equation (1.4), see also in [8]).

2. For every direction $\Omega \in \mathbf{S}^2$

$$\int_0^{2\pi} [F^*((\nu,\tau)_\Omega, E)]'_{\nu=0} \, d\tau = 0, \tag{1.13}$$

where $F^*(\Omega, \phi) = F(\omega, \varphi)$ (see (1.6)) and E is the East direction at the point $(\nu, \tau)_\Omega$ with respect Ω (Theorem 5.1).

Let $F(\omega,\varphi)$ be a nonnegative continuously differentable function defined on $\{(\omega,\varphi):\omega\in\mathbf{S}^2,\varphi\in\mathbf{S}_\omega\}$.

Using (1.11), we construct a function $\overline{F}(\Omega)$ defined on $\mathbf{S}^2$:

$$\overline{F}(\Omega)=\frac{1}{4\pi}\int_0^{2\pi}\left[\int_0^{\frac{\pi}{2}}F((0,\tau)_\Omega,\varphi)\,\cos\varphi\,d\varphi\right]d\tau+$$

$$+\frac{1}{8\pi^2}\int_0^{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}F((0,\tau)_\Omega,\varphi)\,((\pi+2\varphi)\cos\varphi-2\sin^3\varphi)\,d\varphi\right]d\tau-$$

$$-\frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}}\frac{sin\nu}{\cos^2\nu}d\nu\int_0^{2\pi}d\tau\int_0^{2\pi}F((\nu,\tau)_\Omega,\varphi)\sin^3\varphi\,d\varphi \tag{1.14}$$

Note that the last integral converges if the condition (1.13) is satisfied (see (5.7) and (5.8)).

THEOREM 1.3. *A nonnegative continuously differentable function $F(\omega,\varphi)$ defined on $\{(\omega,\varphi):\omega\in\mathbf{S}^2,\varphi\in\mathbf{S}_\omega\}$ represents the projection curvature radius function of some convex body if and only if $F(\omega,\varphi)$ satisfies the conditions (1.12), (1.13) and*

$$\overline{F}_\omega(\varphi)+[\overline{F}_\omega(\varphi)]''_{\varphi\varphi}=F(\omega,\varphi)\quad \textit{for every}\quad \omega\in\mathbf{S}^2\quad \textit{and}\quad \varphi\in\mathbf{S}_\omega. \tag{1.15}$$

where $\overline{F}_\omega(\varphi)$ is the restriction of $\overline{F}(\Omega)$ (given by (1.14)) onto $\mathbf{S}_\omega$.

Note that, in [6] the same problem for centrally symmetrical convex bodies was posed and a necessary and sufficient condition ensuring a positive answer found.

§2. GENERAL FLAG SOLUTION OF (1.4)

We fix $\omega\in\mathbf{S}^2$ and a pole $\mathcal{N}\in\mathbf{S}^2$ and try to solve (1.4) as a differential equation of second order on the circle $\mathbf{S}_\omega$.

We start with two results from [8].

1. For any smooth convex domain D in the plane

$$h(\varphi)=\int_0^\varphi R(\psi)\,\sin(\varphi-\psi)\,d\psi, \tag{2.1}$$

where $h(\varphi)$ is the support function of D with respect to a point $s\in\partial D$. In (2.1) we measure φ from the normal direction at s, $R(\psi)$ is the curvature radius of ∂D at the point with normal ψ.

2. (2.1) is a solution of the following differential equation

$$R(\varphi)=h(\varphi)+h''(\varphi). \tag{2.2}$$

One can easy verify that (also it follows from (2.2) and (2.1))

$$G(\omega,\varphi)=\int_0^\varphi F(\omega,\psi)\,\sin(\varphi-\psi)\,d\psi, \tag{2.3}$$

is a flag solution of the equation (1.4).

Theorem 2.1. Every continuous flag solution of (1.4) has the form

$$g(\omega,\varphi)=\int_0^{\varphi}F(\omega,\psi)\,\sin(\varphi-\psi)\,d\psi+C(\omega)\cos\varphi+S(\omega)\sin\varphi \tag{2.4}$$

whera C_n and S_n are some real coefficients.

