In this article we pose the problem of existence and uniqueness of convex body for which the projection curvature radius function coincides with given function. We find a necessary and sufficient condition that ensures a positive answer to both questions and suggest an algorithm of construction of the body. Also we find a representation of the support function of a convex body by projection curvature radii.

§1. INTRODUCTION

Let $F(\omega)$ be a function defined on the sphere $S^2$. The existence and uniqueness of convex body $B \subset \mathbb{R}^3$ for which the mean curvature radius at a point on $\partial B$ with outer normal direction $\omega$ coincides with given $F(\omega)$ was posed by Christoffel (see [2], [7]). Let $R_1(\omega)$ and $R_2(\omega)$ be the principal radii of curvatures of the surface of the body at the point with normal $\omega \in S^2$. Christoffel problem asked about the existence of $B$ for which

$$R_1(\omega) + R_2(\omega) = F(\omega),$$

(1.1)

The corresponding problem for Gauss curvature $R_1(\omega)R_2(\omega) = F(\omega)$ was posed and solved by Minkovski. W. Blaschke reduced the Christoffel problem to a partial differential equation of second order for the support function (see [7]). A. D. Aleksandrov and A. V. Pogorelov generalized these problems, and proved the existence and uniqueness of convex body for which

$$G(R_1(\omega), R_2(\omega)) = F(\omega),$$

(1.2)

for a class of symmetric functions $G$ (see [2], [9]).

In this paper we generalize the classic problem in a different direction and pose a similar problem for the projection curvature radii of convex bodies (see [4]). By $\mathcal{B}$ we denote the class of convex bodies $B \subset \mathbb{R}^3$. We need some notation.

$S^2$ – the unit sphere in $\mathbb{R}^3$ (the space of spatial directions),

$S_\omega \subset S^2$ – the great circle with pole at $\omega \in S^2$,

$\mathcal{B}(\omega)$ – projection of $B \in \mathcal{B}$ onto the plane containing the origin in $\mathbb{R}^3$ and orthogonal to $\omega$.

$R(\omega, \varphi)$ – curvature radius of $\partial \mathcal{B}(\omega)$ at the point whose outer normal direction is $\varphi \in S_\omega$. 

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Let \( F(\omega, \varphi) \) be a nonnegative function defined on \( \{(\omega, \varphi) : \omega \in \mathbb{S}^2, \varphi \in S_\omega \} \) (the space of “flags” see [1]).

In this article we pose:

Problem 1. existence and uniqueness (up to a translation) of a convex body for which

\[
R(\omega, \varphi) = F(\omega, \varphi) \quad \text{and} \quad (1.3)
\]

Problem 2. construction of that convex body.

It is well known (see [10]) that a convex body \( B \) is determined uniquely by its support function

\[ H(\Omega) = \max \{ \langle \Omega, y \rangle : y \in B \} \]

defined for \( \Omega \in \mathbb{S}^2 \), where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^3 \). Usually one extends \( H(\Omega) \) to a function \( H(x), x \in \mathbb{R}^3 \) using homogeneity: \( H(x) = |x| H(\Omega) \), where \( \Omega \) is the direction of \( \vec{Ox} \) (O is the origin in \( \mathbb{R}^3 \)). Then the definition of convexity of \( H(\Omega) \) is written as

\[
H(x + y) \leq H(x) + H(y) \quad \text{for every} \quad x, y \in \mathbb{R}^3.
\]

Below \( C^k(\mathbb{S}^2) \) denotes the space of \( k \) times continuously differentiable functions in \( \mathbb{S}^2 \). A convex body \( B \) we call \( k \)-smooth if \( H(\Omega) \in C^k(\mathbb{S}^2) \).

Given a function \( H(\Omega) \) defined for \( \Omega \in \mathbb{S}^2 \), by \( H_\omega(\varphi), \varphi \in S_\omega \) we denote the restriction of \( H(\Omega) \) to the circle \( S_\omega \) for \( \omega \in \mathbb{S}^2 \).

Below we show that the Problem 1. is equivalent to the problem of existence of a function \( H(\Omega) \) defined on \( \mathbb{S}^2 \) satisfying the differential equation

\[
H_\omega(\varphi) + [H_\omega(\varphi)]''_{\varphi} = F(\omega, \varphi) \quad \text{for every} \quad \omega \in \mathbb{S}^2 \quad \text{and} \quad \varphi \in S_\omega. \quad (1.4)
\]

Note, that if restrictions of \( H(\Omega) \) satisfies (1.4), then (the extension of) \( H(\Omega) \) is convex.

**Definition 1.1.** If for given \( F(\omega, \varphi) \) there exists \( H(\Omega) \in C^2(\mathbb{S}^2) \) defined on \( \mathbb{S}^2 \) that satisfies (1.4), then \( H(\Omega) \) is called a spherical solution of (1.4).

In (1.4), \( H_\omega(\varphi) \) is a flag function, so we recall the basic concepts associated with flags (in integral geometry the concept of a flag was first systematically employed by R. V. Ambartzumian in [1]).

