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On the denoising problem

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Abstract

In various applications the problem on separation the original signal and the noise arises. In this paper we consider two cases, which naturally arise in applied problems. In the first case, the original signal permits linear prediction by its past behavior. In the second case the original signal is the values of some analytic function at a points from unit disk. In the both cases the noise is assumed to be a stationary process with zero mean value.

Let us note that the first case arises in Physical phenomena consideration. The second case arises in Identification problems for linear systems.

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1 Introduction.

One of the basic problem of Signal Processing Theory is the estimation of the original signal $f(n)$ in presence of an additive noise, i.e.

$$X(n) = f(n) + \xi(n), \quad n = 0, \pm 1, \ldots$$

where $\xi(n), \quad n = 0, \pm 1, \ldots$ is a stationary process, with zero mean value.

Thus, the available informations are the values of observed signal

$$X(n), \quad n = 0, \pm 1, \ldots$$

and the statistical properties of the noise

$$\xi(n), \quad n = 0, \pm 1, \ldots$$

We want to separate $f(n), \quad n = 0, \pm 1, \ldots$ from the observed signal.

Let us note, that this problem is very old, see [10], [12], etc., and there are many results on this subject.

In this problem it is very important to specify the class of original signals.

Here, we consider two cases.

First case: A-priori it is known, that the original signal

$$f(n), \quad n = 0, \pm 1, \ldots$$

permits linear forecasting. This case naturally arises in Physical phenomena consideration.

Second case: The original component $f(n)$ in the observed signal $X(n)$, is the values of a bounded analytic function $F(z), \quad |z|<1$, at some points $z_n, \quad |z_n|<1, \quad n=1,2,\ldots$, i. e.

$$f(n)=F(z_n), \quad n=1,2,\ldots$$

The second case naturally arises in testing and identification problems. The bounded analytic function $F(z), \quad |z|<1$, is the transfer function of some Linear System, which is a subject to be determine.

2 Some classes of signals permitting forecasting

In this section we introduce some classes of signals, which, we will interprete as original signals.

With each bounded signal $f(n), \quad n=0,\pm1,\ldots$ we associate a generalized function $\hat{f}$ defined on the unit circle. It has the following Fourier series expansion

$$\hat{f}(z)=\sum_{n=-\infty}^{\infty} f(n)z^n, \quad |z|=1.$$

Definition 1. We say, that a bounded signal $f(n), \quad n=0,\pm1,\ldots$ permits Linear Forecasting of the other $\sigma>0$, and denote $\{f(n)\}_{n=-\infty}^{\infty}\in LF(\sigma)$, if there is a sequence $h(k), \quad k=1,2,\ldots$ such that

$$|h(k)|\leq Aexp\{-k^{\sigma}\}, \quad k=1,2,\ldots$$

where A is constant number and

$$f(n)=\sum_{k=1}^{\infty} f(n-k)h(k), \quad n=0,\pm1\ldots$$

Definition 2. We write $\{f(n)\}_{n=-\infty}^{\infty}\in LF(\infty)$, if there is a finite sequence of numbers $h(k), \quad k=1,2,\ldots,m$, such that

$$f(n)=\sum_{k=1}^{m} f(n-k)h(k), \quad n=0,\pm1\ldots$$

In the further consideration we need the following classical results.

Theorem (L. Carleson). Let a nontrivial analytic function $F(z), \quad |z|<1$, satisfy the conditions

$$|F^{(n)}(z)|\leq B^n n^{\frac{n}{\alpha}}, \quad |z|<1, \quad n=0,1,\ldots$$

where $0<\alpha<1/2$. Let $F(z)$ vanish at each point of the set $z\in E$ with all its derivatives.

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{dt}{\rho^{\frac{\alpha}{1-\alpha}}(e^{it},E)}<\infty, \tag{1}$$

where $\rho(e^{it},E)$ is the distance of the point e^{it} from the subset E.

This theorem was proved by L. Carleson [2]. Let us note, that the complete description of zero sets for the functions from this class is complicated and one can find the corresponding results in [5].

The following theorem was proved by R. Salinas [11]. Here we give an equivalent version of its original formulation.

Theorem (R. Salinas). Let a bounded analytic function $F(z), \quad |z|<1$, satisfy the conditions

$$|F^{(n)}(z)|\leq B^n n^{\frac{n}{\alpha}}, \quad |z|<1, \quad n=0,1,\ldots \tag{2}$$

where $1/2\leq\alpha$. If at some point $z_0, \quad |z_0|=1$, we have $F^{(n)}(z_0)=0, \quad n=0,1,\ldots$, then $F(z)\equiv 0$.