Proof: Every continuous flag solution of (1.4) is a sum of $G(\omega,\varphi)+g_0(\omega,\varphi)$, where $g_0(\omega,\varphi)$ is a flag solution of the corresponding homogeneous equation

$$H_\omega(\varphi)+[H_\omega(\varphi)]''_{\varphi\varphi}=0 \quad \textit{for every}\ \ \omega\in\mathbf{S}^2 \quad \textit{and}\ \varphi\in\mathbf{S}_\omega. \tag{2.5}$$

We look for the general flag solution of (2.5) as a Fourier series

$$g_0(\omega,\varphi)=\sum_{n=0,1,2,\dots}[C_n(\omega)\cos n\varphi+S_n(\omega)\sin n\varphi]. \tag{2.6}$$

After substitution of (2.6) into (2.5) we obtain that $g_0(\omega,\varphi)$ satisfy (2.5) if and only if it has the form

$$g_0(\omega,\varphi)=C_1(\omega)\cos\varphi+S_1(\omega)\sin\varphi.$$

Theorem 2.1 is proved.

§3. THE CONSISTENCY CONDITION

Now we consider $C=C(\omega)$ and $S=S(\omega)$ in (2.4) as functions of $\omega=(\nu,\tau)$ and try to find $C(\omega)$ and $S(\omega)$ from the condition that $g(\omega,\varphi)$ satisfies (1.7). We write $g(\omega,\varphi)$ in dual coordinates i.e. $g(\omega,\varphi)=g^*(\Omega,\phi)$ and require that $g^*(\Omega,\phi)$ should not depend on ϕ for every $\Omega\in\mathbf{S}^2$, i.e. for every $\Omega\in\mathbf{S}^2$

$$(g^*(\Omega,\phi))'_\phi=(G(\omega,\varphi)+C(\omega)\cos\varphi+S(\omega)\sin\varphi)'_\phi=0, \tag{3.1}$$

where $G(\omega,\varphi)$ was defined in (2.3).

Here and below $(\cdot)'_\phi$ denotes the derivative corresponding to right screw rotation around Ω.

Termwise differentiation with use of expressions (see [5])

$$\tau'_\phi=\frac{\sin\varphi}{\cos\nu},\quad \varphi'_\phi=-\tan\nu\,\sin\varphi,\quad \nu'_\phi=-\cos\varphi, \tag{3.2}$$

after a natural grouping of the summands in (3.1), yields the Fourier series of $-(G(\omega,\varphi))'_\phi$ (a detailed derivation is contained in [5] and [3]). By uniqueness of the Fourier coefficients

$$\begin{cases}(C(\omega))'_\nu+\dfrac{(S(\omega))'_\tau}{\cos\nu}+\tan\nu\,C(\omega)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}A(\omega,\varphi)\,\cos2\varphi\,d\varphi\\ (C(\omega))'_\nu-\dfrac{(S(\omega))'_\tau}{\cos\nu}-\tan\nu\,C(\omega)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}A(\omega,\varphi)\,d\varphi\\ (S(\omega))'_\nu-\dfrac{(C(\omega))'_\tau}{\cos\nu}+\tan\nu\,S(\omega)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}A(\omega,\varphi)\,\sin2\varphi\,d\varphi,\end{cases} \tag{3.3}$$

where

$$A(\omega,\varphi)=\int_0^{\varphi}[F(\omega,\psi)'_\phi\,\sin{(\varphi-\psi)}+F(\omega,\psi)\,\cos{(\varphi-\psi)}\,\varphi'_\phi]\,d\psi. \tag{3.4}$$

§4. AVERAGING

Let H be a continuous spherical solution of (1.4), i.e. restriction of H onto the great circles is a consistent flag solution of (1.4). By Theorem 1.1 there exists a convex body $\mathbf{B}$ with projection curvature radius function $R(\omega,\varphi)=F(\omega,\varphi)$, whose support function is $H(\Omega)$.