A flag is a pair \((\omega, \varphi)\), where \( \omega \in \mathbb{S}^2 \) and \( \varphi \in S_\omega \). To each flag \((\omega, \varphi)\) corresponds a dual flag

\[
(\omega, \varphi) \leftrightarrow (\omega, \varphi)^* = (\Omega, \phi), \quad (1.5)
\]

where \( \Omega \in \mathbb{S}^2 \) is the spatial direction same as \( \varphi \in S_\omega \), while \( \phi \in S_\Omega \) is the direction same as \( \omega \). Given a flag function \( g(\omega, \varphi) \), we denote by \( g^* \) the image of \( g \) defined by

\[
g^*(\Omega, \phi) = g(\omega, \varphi), \quad (1.6)
\]

where \((\omega, \varphi)^* = (\Omega, \phi)\).
**Definition 1.2.** For every $\omega \in S^2$, (1.4) reduces to a differential equation on the circle $S_\omega$. Any continuous function $G(\omega, \varphi)$ that is a solution of (1.4) for every $\omega \in S^2$ we call a flag solution.

**Definition 1.3.** If a flag solution $G(\omega, \varphi)$ satisfies

$$G^*(\Omega, \phi) = G^*(\Omega)$$

(no dependence on the variable $\phi$), then $G(\omega, \varphi)$ is called a consistent flag solution.

There is an important principle: each consistent flag solution $G(\omega, \varphi)$ of (1.4) produces a spherical solution of (1.4) via the map

$$G(\omega, \varphi) \rightarrow G(\Omega, \phi) = G(\Omega) = H(\Omega),$$

and vice versa: restrictions of any spherical solution of (1.4) onto the great circles is a consistent flag solution.

Hence the problem of finding the spherical solutions reduces to finding the consistent flag solutions.

To solve the latter problem, the present paper applies the consistency method first used in [3] and [5] in an integral equations context.

We denote:

- $e[\Omega, \phi]$: the plane containing the origin of $\mathbb{R}^3$, direction $\Omega \in S^2$ and $\phi \in S_\Omega$ (direction determine rotation of the plane around $\Omega$),

- $B[\Omega, \phi]$: projection of $B \in B$ onto the plane $e[\Omega, \phi]$,

- $R^*(\Omega, \phi)$: curvature radius of $\partial B[\Omega, \phi]$ at the point whose outer normal direction is $\Omega$.

It is easy to see that

$$R^*(\Omega, \phi) = R(\omega, \varphi),$$

where $(\Omega, \phi)$ is the flag dual to $(\omega, \varphi)$.

Note, that in the Problem 1. uniqueness (up to a translation) follows from the classical uniqueness result on Christoffel problem, since

$$R_1(\Omega) + R_2(\Omega) = \frac{1}{\pi} \int_0^{2\pi} R^*(\Omega, \phi) d\phi. \tag{1.9}$$

In case $F(\omega, \varphi) \geq 0$ is nonnegative, the equation (1.4) has the following geometrical interpretation. It follows from [4] that homogeneous function $H(x) = |x|H(\Omega)$, where $H(\Omega) \in C^2(S^2)$, is convex if and only if

$$H_\omega(\varphi) + [H_\omega(\varphi)]''_{\varphi, \varphi} \geq 0 \text{ for every } \omega \in S^2 \text{ and } \varphi \in S_\omega, \tag{1.10}$$
where $H_\omega(\varphi)$ is the restriction of $H(\Omega)$ onto $S_\omega$.

So in case $F(\omega, \varphi) \geq 0$, it follows from (1.10), that if $H(\Omega)$ is a spherical solution of (1.4) then its homogeneous extension $H(x) = |x| H(\Omega)$ is convex.

It is well known from convexity theory that if a function $H(x)$ is convex then there is a unique convex body $B \subset \mathbb{R}^3$ with support function $H(x)$.

The support function of each parallel shifts (translation) of that body $B$ will again be a spherical solution of (1.4). By uniqueness, every two spherical solutions of (1.4) differ by a summand $<a, \Omega>$, where $a \in \mathbb{R}^3$. Thus we proved the following theorem.

**Theorem 1.1.** Let $F(\omega, \varphi) \geq 0$ be a nonnegative function defined on $\{(\omega, \varphi) : \omega \in S^2, \varphi \in S_\omega\}$.

If the equation (1.4) has a spherical solution $H(\Omega)$ then there exists a convex body $B$ with projection curvature radius function $F(\omega, \varphi)$, whose support function is $H(\Omega)$. Every spherical solutions of (1.4) has the form $H(\Omega) + <a, \Omega>$, where $a \in \mathbb{R}^3$, each being the support function of a translation of the convex body $B$ by $\overline{O a}$.

The converse statement is also true. It follows from the theory for 2-dimension (see [8]), that the support function $H(\Omega)$ of a convex body $B$ satisfies (1.4) for $F(\omega, \varphi) = R(\omega, \varphi)$, where $R(\omega, \varphi)$ is the projection curvature radius function of $B$.