It turners out, that all the classes $LF(\sigma)$ for $\frac{1}{2}\leq\sigma$, coincide with $LF(\infty)$.

Theorem 1. If $\frac{1}{2}\leq\sigma$, then $LF(\sigma)=LF(\infty)$.

Proof. Since $f \in LF(\sigma)$, $\frac{1}{2} \le \sigma$, so, there is a sequence h_k, $k = 1, 2, \dots$ such that

$$|h(k)| \le Aexp\{-k^{\sigma}\}, \quad k = 1, 2, \dots \tag{3}$$

and

$$f(n) = \sum_{k=1}^{\infty} f(n-k)h(k), \quad n = 0, \pm 1 \dots \tag{4}$$

Let us denote

$$F(rz) = \sum_{n=-\infty}^{\infty} f(n) r^{|n|} z^n, \quad |z| = 1, \quad 0 \le r < 1,$$

and

$$H(rz) = 1 - \sum_{k=1}^{\infty} h_k r^n z^n.$$

The equality (4) may be written in the following form

$$\lim_{r \to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\pi} F(re^{ix}) H(re^{ix}) e^{-inx} dx = 0, \quad n = 0, \pm 1, \dots$$

So, the boundary values of the function $F(z)$ on the unit circle, generates a generalized function with support

$$supp(F) \subset \{z; \quad |z| = 1, \quad H(z) = 0\}.$$

From the inequalities (3) it follows, see [8], p. 26,

$$|H^{(n)}(z)| \le Ce^{an} n^{\frac{n}{\sigma}}, \quad |z| < 1, \quad n = 1, 2, \dots$$

for some constants C and a. Since $\frac{1}{2} \le \sigma$, the function $H(z)$ satisfies conditions of R. Salinas theorem. Consequently, it can vanish at finite number points only, and so $supp(F)$ consists with finite number points. Thus, the bounded signal $f(n)$, permits the following representation

$$f(n) = \sum_{k=1}^{m} c_k e^{ix_k n}, \quad n = 0, \pm 1, \dots,$$

so $f \in LF(\infty)$. Indeed, we have

$$f(n) = \sum_{k=1}^{m} \frac{det(D_k)}{det(D)},$$

where D_k is the following matrix

$$\begin{pmatrix} e^{-ix_1} & \dots & f(n-1) & \dots & e^{-ix_m} \\ e^{-2ix_1} & \dots & f(n-2) & \dots & e^{-2ix_m} \\ \dots & \dots & \dots & \dots & \dots \\ e^{-mix_1} & \dots & f(n-m) & \dots & e^{-mix_m} \end{pmatrix}$$

and D is

$$\begin{pmatrix} e^{-ix_1} & \dots & e^{-ix_k} & \dots & e^{-ix_m} \\ e^{-2ix_1} & \dots & e^{-2ix_k} & \dots & e^{-2ix_m} \\ \dots & \dots & \dots & \dots & \dots \\ e^{-mix_1} & \dots & e^{-mix_k} & \dots & e^{-mix_m} \end{pmatrix}$$

3 Denoising problem for the signals permitting forecasting

In the further considerations we will need the following version of the Law of large numbers, see [6], p. 222, and [9], p.216.

Theorem (Law of Large numbers). Let $\xi_k, \quad k = 0, \pm 1, \ldots$ be a sequence of independent random variables with the same distribution and with the zero mean values. Let $\mathbf{E}(|\xi_0|^p) < \infty$, where $1 \le p \le 2$. Let

$$u(re^{it}) = \sum_{k=-\infty}^{\infty} \xi_k r^{|k|} e^{itk}.$$

Then, for $1/p < \alpha$, by probability one, we have

$$\max_t |u(re^{it})| = O\left(\frac{1}{(1-r)^\alpha}\right), \quad r \to 1-0.$$

Theorem 2. Let $0 < \alpha < \frac{1}{2}$ and a closed subset $E \subset \{z : \quad |z| = 1\}$ satisfy the condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dx}{\rho^{\frac{\alpha}{1-\alpha}}(e^{ix}, E)} < \infty.$$

Let $u(n), \quad n = 0, \pm 1, \ldots$ be a bounded sequence and

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) r^{|n|} e^{ixn}, \quad z = re^{ix}, 0 < r < 1,$$

be an harmonic function. Let the support $supp(U)$ of the generalized function arising by boundary values of $U(z)$, be situated in E.