To calculate $H(\Omega)$ we take $\Omega\in\mathbf{S}^2$ for the pole $\Omega=\mathcal{N}$. Returning to the formula (2.4) for every $\omega=(0,\tau)_\Omega\in\mathbf{S}_\Omega$ we have

$$H(\Omega)=\int_0^{\frac{\pi}{2}}R(\omega,\psi)\,\sin(\frac{\pi}{2}-\psi)\,d\psi+S(\omega), \tag{4.1}$$

We integrate both sides of (4.1) with respect to uniform angular measure $d\tau$ over $[0,2\pi)$ to get

$$2\pi H(\Omega)=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R((0,\tau)_\Omega,\psi)\,\cos\psi\,d\psi\,d\tau+\int_0^{2\pi}S((0,\tau)_\Omega)\,d\tau. \tag{4.2}$$

Now the problem is to calculate

$$\int_0^{2\pi}S((0,\tau)_\Omega)\,d\tau=\overline{S}(0). \tag{4.3}$$

We are going to integrate both sides of (3.3) and (3.4) with respect to $d\tau$ over $[0,2\pi)$.

For $\omega=(\nu,\tau)_\Omega$, where $\nu\in[0,\frac{\pi}{2})$ and $\tau\in(0,2\pi)$ (see (3.5)) we denote

$$\overline{S}(\nu)=\int_0^{2\pi}S((\nu,\tau)_\Omega)\,d\tau, \tag{4.4}$$

$$A(\nu)=\frac{1}{\pi}\int_0^{2\pi}d\tau\int_0^{2\pi}\left[\int_0^{\varphi}[R(\omega,\psi)'_\phi\,\sin{(\varphi-\psi)}+R(\omega,\psi)\,\cos{(\varphi-\psi)}\,\varphi'_\phi]\,d\psi\right]\sin2\varphi\,d\varphi. \tag{4.5}$$

Integrating both sides of (3.3) and (3.4) and taking into account that

$$\int_0^{2\pi}(C(\nu,\tau)_\Omega)'_\tau\,d\tau=0$$

for $\nu\in[0,\frac{\pi}{2})$ we get

$$\overline{S}'(\nu)+\tan\nu\,\overline{S}(\nu)=A(\nu). \tag{4.6}$$

Thus we have differential equation (4.6) for unknown coefficient $\overline{S}(\nu)$.

§5. BOUNDARY CONDITION FOR DIFFERENTIAL EQUATION (4.6)

We have to find $\overline{S}(0)$ given by (4.3). It follows from (4.6) that

$$\left(\frac{\overline{S}(\nu)}{\cos\nu}\right)'=\frac{A(\nu)}{\cos\nu}. \tag{5.1}$$

Integrating both sides of (5.1) with respect to $d\nu$ over $[0, \frac{\pi}{2})$ we obtain

$$\overline{S}(0) = \left.\frac{\overline{S}(\nu)}{\cos\nu}\right|_{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{A(\nu)}{\cos\nu}\, d\nu. \tag{5.2}$$

Now, we are going to calculate $\left.\frac{\overline{S}(\nu)}{\cos\nu}\right|_{\frac{\pi}{2}}$.

It follows from (2.4) that

$$\overline{S}(\nu) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\left[H_\omega(\varphi) - \int_0^{\varphi} R(\omega,\psi)\,\sin(\varphi-\psi)\,d\psi\right]\sin\varphi\,d\varphi\,d\tau =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} H_\omega(\varphi)\sin\varphi\,d\varphi\,d\tau - \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} R(\omega,\psi)\,((2\pi-\psi)\cos\psi + \sin\psi)\,d\psi\,d\tau. \tag{5.3}$$