Before going to the main result, we make some remarks. The purpose of the present paper is to find a necessary and sufficient condition that ensures a positive answer to both Problems 1,2 and suggest an algorithm of construction of the body $B$ by finding a representation of the support function in terms of projection curvature radius function. This happens to be a spherical solution of the equation (1.4).

In this paper the support function of a convex body $B$ is considered with respect to a special choice of the origin $O^*$. It turns out that each 1-smooth convex body $B$ has the special point $O^*$ we will call the centroid of $B$ (see Theorem 6.1). The centroid coincides with the centre of symmetry for centrally symmetrical convex bodies.

For convex bodies $B$ with positive Gaussian curvature one can define the centroid as follows: within $B$ there exists a unique point $O^*$ such that (see Lemma 6.1)

$$\int_{S_\Omega} <\overline{O^* P_\Omega(\tau)}, \Omega> \, d\tau = 0 \quad \text{for every} \quad \Omega \in S^2,$$

where $P_\Omega(\tau)$ is the point on $\partial B$ whose outer normal has the direction $\tau \in S_\Omega$, $d\tau$ is the usual angular
measure on $S_{\Omega}$. The set of points $\{P_\Omega(\tau), \tau \in S_{\Omega}m\}$ we will call the belt of $B$ with normal $\Omega$.

Throughout the paper (in particular, in Theorem 1.2 that follows) we use usual spherical coordinates $\nu, \tau$ for points $\omega \in S^2$ based on a choice of a North Pole $N \in S^2$ and a reference point $\tau = 0$ on the equator $S_N$. We put $\nu = \frac{\pi}{2} - (\omega, N)$ so that the points $(0, \tau)$ lie on the equator $S_N$. The point with coordinates $\nu, \tau$ we will denote by $(\nu, \tau)_N$. On each $S_\omega$ we choose $E=$ the direction East for the reference point and the anticlockwise direction as positive.

Now we describe the main result.

**Theorem 1.2.** The support function of any 3-smooth convex body $B$ with respect to the centroid $O^*$ has the representation

$$H(\Omega) = \frac{1}{4\pi} \int_0^{2\pi} \left[ \int_0^{\pi} R((0, \tau)_\Omega, \varphi) \cos \varphi \, d\varphi \right] d\tau +$$

$$+ \frac{1}{8\pi^2} \int_0^{2\pi} \left[ \int_0^{\pi} R((0, \tau)_\Omega, \varphi) ((\pi + 2\varphi) \cos \varphi - 2\sin^2 \varphi) \, d\varphi \right] d\tau -$$

$$\frac{1}{2\pi^2} \int_0^{\pi} \frac{\sin \nu}{\cos^2 \nu} \, d\nu \int_0^{2\pi} d\tau \int_0^{2\pi} R((\nu, \tau)_\Omega, \varphi) \sin^3 \varphi \, d\varphi \tag{1.11}$$

where $R(\omega, \varphi)$ is the projection curvature radius function of $B$, on $S_\omega$ we measure $\varphi$ from the East direction with respect to $\Omega$. (1.11) is a spherical solution of the equation (1.4) for $F(\omega, \varphi) = R(\omega, \varphi)$.

Remark, that the order of integration in the last integral of (1.11) is important.

Obviously Theorem 1.2 suggests a practical algorithm of reconstruction of convex bodies from projection curvature radius function $R(\omega, \varphi)$ by calculation of the support function $H(\Omega)$.

We turn to Problem 1. Let $R(\omega, \varphi)$ be the projection curvature radius function of a convex body $B$.

Then $F(\omega, \varphi) \equiv R(\omega, \varphi)$ necessarily satisfies the following conditions:

1. $$\int_0^{2\pi} F(\omega, \varphi) \sin \varphi \, d\varphi = \int_0^{2\pi} F(\omega, \varphi) \cos \varphi \, d\varphi = 0, \tag{1.12}$$

for every $\omega \in S^2$ and any reference point on $S_\omega$ (follows from equation (1.4), see also in [8]).

2. For every direction $\Omega \in S^2$

$$\int_0^{2\pi} [F^*((\nu, \tau)_\Omega, E)]'_{\nu=0} d\tau = 0, \tag{1.13}$$

where $F^*((\Omega, \phi) = F(\omega, \varphi)$ (see (1.6)) and $E$ is the East direction at the point $(\nu, \tau)_\Omega$ with respect $\Omega$ (Theorem 5.1).
Let $F(\omega, \varphi)$ be a nonnegative continuously differentable function defined on $\{(\omega, \varphi) : \omega \in S^2, \varphi \in S_\omega\}$.