Then

$$\lim_{r \to 1-} \frac{1}{2\pi}\int_{E_r} U(rz)|dz| = u(0),$$

where

$$E_r = \left\{z; \quad |z| = 1, \quad |U(rz)| > (1-r)^{\frac{\alpha}{3\alpha-2}}\right\}.$$

Proof. It is sufficient to prove

$$\lim_{r \to 1-} \int_{T \setminus E_r} |U(re^{ix})| dx = 0,$$

where $T = \{z : \quad |z| = 1\}$.

Let $I_k, \quad k = 1, 2, \ldots$ be nonintersecting arcs on the unit circle laying out of the closed subset E. Let $\xi_k, \quad \eta_k$ be the end points of the arc I_k.

Let us denote

$$\Delta_k = \{rz; \quad z \in I_k, \quad 0 < r < 1\}.$$

By maximum principle, for arbitrary point $z \in \Delta_k$, we have

$$|U(z)| \le A\left(\frac{1-|z|^2}{|\xi_k - z|^2} + \frac{1-|z|^2}{|\eta_k - z|^2}\right).$$

So, for each $k = 1, 2, \ldots$ we have

$$\int_{I_k \setminus E_r} |U(rz)||dz| = \tag{5}$$

$$= \int_{I_k \setminus E_r} |U(rz)|^{\frac{2-3\alpha}{2-2\alpha}} |U(rz)|^{\frac{\alpha}{2-2\alpha}} |dz| \le$$

$$\le A_1(1-r)^{-\frac{\alpha}{2-2\alpha}} \int_{I_k \setminus E_r} |U(rz)|^{\frac{\alpha}{2-2\alpha}} |dz| \le$$

$$\leq A_2(1-r)^{\frac{1-2\alpha}{1-\alpha}} \int_0^{|I_k\setminus E_r|/(1-r)} (1+x)^{\frac{2\alpha}{2\alpha-2}} dx,$$

where A_1, A_2 are some constants. So,

$$\int_{T\setminus E_r} |U(re^{ix})| dx \leq$$

$$\leq A_2(1-r)^{\frac{1-2\alpha}{1-\alpha}} \sum_{k=1}^{\infty} \int_0^{|I_k\setminus E_r|/(1-r)} (1+x)^{\frac{2\alpha}{2\alpha-2}} dx \leq$$

$$\leq A_2(1-r)^{\frac{1-2\alpha}{1-\alpha}} \left(\sum_{|I_k\setminus E_r|<1-r} \frac{|I_k \setminus E_r|}{1-r} + \sum_{|I_k\setminus E_r|>1-r} \left(\frac{|I_k \setminus E_r|}{1-r} \right)^{\frac{1-2\alpha}{1-\alpha}} \right) \leq$$

$$\leq A_3 \sum_{k=1}^{\infty} |I_k \setminus E_r|^{\frac{1-2\alpha}{1-\alpha}}.$$

For each $k = 1, 2, \dots$ the equality

$$\lim_{r\to 1-} m(I_k \setminus E_r) = 0$$

holds. Since, by theorem's condition

$$\sum_{k=1}^{\infty} |I_k|^{\frac{1-2\alpha}{1-\alpha}} < \infty$$

so, the integral (5) goes to zero, if $r \to 1-0$.

Theorem 3. Let $2/5 < \alpha < 1/2$ and $2/\alpha < p+3 < 5$. Let

$$x(n) = f(n) + \xi(n), \quad n = 0, \pm 1, \dots$$

where

$$\{f(n)\}_{n=-\infty}^{\infty} \in LF(\alpha).$$

We suppose that $supp(F)$ contains only a finite number isolated points, where, in the sense of generalized functions,

$$F(z) = \sum_{n=-\infty}^{\infty} f(n) z^n, \quad |z| = 1.$$

The noise $\xi_k, \quad k = 0, \pm 1, \dots$ is a sequence of independent random variables, which have the same distributions with zero mean values, i.e. $\mathbf{E}(\xi_k) = 0, \quad k = 1, 2, \dots$ and $\mathbf{E}(|\xi_k|^p) < \infty$.

Then, by probability one we have

$$\lim_{r\to 1-} \int_{E_r} X(rz) z^{-n} |dz| = f(n), \quad n = 0, \pm 1, \dots$$

where

$$X(rz) = \sum_{n=-\infty}^{\infty} x(n) r^{|n|} z^n, \quad |z| = 1,$$

and

$$E_r = \{z; \quad |z| = 1, \quad (1-r)^{1/p} |X(rz)| > 1\}.$$

Proof. Let $E = supp(F)$ and $I_k, \quad k = 1, 2, \dots$ be the biggest arcs on the unit circle laying outside of the close subset E. Since $\{f(n)\}_{n=-\infty}^{\infty} \in LF(\alpha)$ so, by L. Carleson's theorem