Let $\varphi \in \mathbf{S}_\omega$ be the direction that corresponds to $\varphi \in [0, 2\pi)$, for $\omega = (\nu, \tau)_\Omega$. As a point of $\mathbf{S}^2$, let φ have spherical coordinates u, t with respect Ω. By the sinus theorem of spherical geometry

$$\cos\nu\,\sin\varphi = \sin u. \tag{5.4}$$

From (5.4) we get

$$(u)'_{\nu=\frac{\pi}{2}} = -\sin\varphi. \tag{5.5}$$

Using (5.5), for a fix τ we write a Taylor expession at a neighbourhood of the point $\nu = \frac{\pi}{2}$:

$$H_{(\nu,\tau)_\Omega}(\varphi) = H((0,\varphi+\tau)_\Omega) + H'_\nu((0,\varphi+\tau)_\Omega)\,\sin\varphi\,(\frac{\pi}{2}-\nu) + o(\frac{\pi}{2}-\nu). \tag{5.6}$$

Similarly, for $\psi \in [0, 2\pi)$ we get

$$R((\nu,\tau)_\Omega,\psi) = R((\frac{\pi}{2},\tau)_\Omega,\psi+\tau) + R'_\nu((\frac{\pi}{2},\tau)_\Omega,\psi+\tau)\,\sin\psi\,(\frac{\pi}{2}-\nu) + o(\frac{\pi}{2}-\nu). \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.3) and taking into account the easy equalities

$$\int_0^{2\pi}\int_0^{2\pi} H((0,\varphi+\tau)_\Omega)\sin\varphi\,d\varphi\,d\tau = 0$$

and

$$\int_0^{2\pi}\int_0^{2\pi} R((\frac{\pi}{2},\tau)_\Omega,\psi+\tau)\,((2\pi-\psi)\cos\psi+\sin\psi)\,d\psi\,d\tau = 0 \tag{5.8}$$

we obtain

$$\lim_{\nu\to\frac{\pi}{2}}\frac{\overline{S}(\nu)}{\cos\nu} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} H'_\nu((0,\varphi+\tau)_\Omega)\,\sin^2\varphi\,d\varphi\,d\tau -$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} R'_\nu((\frac{\pi}{2},\tau)_\Omega,\psi+\tau)\,\sin\psi\,\,((2\pi-\psi)\cos\psi+\sin\psi)\,\,d\psi\,d\tau =$$

$$= \int_0^{2\pi} H'_\nu((0,\tau)_\Omega)\,d\tau - \frac{3}{4}\int_0^{2\pi} [R^*((\nu,\tau)_\Omega, E)]'_{\nu=0} d\tau. \tag{5.9}$$

Theorem 5.1. For every 3*-smooth convex body* $\mathbf{B}$ *and any direction* $\Omega \in \mathbf{S}^2$*, we have*

$$\int_0^{2\pi} [R^*((\nu,\tau)_\Omega, E)]'_{\nu=0}\,d\tau = 0, \tag{5.10}$$

where ν, τ *is the spherical coordinates with respect to* Ω*, where* E *is the East direction at the point* $(\nu,\tau)_\Omega$ *with respect* Ω*.*

Proof. Using spherical geometry, one can prove that (see also (1.4))

$$[R^*((\nu,\tau)_\Omega, E)]'_{\nu=0} = \left[H((\nu,\tau)_\Omega) + H''_{\varphi\varphi}((\nu,\tau)_\Omega)\right]'_{\nu=0} =$$

$$\left[H((\nu,\tau)_\Omega) + H''_{\tau\tau}\frac{1}{\cos^2\nu} - H'_\nu \tan\nu\right]'_{\nu=0} = [H''_{\tau\tau}]'_{\nu=0}, \tag{5.11}$$

where $H(\Omega)$ is the supporet function of $\mathbf{B}$. After integration (5.11) we get

$$\int_0^{2\pi} [R^*((\nu,\tau)_\Omega, E)]'_{\nu=0}\,d\tau = \int_0^{2\pi} [H''_{\tau\tau}]'_{\nu=0}\,d\tau = 0.$$

§6. CENTROID OF A CONVEX BODY

Let $\mathbf{B}$ be a convex body in $\mathbb{R}^3$ and $Q \in \mathbb{R}^3$ be a point. By $H_Q(\Omega)$ we denote the support function of $\mathbf{B}$ with respect to Q.