Using (1.11), we construct a function $\overline{F}(\Omega)$ defined on $S^2$:

$$
\overline{F}(\Omega) = \frac{1}{4\pi} \int_0^{2\pi} \left[ \int_0^{2\pi} F((0, \tau)_\Omega, \varphi) \cos \varphi \, d\varphi \right] \, d\tau + \frac{1}{8\pi^2} \int_0^{2\pi} \left[ \int_0^{2\pi} F((0, \tau)_\Omega, \varphi) ((\pi + 2\varphi) \cos \varphi - 2\sin^3 \varphi) \, d\varphi \right] \, d\tau - \frac{1}{2\pi^2} \int_0^{2\pi} \frac{\sin \nu}{\cos^2 \nu} \, d\nu \int_0^{2\pi} \int_0^{2\pi} F((\nu, \tau)_\Omega, \varphi) \sin^3 \varphi \, d\varphi \, d\tau.
$$

(1.14)

Note that the last integral converges if the condition (1.13) is satisfied (see (5.7) and (5.8)).

**Theorem 1.3.** A nonnegative continuously differentable function $F(\omega, \varphi)$ defined on $\{(\omega, \varphi) : \omega \in S^2, \varphi \in S_\omega\}$ represents the projection curvature radius function of some convex body if and only if $F(\omega, \varphi)$ satisfies the conditions (1.12), (1.13) and

$$
\overline{F}_{\omega}(\varphi) + |\overline{F}_{\omega}(\varphi)|''_{\varphi} = F(\omega, \varphi) \quad \text{for every } \omega \in S^2 \text{ and } \varphi \in S_\omega.
$$

(1.15)

where $\overline{F}_{\omega}(\varphi)$ is the restriction of $\overline{F}(\Omega)$ (given by (1.14)) onto $S_\omega$.

Note that, in [6] the same problem for centrally symmetrical convex bodies was posed and a necessary and sufficient condition ensuring a positive answer found.

**§2. General Flag Solution of (1.4)**

We fix $\omega \in S^2$ and a pole $N \in S^2$ and try to solve (1.4) as a differential equation of second order on the circle $S_\omega$.

We start with two results from [8].

1. For any smooth convex domain $D$ in the plane

$$
h(\varphi) = \int_0^{\varphi} R(\psi) \sin(\varphi - \psi) \, d\psi,
$$

(2.1)

where $h(\varphi)$ is the support function of $D$ with respect to a point $s \in \partial D$. In (2.1) we measure $\varphi$ from the normal direction at $s$, $R(\psi)$ is the curvature radius of $\partial D$ at the point with normal $\psi$.

2. (2.1) is a solution of the following differential equation

$$
R(\varphi) = h(\varphi) + h''(\varphi).
$$

(2.2)

One can easy verify that (also it follows from (2.2) and (2.1))

$$
G(\omega, \varphi) = \int_0^{\varphi} F(\omega, \psi) \sin(\varphi - \psi) \, d\psi,
$$

(2.3)

is a flag solution of the equation (1.4).
Theorem 2.1. Every continuous flag solution of (1.4) has the form

\[ g(\omega, \varphi) = \int_0^\varphi F(\omega, \psi) \sin(\varphi - \psi) \, d\psi + C(\omega) \cos \varphi + S(\omega) \sin \varphi \]  \hspace{1cm} (2.4)

where \( C_n \) and \( S_n \) are some real coefficients.

Proof: Every continuous flag solution of (1.4) is a sum of \( G(\omega, \varphi) + g_0(\omega, \varphi) \), where \( g_0(\omega, \varphi) \) is a flag solution of the corresponding homogeneous equation

\[ H_\omega(\varphi) + [H_\omega(\varphi)]''_{\varphi,\varphi} = 0 \quad \text{for every} \quad \omega \in \mathbb{S}^2 \quad \text{and} \quad \varphi \in S_\omega. \]  \hspace{1cm} (2.5)

We look for the general flag solution of (2.5) as a Fourier series

\[ g_0(\omega, \varphi) = \sum_{n=0,1,2,\ldots} [C_n(\omega) \cos n\varphi + S_n(\omega) \sin n\varphi]. \]  \hspace{1cm} (2.6)

After substitution of (2.6) into (2.5) we obtain that \( g_0(\omega, \varphi) \) satisfy (2.5) if and only if it has the form

\[ g_0(\omega, \varphi) = C_1(\omega) \cos \varphi + S_1(\omega) \sin \varphi. \]

Theorem 2.1 is proved.

§3. THE CONSISTENCY CONDITION

Now we consider \( C = C(\omega) \) and \( S = S(\omega) \) in (2.4) as functions of \( \omega = (\nu, \tau) \) and try to find \( C(\omega) \) and \( S(\omega) \) from the condition that \( g(\omega, \varphi) \) satisfies (1.7). We write \( g(\omega, \varphi) \) in dual coordinates i.e. \( g(\omega, \varphi) = g^*(\Omega, \phi) \) and require that \( g^*(\Omega, \phi) \) should not depend on \( \phi \) for every \( \Omega \in \mathbb{S}^2 \), i.e. for every \( \Omega \in \mathbb{S}^2 \)

\[ (g^*(\Omega, \phi))'_{\phi} = (G(\omega, \varphi) + C(\omega) \cos \varphi + S(\omega) \sin \varphi)'_{\varphi} = 0, \]  \hspace{1cm} (3.1)

where \( G(\omega, \varphi) \) was defined in (2.3).