$$\sum_{k=1}^{\infty} |I_k|^{\frac{1-2\alpha}{1-\alpha}} < \infty.$$

Let $\frac{1}{p} < \sigma < \frac{\alpha}{2-3\alpha}$. It is sufficient to prove, that for the subsets

$$G_r = \{z; \quad |X(rz)| > (1-r)^{-\sigma}\}, \quad 0 < r < 1,$$

by probability one we have

$$\lim_{r\to 1-0} \frac{|G_r|}{(1-r)^\sigma} = 0. \tag{6}$$

Thanks of theorem on Law of Large numbers, by probability one, we have

$$\max_t |u(re^{it})| = O\left(\frac{1}{(1-r)^\sigma}\right), \quad r \to 1-0,$$

where

$$u(rz) = \sum_{n=-\infty}^{\infty} \xi(n) r^{|n|} z^n, \quad |z| = 1,$$

So, (6) follows from the following one

$$\lim_{r\to 1-0} \frac{m(F_r)}{(1-r)^\sigma} = 0, \tag{7}$$

where

$$F_r = \{z; \quad |z| = 1, \quad (1-r)^\sigma |F(rz)| > 1\}.$$

Now, to prove (7) let us note, that for each point $z \in F_r \cap I_k$ we have

$$1 \le (1-r)^\sigma |F(rz)| \le A(1-r)^\sigma \left(\frac{1-|z|^2}{|\xi_k - rz|^2} + \frac{1-|z|^2}{|\eta_k - rz|^2}\right).$$

If

$$|I_k| > (1-r)^{(1+\sigma)/2}$$

then

$$m\left(\left\{z, \quad z \in I_k, \quad 1 \le A(1-r)^\sigma \left(\frac{1-|z|^2}{|\xi_k - rz|^2} + \frac{1-|z|^2}{|\eta_k - rz|^2}\right)\right\}\right) \le$$
$$\le B(1-r)^{(1+\sigma)/2},$$

where ξ_k and η_k are endpoints of the arc I_k. Consequently,

$$m(F_r) \le \sum_{|I_k| < (1-r)^{(1+\sigma)/2}} |I_k| + \sum_{|I_k| > (1-r)^{(1+\sigma)/2}} (1-r)^{(1+\sigma)/2}.$$

This inequality implies (7).

4 Uniqueness theorems for analytic functions

In this section we discuss a uniqueness theorem for some classes of analytic functions, where the conclusion $f(z) \equiv 0$ follows from the condition that $f(z_n)$ goes to zero over some sequence of points $z_n, \quad n = 1, 2, \ldots$. This type results are necessary in investigation of denoising problem, which we will discuss in the next section.

One can find the survey on this type uniqueness results in [5]. A new example is the given below theorem.

At first let us give here some auxiliary definitions and classical results.

Definition 3. Let $h(t), \quad 0 < t$, be a continuous, non-negative function. Let the family of arcs $\{S_k\}_{k=1}^\infty$ cover a given set E in unite disk;

$$E \subseteq \bigcup_{k=1}^\infty S_k.$$

Let us put

$$M_h(E) = \inf \sum_{k=1}^{\infty} h(|S_k|),$$

where $|S|$ is the length of the arc S and the infimum is taken over the family of all covers.

Definition 4. A function $f(t), \quad -\pi < t < \pi$, belongs to Besov space B_2^α if

$$||f||_\alpha^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x-y|^{1+2\alpha}} dxdy < \infty,$$

where $0 < \alpha < 1$.

For arbitrary function $g(t) \in L_1(-\pi, \pi)$ let us denote

$$g^*(t) = \sup_{0<\delta} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} |g(x)| dx,$$

where $g(t)$ is assumed to be continued as a 2π periodic function on $(-\infty, +\infty)$.

For a subset E the quantity

$$C_\alpha(E) = \left(\inf_\mu \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{d\sigma(x) d\sigma(y)}{|x-y|^\alpha} \right)^{-1},$$

where the infimum is taken over the probability measures with support in E, is known as $\alpha-$ capacity of the subset E.

The following lemma is announced in the book [1], p.35.