Theorem 6.1. For a given 1*-smooth convex body* $\mathbf{B}$ *there is a point* $O^* \in \mathbb{R}^3$ *such that*

$$\int_0^{2\pi} [H_{O^*}((\nu,\tau)_\Omega)]'_{\nu=0}\;d\tau = 0 \quad \textit{for every} \quad \Omega \in \mathbf{S}^2, \tag{6.1}$$

where ν, τ *are the spherical coordinates with respect* Ω*.*

Proof. For a given $\mathbf{B}$ and a point $Q \in \mathbb{R}^3$ by $K_Q(\Omega)$ we denote the following function defined on $\mathbf{S}^2$

$$K_Q(\Omega) = \int_0^{2\pi} [H_Q((\nu,\tau)_\Omega)]'_{\nu=0}\;d\tau.$$

$K_Q(\Omega)$ is a continuous odd function with maximum $\overline{K}(Q)$

$$\overline{K}(Q) = \max_{\Omega\in\mathbf{S}^2} K_Q(\Omega).$$

It is easy to see that $\overline{K}(Q) \to \infty$ for $|Q| \to \infty$. Since $\overline{K}(Q)$ is a continuous so there is a point O^* for which

$$\overline{K}(O^*) = \min \overline{K}(Q).$$

Let Ω^* be a (say unique) direction of maximum i.e.

$$\overline{K}(O^*) = \max_{\Omega\in\mathbf{S}^2} K_{O^*}(\Omega) = K_{O^*}(\Omega^*).$$

If $\overline{K}(O^*) = 0$ the theorem is proved. For the case $\overline{K}(O^*) = a > 0$ let O^{**} be the point for which $\overrightarrow{O^*O^{**}} = \varepsilon\,\Omega^*$. It is easy to understand that $H_{O^{**}}(\Omega) = H_{O^*}(\Omega) - \varepsilon(\Omega,\Omega^*)$, hence for a small $\varepsilon > 0$ we find that $\overline{K}(O^{**}) = a - 2\pi\varepsilon$ which is contrary to definition of O^*. So $\overline{K}(O^*) = 0$. For the case where there are two or more directions of maximum one can apply a similar argument. The theorem is proved.

The point O^* we will call *the centroid* of the convex body $\mathbf{B}$. Theorem 6.2 below gives a clearer geometrical interpretation to that concept.

Let $P_\Omega(\tau)$ be the point on $\partial\mathbf{B}$ whose outer normal has the direction $\tau \in \mathbf{S}_\Omega$.

Lemma 6.1. For every 2*-smooth convex body* $\mathbf{B}$ *with positive Gaussian curvature and any direction* $\Omega \in \mathbf{S}^2$*, we have*

$$\int_0^{2\pi} [H_Q((\nu,\tau)_\Omega)]'_{\nu=0}\, d\tau = \int_{\mathbf{S}_\Omega} < \overrightarrow{QP_\Omega(\tau)}, \Omega > d\tau, \tag{6.2}$$

where Q *is a point of* $\mathbb{R}^3$ *and* $d\tau$ *is the usual angular measure on* $\mathbf{S}_\Omega$*).*

Proof. Let $B[\Omega,\tau]$ be the projection of $\mathbf{B}$ onto the plane $e[\Omega,\tau]$ (containing Q and the directions $\Omega \in \mathbf{S}^2$ and $\tau \in \mathbf{S}_\Omega$) and $P^*(\tau)$ be the point on $\partial B[\Omega,\tau]$ with outer normal $(0,\tau)_\Omega$. For the support function of $B[\Omega,\tau]$ (equivalently for the restriction of $H_Q(\Omega)$ onto $e[\Omega,\tau]$) we have

$$[H_Q((\nu,\tau)_\Omega)]'_{\nu=0} = [|\overrightarrow{QP^*(\tau)}|\cos(\nu-\nu_o) + H_{P^*}(\nu)]'_{\nu=0} = |\overrightarrow{QP^*(\tau)}|\sin\nu_o =< \overrightarrow{QP^*(\tau)}, \Omega >, \tag{6.3}$$