Here and below \((\cdot)'_{\phi}\) denotes the derivative corresponding to right screw rotation around \( \Omega \).

Termwise differentiation with use of expressions (see [5])

\[ \tau'_{\phi} = \frac{\sin \varphi}{\cos \nu}, \quad \nu'_{\phi} = -\tan \nu \sin \varphi, \quad \nu'_{\phi} = -\cos \varphi, \]  \hspace{1cm} (3.2)

after a natural grouping of the summands in (3.1), yields the Fourier series of \(-(G(\omega, \varphi))'_{\phi}\) (a detailed derivation is contained in [5] and [3]). By uniqueness of the Fourier coefficients

\[
\begin{align*}
(C(\omega))'_{\nu} + \frac{(S(\omega))'_{\nu}}{\cos \nu} + \tan \nu C(\omega) &= \frac{1}{\pi} \int_0^{2\pi} A(\omega, \varphi) \cos 2\varphi \, d\varphi \\
(C(\omega))'_{\nu} - \frac{(S(\omega))'_{\nu}}{\cos \nu} - \tan \nu C(\omega) &= \frac{1}{2\pi} \int_0^{2\pi} A(\omega, \varphi) \, d\varphi \\
(S(\omega))'_{\nu} - \frac{(C(\omega))'_{\nu}}{\cos \nu} + \tan \nu S(\omega) &= \frac{1}{\pi} \int_0^{2\pi} A(\omega, \varphi) \sin 2\varphi \, d\varphi,
\end{align*}
\]  \hspace{1cm} (3.3)
where
\[ A(\omega, \varphi) = \int_{0}^{\varphi} [F(\omega, \psi)_{\psi} \sin(\varphi - \psi) + F(\omega, \psi) \cos(\varphi - \psi) \varphi_{\psi}] \, d\psi. \] (3.4)

\section*{4. Averaging}

Let \( H \) be a continuous spherical solution of (1.4), i.e. restriction of \( H \) onto the great circles is a consistent flag solution of (1.4). By Theorem 1.1 there exists a convex body \( B \) with projection curvature radius function \( R(\omega, \varphi) = F(\omega, \varphi) \), whose support function is \( H(\Omega) \).

To calculate \( H(\Omega) \) we take \( \Omega \in S^2 \) for the pole \( \Omega = N \). Returning to the formula (2.4) for every \( \omega = (0, \tau)_{\Omega} \in S_{\Omega} \) we have
\[ H(\Omega) = \int_{0}^{\varphi} R(\omega, \psi) \sin(\frac{\pi}{2} - \psi) \, d\psi + S(\omega), \] (4.1)

We integrate both sides of (4.1) with respect to uniform angular measure \( d\tau \) over \([0, 2\pi]\) to get
\[ 2\pi H(\Omega) = \int_{0}^{2\pi} \int_{0}^{\varphi} R((0, \tau)_{\Omega}, \psi) \cos \psi \, d\psi \, d\tau + \int_{0}^{2\pi} S((0, \tau)_{\Omega}) \, d\tau. \] (4.2)

Now the problem is to calculate
\[ \int_{0}^{2\pi} S((0, \tau)_{\Omega}) \, d\tau = \overline{S}(0). \] (4.3)

We are going to integrate both sides of (3.3) and (3.4) with respect to \( d\tau \) over \([0, 2\pi]\).

For \( \omega = (\nu, \tau)_{\Omega} \), where \( \nu \in [0, \frac{\pi}{2}] \) and \( \tau \in (0, 2\pi) \) (see (3.5)) we denote
\[ \overline{S}(\nu) = \int_{0}^{2\pi} S((\nu, \tau)_{\Omega}) \, d\tau, \] (4.4)
\[ A(\nu) = \frac{1}{\pi} \int_{0}^{2\pi} d\tau \int_{0}^{\frac{\pi}{2}} \left[ R(\omega, \psi)_{\psi} \sin(\varphi - \psi) + R(\omega, \psi) \cos(\varphi - \psi) \varphi_{\psi} \right] \sin 2\varphi \, d\varphi. \] (4.5)

Integrating both sides of (3.3) and (3.4) and taking into account that
\[ \int_{0}^{2\pi} (C(\nu, \tau)_{\Omega})_{\tau} \, d\tau = 0 \]
for \( \nu \in [0, \frac{\pi}{2}] \) we get
\[ \overline{S}'(\nu) + \tan \nu \overline{S}(\nu) = A(\nu). \] (4.6)

Thus we have differential equation (4.6) for unknown coefficient \( \overline{S}(\nu) \).