Lemma 1. Let E be Borelian set and $C_\alpha(E) = 0$, where $0 < \alpha < 1$. Let $0 \le h(r), \quad r > 0$, be an increasing function and

$$\int_0 \frac{dh(r)}{r^\alpha} < \infty.$$

Then

$$M_h(E) = 0.$$

Lemma 2. Let $f(x) \in B_2^\alpha$ and $0 < \beta \le \alpha$. Then there is a function $g(x) \in B_2^{\alpha-\beta}$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-|z|^2}{|z-e^{it}|^2} f(t) dt =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(t)}{(z-e^{it})^{1-\beta}} dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(t)}{(\bar{z}-e^{it})^{1-\beta}} dt, \quad |z| < 1.$$

Proof. Let to the function $f(x)$ correspond the following Fourier series

$$f(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{ikx}.$$

Since $f(x) \in B_2^\alpha$, so

$$\sum_{k=-\infty}^{\infty} |a_k|^2 |k|^{2\alpha} < \infty.$$

We can write

$$F(re^{ix}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ikx} =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\frac{\Gamma(1+|k|-\beta)}{\Gamma(1-\beta)\Gamma(1+|k|)}r^{|k|}e^{ik(x-t)}\right)g(t)dt,$$

where

$$g(t) = \sum_{k=-\infty}^{\infty} a_k \frac{\Gamma(1-\beta)\Gamma(1+|k|)}{\Gamma(1+|k|-\beta)}e^{ixk}.$$

Lemma 3. Let $f(x) \in B_2^{\alpha}$, where $0 < \alpha < 1$ and $0 < \beta \leq \alpha$. Then there is a subset $F \in \{z; |z| = 1\}$ with

$$C_{\alpha-\beta}(F) = 0$$

end for each $e^{ix} \notin F$ there is a number $A(x)$ such that

$$|F(z_1) - F(z_2)| < A(x)|z_1 - z_2|^{\alpha-\beta},$$

where

$$|e^{ix} - z_j| < 2(1 - |z_j|), \quad j = 1, 2,$$

holds.

Here $F(z)$ is harmonic function with the boundary values $f(x)$, i.e.

$$\lim_{r\to 1-} F(re^{ix}) = f(x).$$

Proof. Thanks of the previous lemma there is a function $g(x) \in B_2^{\alpha-\beta}$ such that

$$\frac{\partial F(z)}{\partial z} = \frac{1-\beta}{2\pi}\int_{-\pi}^{\pi}\frac{g(t)}{(z-e^{it})^{2-\beta}}dt.$$

Taking into account the inequalities

$$1 - |z| < |e^{it} - z|, \quad \left|e^{it} - \frac{z}{|z|}\right| \leq 2|e^{it} - z|,$$

for each $|z| < 1$, we have

$$\frac{(1-|z|)^{1-\beta}}{1-\beta}\left|\frac{\partial F(z)}{\partial z}\right| \leq$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-|z|)^{1-\beta}}{|z-e^{it}|^{2-\beta}}|g(t)|dt \leq$$

$$\leq \frac{1}{2\pi}\int_{|e^{it}-z/|z||\leq y}\frac{|g(t)|}{(1-|z|)}dt + \frac{1}{2\pi}\int_{|e^{it}-z/|z||>y}\frac{(1-|z|)^{1-\beta}}{|e^{it}-z/|z||^{2-\beta}}|g(t)|dt.$$

If $z, \quad |e^{ix} - z| \leq 2(1 - |z|)$, then putting $y = (1 - |z|)$ we get

$$(1-|z|)^{1-\beta}\left|\frac{\partial F(z)}{\partial z}\right| \leq$$

$$\leq \frac{1}{2\pi}\int_{|t-x|\leq(1-|z|)}\frac{|g(t)|}{(1-|z|)}dt + (1-|z|)^{1-\beta}\int_{1-|z|}^{2}\frac{1}{y^{2-\beta}}d\left(\int_{x-y}^{x+y}|g(t)|dt\right) \leq$$

$$\leq g^*(t) + (1-|z|)^{1-\beta}\int_{1-|z|}^{2}\frac{1}{y^{3-\beta}}\left(\int_{x-y}^{x+y}|g(t)|dt\right)dy.$$

Consequently,

$$(1-|z|)^{1-\beta}\left|\frac{\partial F(z)}{\partial z}\right| \leq g^*(x).$$

Let l be the linear interval with end points $z_1, z_2 \in \{z, \quad |e^{ix}-z| \leq 2(1-|z|)\}$. We have

$$|F(z_1)-F(z_2)| \leq \left|\int_l \frac{\partial F(z)}{\partial z} dz\right| \leq$$

$$\leq g^*(x)\int_l \frac{|dz|}{(1-|z|)^{1-\alpha+\beta}} \leq g^*(x)|z_1-z_2|^{\alpha-\beta}.$$

It is sufficient to note, that

$$C_{\alpha-\beta}(F)=0,$$

where $F=\{x; \quad g^*(x)=\infty\}$.

The proof of the following lemma may be found in [14].