where $H_{P^*}(\nu)$ is the support function of $B[\Omega,\tau]$ with respect to the point $P^*(\tau) \in \partial B[\Omega,\tau]$ and $(\nu_0,\tau)_\Omega$ is the direction of $\overrightarrow{QP^*(\tau)}$. The statment $[H_{P^*}(\nu)]'_{\nu=0} = 0$ was proved in [5]. Integrating (6.3) and taking into account that $< \overrightarrow{QP^*(\tau)}, \Omega >=< Q\vec{P_\Omega}(\tau), \Omega >$ we get (6.2).

Theorem 6.1 and Lemma 6.1 imply the following Theorem.

Theorem 6.2. For a smooth convex body $\mathbf{B}$ *with positive Gaussian curvature we have*

$$\int_0^{2\pi} < \overrightarrow{O^*P_\Omega(\tau)}, \Omega > d\tau = 0 \quad \textit{for every} \quad \Omega \in \mathbf{S}^2, \tag{6.4}$$

where O^* *is the centroid of* $\mathbf{B}$*.*

One can consider the last statement as a definition of the centroid of $\mathbf{B}$.

§7. A REPRESENTATION FOR SUPPORT FUNCTION OF CONVEX BODIES

Let O^* be the centroid of the convex body $\mathbf{B}$ (see §6). Now we take O^* for the origin of $\mathbb{R}^3$. Below $H_{O^*}(\Omega)$ we will simply denote by $H(\Omega)$.

By Theorem 6.1, Theorem 5.1 and Lemma 6.1 we have the boundary condition (see (5.9))

$$\frac{\overline{S}(\nu)}{\cos\nu}\Big|_{\frac{\pi}{2}} = 0. \tag{7.1}$$

Substituting (5.2) into (4.2) we get

$$2\pi H(\Omega) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} R((0,\tau)_\Omega,\psi)\,\cos\psi\,d\psi\,d\tau - \int_0^{\frac{\pi}{2}}\frac{A(\nu)}{\cos\nu}\,d\nu = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} R((0,\tau)_\Omega,\psi)\,\cos\psi\,d\psi\,d\tau-$$

$$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\nu}{\cos\nu}\int_0^{2\pi}d\tau\int_0^{2\pi}\left[\int_0^{\varphi}[R(\omega,\psi)'_\phi\,\sin(\varphi-\psi)+R(\omega,\psi)\,\cos(\varphi-\psi)\,\varphi'_\phi]\,d\psi\right]\sin 2\varphi\,d\varphi.$$

Using expressions (3.2) and integrating by $d\varphi$ yields

$$2\pi\,H(\Omega) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} R((0,\tau)_\Omega,\psi)\,\cos\psi\,d\psi\,d\tau+$$

$$+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\nu}{\cos\nu}\int_0^{2\pi}d\tau\int_0^{2\pi}[R(\omega,\psi)'_\nu\,I + R(\omega,\psi)\,\tan\nu\,II]\,d\psi, \tag{7.2}$$

where

$$II = \int_\psi^{2\pi}\sin 2\varphi\,\cos(\varphi-\psi)\,\sin\varphi\,d\varphi = \left[\frac{(2\pi-\psi)\cos\psi}{4}+\frac{\sin\psi(1+\sin^2\psi)}{4}-\sin^3\psi\right],$$

and

$$I = \int_\psi^{2\pi}\sin 2\varphi\,\sin(\varphi-\psi)\,\cos\varphi\,d\varphi = \left[\frac{(2\pi-\psi)\cos\psi}{4}+\frac{\sin\psi(1+\sin^2\psi)}{4}\right]. \tag{7.3}$$