\section*{5. Boundary Condition for Differential Equation (4.6)}

We have to find \( \overline{S}(0) \) given by (4.3). It follows from (4.6) that
\[ \frac{(\overline{S}(\nu))'}{(\cos \nu)} = \frac{A(\nu)}{\cos \nu}. \] (5.1)
Integrating both sides of (5.1) with respect to $d\nu$ over $[0, \pi/2]$ we obtain

$$\overline{S}(0) = \frac{\overline{S}(\nu)}{\cos \nu} \bigg|_{\pi/2} - \int_0^{\pi/2} A(\nu) d\nu.$$  \hspace{1cm} (5.2)

Now, we are going to calculate $\frac{\overline{S}(\nu)}{\cos \nu}$.

It follows from (2.4) that

$$\overline{S}(\nu) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ H_\omega(\varphi) - \int_0^\varphi R(\omega, \psi) \sin(\varphi - \psi) d\psi \right] \sin \varphi d\varphi d\tau =$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H_\omega(\varphi) \sin \varphi d\varphi d\tau - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} R(\omega, \psi) ((2\pi - \psi) \cos \varphi + \sin \varphi) d\psi d\tau.$$  \hspace{1cm} (5.3)

Let $\varphi \in S_\omega$ be the direction that corresponds to $\varphi \in [0, 2\pi]$, for $\omega = (\nu, \tau)$. As a point of $S^2$, let $\varphi$ have spherical coordinates $u, t$ with respect $\Omega$. By the sinus theorem of spherical geometry

$$\cos \nu \sin \varphi = \sin u.$$  \hspace{1cm} (5.4)

From (5.4) we get

$$\left(\frac{d}{d\nu} \right)^t = -\sin \varphi.$$  \hspace{1cm} (5.5)

Using (5.5), for a fix $\tau$ we write a Taylor expression at a neighbourhood of the point $\nu = \frac{\pi}{2}$:

$$H_{(\nu, \tau)}(\varphi) = H((0, \varphi + \tau)_{\Omega}) + H'_{\nu}((0, \varphi + \tau)_{\Omega}) \sin \varphi \left(\frac{\pi}{2} - \nu\right) + o\left(\frac{\pi}{2} - \nu\right).$$  \hspace{1cm} (5.6)

Similarly, for $\psi \in [0, 2\pi]$ we get

$$R((\nu, \tau)_{\Omega}, \psi) = R((\frac{\pi}{2}, \tau)_{\Omega}, \psi + \tau) + R'_{\nu}((\frac{\pi}{2}, \tau)_{\Omega}, \psi + \tau) \sin \psi \left(\frac{\pi}{2} - \nu\right) + o\left(\frac{\pi}{2} - \nu\right).$$  \hspace{1cm} (5.7)

Substituting (5.6) and (5.7) into (5.3) and taking into account the easy equalities

$$\int_0^{2\pi} \int_0^{2\pi} H((0, \varphi + \tau)_{\Omega}) \sin \varphi d\varphi d\tau = 0$$

and

$$\int_0^{2\pi} \int_0^{2\pi} R((\frac{\pi}{2}, \tau)_{\Omega}, \psi + \tau) ((2\pi - \psi) \cos \varphi + \sin \varphi) d\psi d\tau = 0$$  \hspace{1cm} (5.8)

we obtain

$$\lim_{\nu \to \frac{\pi}{2}} \frac{\overline{S}(\nu)}{\cos \nu} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H'_{\nu}((0, \varphi + \tau)_{\Omega}) \sin^2 \varphi d\varphi d\tau -$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} R'_{\nu}((\frac{\pi}{2}, \tau)_{\Omega}, \psi + \tau) \sin \psi ((2\pi - \psi) \cos \varphi + \sin \varphi) d\psi d\tau.$$
\[
= \int_0^{2\pi} H'_\nu((0,\tau)\Omega) \, d\tau - \frac{3}{4} \int_0^{2\pi} [R^*((\nu,\tau)\Omega, E)]'_\nu \, d\tau.
\] (5.9)

**Theorem 5.1.** For every 3-smooth convex body \( \mathbf{B} \) and any direction \( \Omega \in S^2 \), we have
\[
\int_0^{2\pi} [R^*((\nu,\tau)\Omega, E)]'_\nu \, d\tau = 0,
\] (5.10)

where \( \nu, \tau \) is the spherical coordinates with respect to \( \Omega \), where \( E \) is the East direction at the point \( (\nu,\tau)\Omega \) with respect \( \Omega \).

**Proof.** Using spherical geometry, one can prove that (see also (1.4))
\[
[R^*((\nu,\tau)\Omega, E)]'_\nu = [H((\nu,\tau)\Omega) + H''_{\nu \nu}((\nu,\tau)\Omega)]'_\nu =
\left[H((\nu,\tau)\Omega) + \frac{1}{\cos^2 \nu} - H'_\nu \tan \nu \right]'_{\nu=0} = [H''_{\tau \tau}]'_{\nu=0},
\] (5.11)

where \( H(\Omega) \) is the support function of \( \mathbf{B} \). After integration (5.11) we get
\[
\int_0^{2\pi} [R^*((\nu,\tau)\Omega, E)]'_\nu \, d\tau = \int_0^{2\pi} [H''_{\tau \tau}]'_{\nu=0} \, d\tau = 0.
\]

### §6. CENTROID OF A CONVEX BODY

Let \( \mathbf{B} \) be a convex body in \( \mathbb{R}^3 \) and \( Q \in \mathbb{R}^3 \) be a point. By \( H_Q(\Omega) \) we denote the support function of \( \mathbf{B} \) with respect to \( Q \).