Lemma 4. Let $g(t), \quad 0<t$, be positive and nondecreasing function. Let $u(z)$ be non-negative harmonic function defined on the unit disk. Then, the subset

$$F=\left\{\xi; |\xi|=1, \quad \sup_{z\in\Delta(\xi)} \frac{u(z)}{g(1-|z|)}=\infty\right\}$$

has zero Hausdorff's measure, i.e.

$$M_{h(t)}(F)=0,$$

where $h(t)=tg(t)$.

Theorem 5. Let $\{\xi_n\}$ be a sequence in the unit disk with

$$\lim_{n\to\infty}|\xi_n|=1.$$

Let E be a subset of unit circle such that for some continuous function $0\leq h(t), \quad 0<t$, satisfying the condition

$$\lim_{t\to 0+}\frac{h(t)}{t\log\frac{1}{t}}=0$$

we have $M_h(E)>0$.

Let for each point $y\in E$ there is a subsequence $\{\xi_{n_k}\}$ such that

$$\left|y-\frac{\xi_{n_k}}{\xi_{n_k}}\right|<2(1-|\xi_{n_k}|), \quad k=1,2,...$$

Let $0\leq\alpha<1$ be a fixed number and $f(z)$ is an analytic function with

$$\int_0^1\int_{-\pi}^{\pi}|f'(z)|^2(1-|z|)^\alpha dxdy<\infty,$$

and

$$\lim_{n\to\infty}f(\xi_n)=0 \tag{8}$$

then $f(z)\equiv 0$.

Proof. At first let us note, that instead of (8) we can assume

$$\lim_{n\to\infty}\frac{f(\xi_n)}{(1-|\xi_n|)^{\alpha-\beta}}=0,$$

where $0<\beta<\alpha$ is a constant. Indeed, By lemma 3 there is a subset F_1, with

$$C_{\alpha-\beta}(F_1)=0,$$

end for each $x \notin F_1$ there is a number $A(x) < \infty$ such that for arbitrary points z_1, z_2 from the unite disk, satisfying the condition

$$|e^{ix} - z_j| < 2(1 - |z_j|), \quad j = 1, 2,$$

we have

$$|f(z_1) - f(z_2)| < A(x)|z_1 - z_2|^{\alpha-\beta}.$$

In particularly, if $x \in E \setminus F_1$ then

$$|f(\xi_n)| < A(x)(1 - |\xi_n|)^{\alpha-\beta}$$

for each point ξ_n satisfying the condition

$$|e^{ix} - \xi_n| < 2(1 - |\xi_n|).$$

Thanks of lemma 1 we have

$$M_h(F_1) = 0$$

and so,

$$M_h(E \setminus F_1) > 0.$$

For each point $z, \quad |z| < 1$, let us denote

$$C(z) = \{w; \quad |w| = 1, \quad |w - z| \leq 2(1 - |z|)\}.$$

The zeros $\Lambda = \{z_k\}_{k=1}^{\infty}$ of our function $f(z)$ satisfy condition

$$\sum_{k=1}^{\infty}(1 - |z_k|) < \infty$$

and, (see [13]), we have also

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \rho_{\frac{1+\alpha}{2}}(e^{ix}, \Lambda)dx > -\infty,$$

where

$$\rho_\sigma(\xi, \Lambda) = \inf_{z \in \Lambda} \frac{|\xi - z|}{(1 - |z|)^\sigma}.$$

Let us consider a new function

$$G(w) = \sum_{k=1}^{\infty} \chi_k(w), \quad w \in \partial D,$$

where $\chi_k(w)$ is the characteristic function of the arc $C(z_k)$.

We want to prove that the subset

$$F_2 = \{w; \quad w \in \partial D, \quad G(w) = +\infty\}$$

satisfies the condition

$$M_{t \log 1/t}(F_2) = 0. \tag{9}$$

Let us suppose $M_{t \log 1/t}(F_2) > 0$. Then, see [1], p. 18, there is a compact subset $F \subset F_2$ for which

$$M_{t \log 1/t}(F) > 0.$$

For each natural N, the family of subsets $C(z_k), \quad k = N, N+1, \ldots$ cover F. By Alfor's theorem, see [1], from that family of arcs we can choose a finite number, which cover F and have a finite multiplicity less A, where A is an absolute constant. Let

$$C(z_{k_1}), \ldots, C(z_{k_m})$$

be the constructed subfamily, which cover the set F.