Integrating by parts (7.2) we get

$$2\pi\,H(\Omega) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} R((0,\tau)_\Omega,\psi)\,\cos\psi\,d\psi\,d\tau - \frac{1}{\pi}\int_0^{\frac{\pi}{2}}d\nu\int_0^{2\pi}d\tau\int_0^{2\pi}R(\omega,\psi)\frac{\sin\nu\,\sin^3\psi}{\cos^2\nu}\,d\psi-$$

$$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}d\tau\int_0^{2\pi}R((0,\tau)_\Omega,\psi)I\,d\psi + \lim_{a\to\frac{\pi}{2}}\frac{1}{\pi\cos a}\int_0^{\frac{\pi}{2}}d\tau\int_0^{2\pi}R((a,\tau)_\Omega,\psi)\,I\,d\psi. \tag{7.4}$$

Using (5.7), Theorem 5.1 and taking into account that

$$\int_0^{2\pi} I\,d\psi = 0$$

we get

$$2\pi\,H(\Omega) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} R((0,\tau)_\Omega,\psi)\,\cos\psi\,d\psi\,d\tau-$$

$$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}d\nu\int_0^{2\pi}d\tau\int_0^{2\pi}R(\omega,\psi)\frac{\sin\nu\,\sin^3\psi}{\cos^2\nu}\,d\psi - \frac{1}{\pi}\int_0^{\frac{\pi}{2}}d\tau\int_0^{2\pi}R((0,\tau),\psi)I\,d\psi. \tag{7.5}$$

From (7.5), by (1.12) we obtain (1.11). Theorem 1.2 is proved.

§8. PROOF OF THEOREM 1.3

Proof. Necessity: let $F(\omega,\varphi)$ be the projection curvature radius function of a convex body $\mathbf{B}$, then it satisfies (1.12) (see [8]), the condition (1.13) (Theorem 5.1) and the condition (1.14) (Theorem 1.2). Sufficiently: let $F(\omega,\psi)$ be a nonnegative continuous differentable function satisfing the conditions (1.12), (1.13), (1.15). By means of (1.14) we construct the function $\overline{F}(\Omega)$ defined on $\mathbf{S}^2$ as in (1.14). According to (1.15), $\overline{F}(\Omega)$ is a convex function hence there exists a convex body $\mathbf{B}$ with support function $\overline{F}(\Omega)$. The same (1.15) implies that $F(\omega,\varphi)$ is the projection curvature radius of $\mathbf{B}$.

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REFERENCES

1. R. V. Ambartzumian, "Factorization Calculus and Geometrical Probability", Cambridge Univ. Press, Cambridge, 1990.
2. A. D. Alexandrov, "Uniqueness theorems for surfaces in the large"[in Russian], Vesti, LGU, no. 19, 1956.
3. R. H. Aramyan, "An approach to generalized Funk equations, I" [in Russian], Izv. Akad. Nauk Armenii. Matematika, [English translation: Journal of Contemporary Math. Anal. (Armenian Academy of Sciences)], vol. 36, no. 1, pp. 47 – 58, 2001.
4. R. H. Aramyan, "Curvature radii of planar projections of convex bodies in R^n" [in Russian], Izv. Akad. Nauk Armenii. Matematika, [English translation: Journal of Contemporary Math. Anal. (Armenian Academy of Sciences)], vol. 37, no. 1, pp. 2 – 14, 2002.
5. R. H. Aramyan, "Generalized Radon transform with an application in convexity theory" [in Russian], Izv. Akad. Nauk Armenii. Matematika, [English translation: Journal of Contemporary Math. Anal. (Armenian Academy of Sciences)], vol. 38, no. 3, 2003.
6. R. H. Aramyan, "Reconstruction of centrally symmetrical convex bodies by projection curvature radii" [in print].
7. I. Ya. Bakelman, A. L. Verner, B. E. Kantor, "Differential Geometry in the Large"[in Russian], Nauka, Moskow, 1973.
8. W. Blaschke, "Kreis und Kugel" (Veit, Leipzig), 2nd Ed. De Gruyter, Berlin, 1956.
9. A. V. Pogorelov, "Exterior Geometry of Convex Surfaces" [in Russian], Nauka, Moscow, 1969.
10. W. Wiel, R. Schneider, "Zonoids and related Topics", in Convexity and its Applications, Ed. P. Gruber and J. Wills, Birkhauser, Basel, 1983.

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