**Theorem 6.1.** For a given 1-smooth convex body \( \mathbf{B} \) there is a point \( O^* \in \mathbb{R}^3 \) such that
\[
\int_0^{2\pi} [H_{O^*}((\nu,\tau)\Omega)]'_\nu \, d\tau = 0 \quad \text{for every} \quad \Omega \in S^2,
\] (6.1)

where \( \nu, \tau \) are the spherical coordinates with respect \( \Omega \).

**Proof.** For a given \( \mathbf{B} \) and a point \( Q \in \mathbb{R}^3 \) by \( K_Q(\Omega) \) we denote the following function defined on \( S^2 \)
\[
K_Q(\Omega) = \int_0^{2\pi} [H_Q((\nu,\tau)\Omega)]'_\nu \, d\tau.
\]

\( K_Q(\Omega) \) is a continuous odd function with maximum \( \overline{K}(Q) \)
\[
\overline{K}(Q) = \max_{\Omega \in S^2} K_Q(\Omega).
\]

It is easy to see that \( \overline{K}(Q) \to \infty \) for \( |Q| \to \infty \). Since \( \overline{K}(Q) \) is a continuous so there is a point \( O^* \) for which
\[
\overline{K}(O^*) = \min \overline{K}(Q).
\]
Let $\Omega^*$ be a (say unique) direction of maximum i.e.,

$$\mathbf{K}(O^*) = \max_{\Omega \in S^2} K_{O^*}(\Omega) = K_{O^*}(\Omega^*).$$

If $\mathbf{K}(O^*) = 0$ the theorem is proved. For the case $\mathbf{K}(O^*) = a > 0$ let $O^{**}$ be the point for which $\overrightarrow{O^*O^{**}} = \varepsilon \overrightarrow{O^*}$. It is easy to understand that $H_{O^{**}}(\Omega) = H_{O^*}(\Omega) - \varepsilon(\Omega, \Omega^*)$, hence for a small $\varepsilon > 0$ we find that $\mathbf{K}(O^{**}) = a - 2\pi \varepsilon$ which is contrary to definition of $O^*$. So $\mathbf{K}(O^*) = 0$. For the case where there are two or more directions of maximum one can apply a similar argument. The theorem is proved.

The point $O^*$ we will call the centroid of the convex body $B$. Theorem 6.2 below gives a clearer geometrical interpretation to that concept.

Let $P_{\Omega}(\tau)$ be the point on $\partial B$ whose outer normal has the direction $\tau \in S^2$.

**Lemma 6.1.** For every 2-smooth convex body $B$ with positive Gaussian curvature and any direction $\Omega \in S^2$, we have

$$\int_0^{2\pi} [H_Q((\nu, \tau)_{\Omega})]_{\nu = 0} d\tau = \int_{S^2} \langle Q P_{\Omega}(\tau), \Omega \rangle > d\tau,$$

where $Q$ is a point of $\mathbb{R}^3$ and $d\tau$ is the usual angular measure on $S^2$.

**Proof.** Let $B[\Omega, \tau]$ be the projection of $B$ onto the plane $e[\Omega, \tau]$ (containing $Q$ and the directions $\Omega \in S^2$ and $\tau \in S_1$) and $P^*(\tau)$ be the point on $\partial B[\Omega, \tau]$ with outer normal $(0, \tau)_{\Omega}$. For the support function of $B[\Omega, \tau]$ (equivalently for the restriction of $H_Q(\Omega)$ onto $e[\Omega, \tau]$) we have

$$[H_Q((\nu, \tau)_{\Omega})]_{\nu = 0} = [Q P^*(\tau)] \cos(\nu - \nu_0) + H_P^*(\nu)]_{\nu = 0} = [Q P^*(\tau)] \sin \nu_0 = \langle Q P^*(\tau), \Omega \rangle,$$

where $H_P^*(\nu)$ is the support function of $B[\Omega, \tau]$ with respect to the point $P^*(\tau) \in \partial B[\Omega, \tau]$ and $(\nu_0, \tau)_{\Omega}$ is the direction of $Q P^*(\tau)$. The statement $[H_P^*(\nu)]_{\nu = 0} = 0$ was proved in [5]. Integrating (6.3) and taking into account that $\langle Q P^*(\tau), \Omega \rangle = \langle Q P_{\Omega}(\tau), \Omega \rangle$ we get (6.2).

Theorem 6.1 and Lemma 6.1 imply the following Theorem.

**Theorem 6.2.** For a smooth convex body $B$ with positive Gaussian curvature we have

$$\int_0^{2\pi} \langle O^* P_{\Omega}(\tau), \Omega \rangle > d\tau = 0 \text{ for every } \Omega \in S^2,$$

where $O^*$ is the centroid of $B$.