We can write

$$\left(1-\frac{1+\alpha}{2\sigma}\right)\sum_{j=1}^{m}|C(z_{n_j})|\log\frac{e}{|C(z_{n_j})|}\le$$
$$\le\left(1-\frac{1+\alpha}{2\sigma}\right)\sum_{j=1}^{m_1}\int_{C(z_{n_j})}\log^+\left(\frac{1}{|w-z_{n_j}|}\right)|dw|\le$$
$$\le\sum_{j=1}^{m_1}\int_{C(z_{n_j})}\log^+\left(\frac{(1-|z_{n_j}|)^{(1+\alpha)/2}}{|w-z_{n_j}|}\right)|dw|\le$$
$$\le A\int_{Q_N}\log^+\left(\sup_k\frac{(1-|z_k|)^{(1+\alpha)/2}}{|w-z_k|}\right)|dw|,$$

where

$$Q_N=\bigcup_{j=N}^{\infty}C(z_j).$$

Letting N to go infinity we get

$$M_{t\log 1/t}(F)=0.$$

The received contradiction proves (9). Consequently,

$$M_h(F_2)=0.$$

Thus, for each point $e^{ix}\in E\setminus F_2$ in domain

$$\{z;\quad |e^{ix}-z|<2(1-|z|)\}$$

there are only finite number zeros of the function $f(z)$.

By F. Riesz theorem we have the representation

$$f(z)=B(z)F(z),$$

where $B(z)$ is the Blashke product constructed by zeros $\{w_n\}$ of the function $f(z)$ and $F(z)\in H^\infty$, which has no zeros at all.

Let us denote

$$v(z)=\sum_{n=1}^{\infty}\frac{1-|z|^2}{\left|z-\frac{w_n}{|w_n|}\right|}(1-|w_n|),\quad |z|<1.$$

For arbitrary two points $z,\quad w$ from the unit disk we have

$$-\frac{(1-|z|^2)(1-|w|^2)}{|z-w|^2}\le-\log\left(1+\frac{(1-|z|^2)(1-|w|^2)}{|z-w|^2}\right)=\log\left|\frac{w-z}{1-\bar wz}\right|^2.$$

Let y be a point on the unit circle and n_0 be a natural number. Let for each $n=n_0,n_0+1,\ldots$ the inequalities

$$\left|y-\frac{w_n}{|w_n|}\right|\ge 4(1-|w_n|),\quad n=1,2,\ldots$$

hold. Then for arbitrary point $z,\quad |z|<1$, satisfying the condition

$$\left|y-\frac{z}{|z|}\right|\le 2(1-|z|).$$

there is a constant $C>0$ such that

$$-Cv(z)\le\log|B(z)|.$$

Thanks of the lemma 4, applied to the function $v(z)\ge 0$, we get that there is a subset $F_3\in\partial D$ with

$$M_{h(t)}(F_3)=0,$$

where $h(t) = tg(t)$ and for each point $y \in \partial D \setminus F_3$ we have

$$\sup \left\{ \frac{v(z)}{g(1-|z|)}; |z| < 1, \quad \left| y - \frac{z}{|z|} \right| \leq 2(1-|z|) \right\} < +\infty.$$

Consequently, there is a subset F_4 with

$$M_{h(t)}(F_4) = 0$$

and for each point $y \notin E \setminus F_4$ we have

$$\sup \left\{ \frac{|f(z)|}{(1-|z|)^{1-\alpha}}; \quad |z| < 1, \quad \left| y - \frac{z}{|z|} \right| \leq 2(1-|z|) \right\} < \infty.$$

These remarks contradict to the theorem conditions since

$$E \setminus (F_1 \cup F_2 \cup F_3 \cup F_4) \neq \emptyset.$$

5 Denoising problem for analytic functions

In this section we consider the following problem: let $f(z)$ be a bounded analytic function and $\{z_n\}_{n=1}^{\infty}$ be a sequence from the unit disk. Let we can calculate empirically the values of this function at the points $\{z_n\}_{n=1}^{\infty}$ with some error, i.e.

$$w_n = f(z_n) + \epsilon_n, \quad n = 1, \ldots,$$

where $\epsilon_n, \quad n = 1, \ldots$ is a sequence of independent random quantities with the same distribution and with mean value equal to zero. The following problem naturally arises: is it possible to choose the points $\{z_n\}_{n=1}^{\infty}$ in such a way that by observed quantities $w_n, \quad n = 1, 2, \ldots$ it will be possible to restore the function $f(z)$ by probability one?

The relation of this problem with the identification problen for linear bounded systems one can find in [16].

Here we give some classical results of Shizuo Kakutani, see[8], which play a principal role to answer this question.

Let Ω be an arbitrary set and let σ be a $\sigma-$field of subsets of Ω. Let $\Re(\sigma)$ be the family of all countably additive measures $\mu(d\omega)$ defined on σ for which $\mu(\Omega) = 1$.

Definition 5. Two measures $\mu, \nu \in \Re(\sigma)$ called orthogonal (notation $\mu \perp \nu$) if there are disjoint subsets $B, B' \in \sigma$ such that

$$\mu(B) = \nu(B') = 1.$$

Let $\mu, \nu \in \Re(\sigma)$ be measures on (Ω, σ). For arbitrary measure $\tau \in \Re(\sigma)$ such that μ and ν are absolutely continuous in respect to τ, let us denote

$$\rho(\mu, \nu) = \int_{\Omega} \sqrt{\frac{\mu(d\omega)}{\tau(d\omega)}} \sqrt{\frac{\nu(d\omega)}{\tau(d\omega)}} \tau(d\omega).$$

This integral doesn't depend upon the choice of the measure τ. That is way the following E.Hellinger's notation

$$\rho(\mu, \nu) = \int_{\Omega} \sqrt{\mu(d\omega)\nu(d\omega)}$$

is natural.

Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ be two family of probability measures on $\mathbf{C}$. Let us denote by $\mu = \mu_1 \times \mu_2 \times \ldots, \quad \nu = \nu_1 \times \nu_2 \times \ldots.$ the infinite direct products.

It is easy to see that if for some k_0 we have $\mu_{k_0} \perp \nu_{k_0}$ then $\mu \perp \nu$. The case $\mu_k \sim \nu_k, \quad k = 1, \ldots$ was considered in [12].

Theorem (S. Kakutani). Let $\mu_k \sim \nu_k$ for all $k = 1, \ldots$. Then the measures μ and ν are equivalent if and only if

$$\prod_{k=1}^{\infty} \rho(\mu_k, \nu_k) > 0.$$

Otherwise those measures are orthogonal, i.e. $\mu \perp \nu$.

Here, we need only the following particular case of S. Kakutani's Theorem. Let $f(z), g(z) \quad |z| < 1$, be bounded analytic functions and $z_k, \quad k = 1, 2, \ldots$ be points in the unite disk. Let

$$d\mu_k(z) = P(z - f(z_k))dxdy, \quad z = x + iy$$

and

$$d\nu_k(z) = P(z - g(z_k))dxdy,$$

where $P(z) \geq 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(z)dxdy = 1.$$

We have

$$\rho(d\mu_k, d\nu_k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{P(z - f(z_k))P(z - g(z_k))}dxdy.$$

and

$$1 - \rho(d\mu_k, d\nu_k) =$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\sqrt{P(z - f(z_k))} - \sqrt{P(z - g(z_k))} \right)^2 dxdy \geq A|f(z_k) - g(z_k)|^2$$

for some number $A > 0$.

The corresponding infinite products of measures are orthogonal if

$$\sum_{k=1}^{\infty} |f(z_k) - g(z_k)|^2 = \infty.$$

So, if we have

$$w_n = H(z_n) + \epsilon_n, \quad n = 1, \ldots,$$

then by probability one, it is possible to identify $H(z)$ with some $f(z)$, if the points $\{z_n\}_{n=1}^{\infty}$ are possible to choose in such a way, that from the condition $f(z), g(z) \in H^{\infty}$ and

$$\sum_{k=1}^{\infty} |f(z_k) - g(z_k)|^2 < \infty$$

it follows $f(z) \equiv g(z)$. This note pernits to formulate the following result.

Theorem 6. Let $0 \leq \alpha < 1$ be a fixed number.

Let $\{z_n\}$ be a sequence in the unit disk with

$$\lim_{n \to \infty} |z_n| = 1.$$

Let E be a subset of unit circle such that for some continuous function $0 \leq h(t), \quad 0 < t$, satisfying the condition

$$\lim_{t \to 0+} \frac{h(t)}{t \log \frac{1}{t}} = 0,$$

we have

$$M_h(E) > 0.$$

Let for each point $y \in E$ there is a subsequence $\{z_{n_k}\}$ such that

$$\left| y - \frac{z_{n_k}}{z_{n_k}} \right| < 2(1 - |z_{n_k}|), \quad k = 1, 2, ...$$

Let we can observe the quantities

$$X_n = S(z_n) + \xi_n, \quad n = 1, 2, \ldots$$

where $\xi_n, \quad n = 1, 2, \ldots$ are independent random variables with the same absolutely continuous distributions and zero mean values.

Let $S(z)$ be a bounded analytic function and

$$\int_0^1 \int_{-\pi}^{\pi} |S'(z)|^2 (1 - |z|)^\alpha dxdy < \infty, \quad 0 < \alpha < 1.$$

Then, by $X_n, \quad n = 1, 2, \ldots$ it is possible to restore the function $S(z)$ by probability one.

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