One can consider the last statement as a definition of the centroid of $B$. 
Let $O^*$ be the centroid of the convex body $B$ (see §6). Now we take $O^*$ for the origin of $\mathbb{R}^3$. Below $H_{O^*}(\Omega)$ we will simply denote by $H(\Omega)$.

By Theorem 6.1, Theorem 5.1 and Lemma 6.1 we have the boundary condition (see (5.9))

$$\frac{S(\nu)}{\cos \nu} \bigg|_{\psi} = 0. \quad (7.1)$$

Substituting (5.2) into (4.2) we get

$$2\pi H(\Omega) = \int_0^{2\pi} \int_0^\phi R(0, \tau|\Omega, \psi) \cos \psi d\psi d\tau - \int_0^\phi A(\nu) \cos \nu d\nu = \int_0^{2\pi} \int_0^\phi R(0, \tau|\Omega, \psi) \cos \psi d\psi d\tau -$$

$$- \frac{1}{\pi} \int_0^\phi \frac{d\nu}{\cos \nu} \int_0^{2\pi} d\tau \int_0^{2\pi} \left[ \int_0^\nu \left[ R(\omega, \psi') \sin (\varphi - \psi) + R(\omega, \psi) \cos (\varphi - \psi) \varphi' d\psi \right] \sin 2\varphi d\varphi \right].$$

Using expressions (3.2) and integrating by $d\varphi$ yields

$$2\pi H(\Omega) = \int_0^{2\pi} \int_0^\phi R(0, \tau|\Omega, \psi) \cos \psi d\psi d\tau +$$

$$+ \frac{1}{\pi} \int_0^\phi \frac{d\nu}{\cos \nu} \int_0^{2\pi} d\tau \int_0^{2\pi} \left[ R(\omega, \psi') I + R(\omega, \psi) \tan \nu II \right] d\psi, \quad (7.2)$$

where

$$II = \int_0^{2\pi} \sin 2\varphi \cos (\varphi - \psi) \sin \varphi d\varphi = \left[ \frac{(2\pi - \psi) \cos \psi}{4} + \frac{\sin \psi (1 + \sin^2 \psi)}{4} - \sin^3 \psi \right],$$

and

$$I = \int_0^{2\pi} \sin 2\varphi \sin (\varphi - \psi) \cos \varphi d\varphi = \left[ \frac{(2\pi - \psi) \cos \psi}{4} + \frac{\sin \psi (1 + \sin^2 \psi)}{4} \right]. \quad (7.3)$$

Integrating by parts (7.2) we get

$$2\pi H(\Omega) = \int_0^{2\pi} \int_0^\phi R((0, \tau)|\Omega, \psi) \cos \psi d\psi d\tau - \frac{1}{\pi} \int_0^\phi \frac{d\nu}{\cos \nu} \int_0^{2\pi} d\tau \int_0^{2\pi} R(\omega, \psi) \sin \nu \sin^3 \psi \cos^2 \nu d\psi -$$

$$- \frac{1}{\pi} \int_0^\phi d\nu \int_0^{2\pi} d\tau \int_0^{2\pi} R((0, \tau)|\Omega, \psi) I d\psi + \lim_{\alpha \to \pi} \frac{1}{\pi \cos \alpha} \int_0^\phi d\tau \int_0^{2\pi} R((\alpha, \tau)|\Omega, \psi) I d\psi. \quad (7.4)$$

Using (5.7), Theorem 5.1 and taking into account that

$$\int_0^{2\pi} I d\psi = 0$$

we get

$$2\pi H(\Omega) = \int_0^{2\pi} \int_0^\phi R((0, \tau)|\Omega, \psi) \cos \psi d\psi d\tau -$$

$$- \frac{1}{\pi} \int_0^\phi d\nu \int_0^{2\pi} d\tau \int_0^{2\pi} R(\omega, \psi) \sin \nu \sin^3 \psi \cos^2 \nu d\psi - \frac{1}{\pi} \int_0^\phi d\tau \int_0^{2\pi} R((0, \tau)|\psi) I d\psi. \quad (7.5)$$

From (7.5), by (1.12) we obtain (1.11). Theorem 1.2 is proved.
§8. PROOF OF THEOREM 1.3

Proof. Necessity: let $F(\omega, \varphi)$ be the projection curvature radius function of a convex body $B$, then it satisfies (1.12) (see [8]), the condition (1.13) (Theorem 5.1) and the condition (1.14) (Theorem 1.2).

Sufficiently: let $F(\omega, \psi)$ be a nonnegative continuous differentiable function satisfying the conditions (1.12), (1.13), (1.15). By means of (1.14) we construct the function $\overline{F}(\Omega)$ defined on $S^2$ as in (1.14). According to (1.15), $\overline{F}(\Omega)$ is a convex function hence there exists a convex body $B$ with support function $\overline{F}(\Omega)$.

The same (1.15) implies that $F(\omega, \varphi)$ is the projection curvature radius of $B$.

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